A direct proof of the Yff's conjecture

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Abstract: *In this paper it is given proof Yff's conjecture using convexity arguments*.

Let the triangle ABC and the point $\Omega \in Int\triangle ABC$ the have

Theorem: $2\omega \leq \sqrt[3]{ABC}$ $\quad$ (*the Yff's inequality*) (1)

$\sphericalangle \Omega AB = \sphericalangle \Omega BC = \sphericalangle \Omega CA = \omega$

Proof: Following the sin theorem in the triangles: $\Omega AB, \Omega BC$ and ΩCA to obtain:

$\frac{\sin\omega}{\sin(B-\omega)} \bullet \frac{\sin\omega}{\sin(C-\omega)} \bullet \frac{\sin\omega}{\sin(A-\omega)} = \frac{\Omega B}{\Omega A} \bullet \frac{\Omega C}{\Omega B} \bullet \frac{\Omega A}{\Omega C} = 1$ to rezult::

$$\sin^3\omega = \sin(A-\omega)\sin(B-\omega)\sin(C-\omega) \qquad (2)$$

let $f(x) = \ln[\sin(e^x - \omega)]$; $x \in (\ln\omega; \infty)$

to obtain: $f'(x) = \frac{e^x\cos(e^x-\omega)}{\sin(e^x-\omega)}$

and $f''(x) = \frac{(e^x\cos(e^x-\omega)-e^{2x}\sin(e^x-\omega))\bullet\sin(e^x-\omega)-e^{2x}\cos^2(e^x-\omega)}{\sin^2(e^x-\omega)} =$

$\frac{e^x\cos(e^x-\omega)\sin(e^x-\omega)-e^{2x}\left[\sin^2(e^x-\omega)+\cos^2(e^x-\omega)\right]}{\sin^2(e^x-\omega)} =$

$= \frac{e^x[\cos(e^x-\omega)\sin(e^x-\omega)-e^x]}{\sin^2(e^x-\omega)} \leq 0$

let $g(x) = \cos(e^x-\omega)\sin(e^x-\omega) - e^x$ and

$g'(x) = e^x \bullet \cos^2(e^x-\omega) - e^x \bullet \sin^2(e^x-\omega) - e^x =$

$= e^x[\cos(2e^x - 2\omega) - 1] \leq 0$ to rezult g it's decreasing and $x \geq \ln\omega$ to obtain:

$g(x) \leq g(\ln\omega) = \cos 0 \sin 0 - \omega = -\omega < 0$, because $e^{\ln\omega} = \omega$

to rezult of the function f it's concave , acording the Jensen inequality to obtain:

$$\ln\sin(e^x-\omega) + \ln\sin(e^y-\omega) + \ln\sin(e^z-\omega) \leq 3 \bullet \ln\sin(e^{\frac{x+y+z}{3}} - \omega) \quad ; \qquad (3)$$

for x=ln$A$, y=ln$B$ and z=ln$C$ to obtain the inequality:

$$\sin(A-\omega)\sin(B-\omega)\sin(C-\omega) \leq \sin^3(\sqrt[3]{ABC} - \omega) \qquad (4)$$

or ,$\sin^3\omega \le \sin^3(\sqrt[3]{ABC} - \omega)$ to rezult $\sin\omega \le \sin^3(\sqrt[3]{ABC} - \omega)$

because the function sin it's increasing for $x \in \left[0; \frac{\pi}{2}\right]$

and well-known $\omega \le \frac{\pi}{6}$ and $\sqrt[3]{ABC} - \omega \in (0; \frac{A+B+C}{3} - \omega) \subset (0; \frac{\pi}{2})$

to obtain: $\omega \le \sqrt[3]{ABC} - \omega$ or $2\omega \le \sqrt[3]{ABC}$ the Yff's inequality

References:

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