A mathematical model of the quark and lepton mixing angles

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Abstract

A single mathematical model encompassing both quark and lepton mixing is described. This model exploits the fact that when a 3×3 rotation matrix whose elements are squared is subtracted from its transpose, a matrix is produced whose non-diagonal elements have a common absolute value, where this value is an intrinsic property of the rotation matrix. For the traditional CKM quark mixing matrix with its second and third rows interchanged (i.e., c − t interchange), this value equals one-third the corresponding value for the leptonic matrix (roughly, 0.05 versus 0.15). By imposing this and two additional related constraints on mixing, and letting leptonic φ_{23} be maximal, a framework is defined possessing just two free parameters. A mixing model is then specified using values for these two parameters that derive from the solution to a simple equation, where this solution also accurately reproduces the fine structure constant. The resultant model, which is entirely free from parameters adjusted to fit the mixing data, possesses the following angles $\theta_{23} = 2.367442°$, $\theta_{13} = 0.190986°$, $\theta_{12} = 12.920966°$, $\varphi_{23} =$ maximal, $\varphi_{13} = 0.013665°$, and $\varphi_{12} = 33.210911°$, which fit the experimental quark and lepton mixing angles. At the time of its introduction in 2007, this model had a 7.0 σ disagreement with the value for $|V_{ub}|$, whereas a revised value for $|V_{ub}|$ from the same source now yields a disagreement of just 1.6 σ.

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I. INTRODUCTION

The phenomenon of mixing [1–3] has been explored with the aid of a wide variety of physical models [4]. However, an alternative approach to understanding mixing is available: the mathematics of rotation matrices, on the one hand, and the quark and lepton mixing data, on the other, may be analyzed apart from the Standard Model to see what they can reveal about each other. As this article will show, even this limited approach may produce results worthy of note. Specifically, we will demonstrate that if a 3×3 rotation matrix whose elements are squared is subtracted from its transpose, a matrix is produced whose non-diagonal elements possess a common absolute value, where this value is an intrinsic property of the rotation matrix. For the mixing matrices determined by experiment this value in the leptonic sector measures three times its value in the quark sector.

However, generally speaking, the above property has not been discussed in the mixing literature. Why should this be so? It is perhaps because it is only if one builds the CKM quark mixing matrix with its c- and t-quarks interchanged relative to convention, that this distinguishing property for leptons (roughly 0.15) [3] is readily seen as three times that for quarks (roughly 0.05) [2]. When the quark mixing matrix is built in the traditional manner, no such obvious relation presents itself. Of course, the above relation might be coincidental, but given that the traditional assignment of the c-quark to the 2^{nd} generation and the t-quark to the 3^{rd} is arbitrary, there is no reason such an exchange should not be made.

In this article, a single mathematical model encompassing both quark and lepton mixing will be described. The model is defined with the aid of, and distinguished by, three constraints that each exploit the property described above. This allows all six model angles to arise naturally within a common mathematical framework. This method follows earlier work by the author, which predicted mixing angles identical to those of this article [5]. At the time of its introduction in 2007, this model had a 7.0 σ disagreement with the value for $|V_{ub}|$, whereas a revised value for $|V_{ub}|$ from the same source now yields a disagreement of just 1.6 σ.

II. QUARK AND LEPTON MIXING MATRICES

A mixing matrix without phase is a merely a 3×3 rotation matrix. Let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{1}$$

be such a matrix, so that squaring its elements gives

$$\begin{bmatrix} r_{11}^2 & r_{12}^2 & r_{13}^2 \\ r_{21}^2 & r_{22}^2 & r_{23}^2 \\ r_{31}^2 & r_{32}^2 & r_{33}^2 \end{bmatrix} \quad .$$

Given that a rotation matrix with its elements squared has rows and columns that sum to one, the above matrix can also be written

$$\begin{bmatrix} r_{11}^2 & 1 - r_{11}^2 - r_{13}^2 & r_{13}^2 \\ 1 - r_{11}^2 - r_{31}^2 & r_{11}^2 + r_{13}^2 + r_{31}^2 + r_{33}^2 - 1 & 1 - r_{13}^2 - r_{33}^2 \\ r_{31}^2 & 1 - r_{31}^2 - r_{33}^2 & r_{33}^2 \end{bmatrix} \quad .$$

When this matrix is subtracted from its transpose it gives

$$\begin{bmatrix} 0 & r_{31}^2 - r_{13}^2 & r_{13}^2 - r_{31}^2 \\ r_{13}^2 - r_{31}^2 & 0 & r_{31}^2 - r_{13}^2 \\ r_{31}^2 - r_{13}^2 & r_{13}^2 - r_{31}^2 & 0 \end{bmatrix} \quad ,$$

a matrix whose non-diagonal elements all equal

$$\pm \left(r_{31}^2 - r_{13}^2 \right) \quad .$$

It follows that $r_{31}^2 - r_{13}^2$ is an intrinsic property of the rotation matrix R.

Now assume that R is produced by rotations through the angles ψ_{23}, ψ_{13}, and ψ_{12}, so that

$$R = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix} \quad , \tag{2}$$

where $s_{12} \equiv \sin\psi_{12}$, $c_{12} \equiv \cos\psi_{12}$, etc. For R define

$$\begin{aligned}\Delta P_{\psi_{23},\psi_{13},\psi_{12}} &= r_{31}^2 - r_{13}^2 \\ &= (s_{12}s_{23} - c_{12}c_{23}s_{13})^2 - (s_{13})^2 \quad ,\end{aligned} \tag{3}$$

its degree of "coupling asymmetry."

Additionally, define the quark mixing matrix [2]

$$V = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix} \quad , \tag{4}$$

where, now, $s_{12} \equiv \sin\theta_{12}$, $c_{12} \equiv \cos\theta_{12}$, etc., and where θ_{23}, θ_{13}, and θ_{12} are the quark mixing angles.

Similarly, define the leptonic mixing matrix [3]

$$U = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{bmatrix} \quad , \tag{5}$$

where $s_{12} \equiv \sin\varphi_{12}$, $c_{12} \equiv \cos\varphi_{12}$, etc., and where φ_{23}, φ_{13}, and φ_{12} are the leptonic mixing angles. Note that, above, phase is omitted for both matrices.

With the aid of the above definitions we can now illustrate how $\Delta P_{\psi_{23},\psi_{13},\psi_{12}}$ will be exploited in this article; note that in this section the values of all matrix elements are approximate. Consider first the mixing matrix that arises if the usual CKM quark mixing matrix with its elements squared [2]

$$\begin{array}{c c} & \begin{array}{ccc} d & s & b \end{array} \\ \begin{array}{c} u \\ c \\ t \end{array} & \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0.05 & 0.95 & 0 \\ 0 & 0 & 1.00 \end{bmatrix} \end{array}$$

has its second and third rows (i.e., its c- and t-quarks) interchanged

$$\begin{array}{c c} & \begin{array}{ccc} d & s & b \end{array} \\ \begin{array}{c} u \\ t \\ c \end{array} & \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0 & 1.00 \\ 0.05 & 0.95 & 0 \end{bmatrix} \end{array}$$

(equivalent to applying an $\pi/2$ offset to θ_{23}). Subtracting this second matrix from its transpose gives

$$\begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0 & 1.00 \\ 0.05 & 0.95 & 0 \end{bmatrix} - \begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.05 & 0 & 0.95 \\ 0 & 1.00 & 0 \end{bmatrix} = \begin{bmatrix} 0 & +0.05 & -0.05 \\ -0.05 & 0 & +0.05 \\ +0.05 & -0.05 & 0 \end{bmatrix} \tag{6}$$

where

$$\Delta P_{\frac{\pi}{2}+\theta_{23},\theta_{13},\theta_{12}} = 0.05 \quad . \tag{7}$$

In the same way, consider the mixing matrix that arises if the usual leptonic mixing matrix with its elements squared [3]

$$\begin{array}{cc} & \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} & \begin{bmatrix} 0.70 & 0.30 & 0 \\ 0.15 & 0.35 & 0.50 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \end{array}$$

has its second and third rows (i.e., ν_μ and ν_τ) interchanged

$$\begin{array}{cc} & \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{array}{c} \nu_e \\ \nu_\tau \\ \nu_\mu \end{array} & \begin{bmatrix} 0.70 & 0.30 & 0 \\ 0.15 & 0.35 & 0.50 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} \end{array}$$

(equivalent to applying an offset of, say, $-\pi/2$ to φ_{23}). Subtracting *either* of these matrices from its transpose gives

$$\begin{bmatrix} 0.70 & 0.30 & 0 \\ 0.15 & 0.35 & 0.50 \\ 0.15 & 0.35 & 0.50 \end{bmatrix} - \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.30 & 0.35 & 0.35 \\ 0 & 0.50 & 0.50 \end{bmatrix} = \begin{bmatrix} 0 & +0.15 & -0.15 \\ -0.15 & 0 & +0.15 \\ +0.15 & -0.15 & 0 \end{bmatrix} \tag{8}$$

where

$$\Delta P_{\varphi_{23}-\frac{\pi}{2},\varphi_{13},\varphi_{12}} = 0.15 \quad . \tag{9}$$

(So, as it happens, for the leptonic mixing matrix, interchanging the second and third rows has little or no effect on ΔP.)

In this way, the calculation of ΔP for quark and lepton mixing leads to Eqs. (7) and (9), which combine to suggest the relation

$$\Delta P_{\varphi_{23}-\frac{\pi}{2},\varphi_{13},\varphi_{12}} = 3 \times \Delta P_{\frac{\pi}{2}+\theta_{23},\theta_{13},\theta_{12}} \quad . \tag{10}$$

This, in turn, raises the question of whether Eq. (10) constitutes a precise physical law.

III. IMPOSING SIX INDEPENDENT CONSTRAINTS ON THE SIX MIXING ANGLES

We will now show how to specify a particular set of six mixing angles by imposing on them six independent constraints, three of which will assume the form of Eq. (10).

Firstly, let

$$\varphi_{23} = \frac{\pi}{4} + \frac{\pi}{2} \quad , \tag{11}$$

and note that the addition of the offset $\pi/2$ to φ_{23} distinguishes its use here from convention [3].

Secondly, following Eq. (10), let

$$\Delta P_{\varphi_{23}-\frac{\pi}{2},\varphi_{13},\varphi_{12}} = N \times \Delta P_{\frac{\pi}{2}+\theta_{23},\theta_{13},\theta_{12}} \tag{12}$$

to constrain all six mixing angles, where $N = 3$.

Lastly, generate four mixing angles with the aid of g_{12} and g_{13}, model parameters whose possible values will be examined later in detail. Their subscripts were chosen because g_{12} helps define the mixing angles φ_{12} and θ_{12}, whereas g_{13} helps define the mixing angles φ_{13} and θ_{13}:

$$\sin\varphi_{12} = \sqrt{g_{12}N} \quad , \tag{13}$$

$$\sin\theta_{12} = \sqrt{g_{12}} \times \sin\varphi_{23} \quad , \tag{14}$$

$$\sin\theta_{13} = \sqrt{g_{13}/N} \quad , \tag{15}$$

$$\sin\varphi_{13} = \sqrt{g_{13}} \times \sin\theta_{23} \quad . \tag{16}$$

Equations (11)–(16) together supply the six constraints needed to determine the quark and lepton mixing angles. Given that φ_{23} has its value explicitly assigned by Eq. (11), the three angles specifed by Eqs. (13)–(15) are readily calculated. In contrast, Eq. (16) must be solved simultaneously with Eq. (12).

It is important to recognize that Eqs. (13)–(16) were chosen only because they automatically impose the following additional two constraints on quark and lepton coupling asymmetry

$$\Delta P_{\varphi_{23},0,\varphi_{12}} = N \times \Delta P_{\frac{\pi}{2},0,\theta_{12}} \quad , \tag{17}$$

$$\Delta P_{-\frac{\pi}{2},\varphi_{13},0} = N \times \Delta P_{\theta_{23},\theta_{13},0} \quad . \tag{18}$$

It is these twin constraints in combination with the related constraint, Eq. (12), that constitute the conceptual key to the parameterization described by this article.

Below, the constraints imposed by Eqs. (12), (17), and (18) are expressed in a way that makes it easier to compare the angles they employ. Observe, particularly, that the angles of the first row equal the sum of the angles of the second and third rows:

$$\begin{array}{llll}
\Delta P \text{ for } (\ \varphi_{23} - \frac{\pi}{2},\ \ \varphi_{13},\ \ \varphi_{12}\ ) & \text{equals } N \text{ times } \Delta P \text{ for } (\ \frac{\pi}{2} + \theta_{23},\ \ \theta_{13},\ \ \theta_{12}\ ) & , \\
\Delta P \text{ for } (\ \varphi_{23}\ \ \ \ \ \ \ ,\ \ 0,\ \ \varphi_{12}\ ) & \text{equals } N \text{ times } \Delta P \text{ for } (\ \frac{\pi}{2}\ \ \ \ \ \ \ ,\ \ 0,\ \ \theta_{12}\ ) & , \\
\Delta P \text{ for } (\ \ \ \ \ \ \ - \frac{\pi}{2},\ \ \varphi_{13},\ \ 0\ ) & \text{equals } N \text{ times } \Delta P \text{ for } (\ \ \ \ \ \ \ \theta_{23},\ \ \theta_{13},\ \ 0\ ) & .
\end{array}$$

Finally, consider that, above, the value for ΔP in the lepton sector consistently equals N times ΔP in the quark sector, just as the electric charge of the lepton sector is N times that of the quark sector

$$-1 + 0 = N \times \left(\frac{1}{3} + -\frac{2}{3} \right) \quad , \tag{19}$$

where $N = 3$. Given that $N = 3$ reflects the number of quark colors needed to balance the amount of electric charge possessed by the two sectors, it is reasonable to conjecture that it may fulfill an equivalent role with regard to coupling asymmetry. That is to say, it may balance the amount of coupling asymmetry possessed by the leptonic sector against that possessed by the quark sector.

IV. THE SECONDARY COUPLING CONSTRAINTS

To see how Eqs. (17) and (18) derive from Eqs. (13)–(16), consider that according to Eq. (3)

$$\Delta P_{\psi_{23},\psi_{13},\psi_{12}} = (\sin\psi_{12}\sin\psi_{23} - \cos\psi_{12}\cos\psi_{23}\sin\psi_{13})^2 - \sin^2\psi_{13} \quad . \tag{20}$$

Substitution reveals that

$$\Delta P_{\varphi_{23},0,\varphi_{12}} = \sin^2\varphi_{12}\sin^2\varphi_{23} \quad , \tag{21}$$

$$\Delta P_{\frac{\pi}{2},0,\theta_{12}} = \sin^2\theta_{12} \quad , \tag{22}$$

$$\begin{aligned}\Delta P_{\theta_{23},\theta_{13},0} &= (-\cos\theta_{23}\sin\theta_{13})^2 - \sin^2\theta_{13} \\ &= \sin^2\theta_{13}\times(\cos^2\theta_{23}-1) \quad , \\ &= -\sin^2\theta_{13}\sin^2\theta_{23} \quad , \end{aligned} \tag{23}$$

$$\Delta P_{-\frac{\pi}{2},\varphi_{13},0} = -\sin^2\varphi_{13} \quad . \tag{24}$$

Equations (13) and (21) give

$$\frac{\Delta P_{\varphi_{23},0,\varphi_{12}}}{\sin^2\varphi_{23}} \times \frac{1}{N} = g_{12} \quad , \tag{25}$$

and Eqs. (14) and (22) give

$$\frac{\Delta P_{\frac{\pi}{2},0,\theta_{12}}}{\sin^2\varphi_{23}} = g_{12} \quad , \tag{26}$$

so that Eqs. (25) and (26) recover Eq. (17). Equations (15) and (23) give

$$\frac{\Delta P_{\theta_{23},\theta_{13},0}}{\sin^2\theta_{23}} \times N = -g_{13} \quad , \tag{27}$$

and Eqs. (16) and (24) give

$$\frac{\Delta P_{-\frac{\pi}{2},\varphi_{13},0}}{\sin^2\theta_{23}} = -g_{13} \quad , \tag{28}$$

so that Eqs. (27) and (28) recover Eq. (18).

In this way Eqs. (13)–(16) assure that various g_{12} and g_{13} produce mixing angles that fulfill the *secondary coupling constraints* represented by Eqs (17) and (18). However, the values chosen for g_{12} and g_{13} must also fulfill Eq. (12), the *primary coupling constraint*, while simultaneously fulfilling Eq. (16).

V. THE MIXING MODEL

With the aid of the set of Eqs. (11)–(16), one need only specify g_{12} and g_{13} in order to single out a particular set of six mixing angles. As trial values, let

$$g_{12} = \frac{1}{9} \tag{29}$$

and

$$g_{13} = 0 \quad . \tag{30}$$

Observe that in the leptonic sector these values produce the following magnitudes

$$U_{g_{12}=\frac{1}{9}, g_{13}=0} = \begin{array}{cc} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} & \\ \begin{bmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ \sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \\ \sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \end{bmatrix} & \begin{array}{c} \nu_e \\ \nu_\tau \\ \nu_\mu \end{array} \end{array} \quad , \tag{31}$$

which equal those of the well known "tri-bimaximal" mixing matrix [6]. Given that φ_{23} is defined with a $\pi/2$ offset relative to convention (see Eq. (11)), ν_μ and ν_τ are interchanged above [3]. Equations (11)–(16) reveal that $\theta_{23} = \theta_{13} = \varphi_{13} = 0$.

As it turns out, $g_{12} = \frac{1}{9}$ produces a quark mixing matrix whose θ_{12} is too large, but if we alter g_{12} so that

$$g_{12} = \frac{1}{10} \tag{32}$$

and

$$g_{13} = 0 \tag{33}$$

the following magnitudes for the quark mixing matrix

$$V_{g_{12}=\frac{1}{10}, g_{13}=0} = \begin{array}{cc} \begin{array}{ccc} d & s & b \end{array} & \\ \begin{bmatrix} \sqrt{0.95} & \sqrt{0.05} & 0 \\ \sqrt{0.05} & \sqrt{0.95} & 0 \\ 0 & 0 & 1.00 \end{bmatrix} & \begin{array}{c} u \\ c \\ t \end{array} \end{array} \tag{34}$$

and leptonic mixing matrix

$$U_{g_{12}=\frac{1}{10}, g_{13}=0} = \begin{array}{cc} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} & \\ \begin{bmatrix} \sqrt{0.70} & \sqrt{0.30} & 0 \\ \sqrt{0.15} & \sqrt{0.35} & \sqrt{0.50} \\ \sqrt{0.15} & \sqrt{0.35} & \sqrt{0.50} \end{bmatrix} & \begin{array}{c} \nu_e \\ \nu_\tau \\ \nu_\mu \end{array} \end{array} \tag{35}$$

are produced. Note that the above matrix elements are exact and that ν_μ and ν_τ are interchanged. Equations (11)–(16) again require that $\theta_{23} = \theta_{13} = \varphi_{13} = 0$.

The above mixing matrices prove close enough to their corresponding experimental values [2, 3, 7] to suggest that with the right value for g_{13} it might be possible to generate matrices that fit all six mixing angles.

VI. ASSIGNING VALUES TO g_{12} AND g_{13}

Although the values for g_{12} and g_{13} may be adjusted to fit experiment, it is perhaps significant that a key solution to the following equation

$$\begin{aligned} \frac{(A^3 - x^3)}{3^3} + \left(A^2 - x^3\right) &= \frac{(A-y)^3}{3^3} + (A-y)^2 \\ &= \frac{1}{\alpha} \end{aligned} \tag{36}$$

closely reproduces the value of the fine structure constant α, while simultaneously producing values for x and y that are suitable for g_{12} and g_{13}, respectively.

To see how this is so, note that for Eq. (36) the expression on the left is very similar to that on the right, where, for $A \gg 1$, solutions exist where x^3 and y assume small, but slightly different, positive values. It is logical, therefore, to consider how x and y interrelate.

As it turns out

$$y \approx \left(\frac{28}{18+A}\right)\left(\frac{1}{3A}\right)\left(x^3\right) \tag{37}$$

closely approximates y for small x^3. So, by way of example, for

$$A = 9 \tag{38}$$

and

$$x = \frac{1}{A} = \frac{1}{9} \tag{39}$$

Eq. (37) gives

$$y \approx \left(\frac{28}{18+9}\right)\left(\frac{1}{3\times 9}\right)\left(\frac{1}{9^3}\right) = \frac{1}{18980.03\ldots} \quad ,$$

whereas Eq. (36) requires only this slightly different value

$$y = \frac{1}{18979.96\ldots} \quad .$$

Now, the key point is that in Eq. (37) the expression $28/\left(18+A\right)$ equals unity only for

$$A = 10 \quad . \tag{40}$$

It follows that if

$$x = \frac{1}{A} = \frac{1}{10} \tag{41}$$

then Eq. (36) simplifies to

$$y \approx \left(\frac{28}{18+10}\right)\left(\frac{1}{3\times 10}\right)\left(\frac{1}{10^3}\right) \quad , \tag{42}$$

which reduces to

$$y \approx \frac{1}{30000} \quad ,$$

while Eq. (36) actually requires that

$$y = \frac{1}{29999.93\ldots} \quad .$$

We can then take advantage of the above special solution to define

$$g_{12} = x \quad , \tag{43}$$

and

$$g_{13} = y \quad , \tag{44}$$

the values that the mixing model will exploit to generate five of the six experimental mixing angles, with the remaining angle predefined as $\varphi_{23} = \frac{\pi}{4} + \frac{\pi}{2}$.

It follows that the above constants are not freely adjusted to fit the mixing data, but instead arise naturally from the above special solution to Eq. (36). But, as noted earlier, Eq. (36) *simultaneously* produces the following excellent approximation of the fine structure constant reciprocal

$$\frac{10^3 - 10^{-3}}{3^3} + 10^2 - 10^{-3} = \frac{999.999}{3^3} + 99.999 = 137.036 \quad , \tag{45}$$

or, equivalently, from the right side of Eq. (36)

$$\left(\frac{10 - 1/29999.93\ldots}{3}\right)^3 + (10 - 1/29999.93\ldots)^2 = 137.036 \quad . \tag{46}$$

(The above value for the fine structure constant inverse differs only slightly from its 2006 CODATA value of 137.035 999 679, with an error of about 2 parts per billion; the fine structure constant reciprocal is known to 0.68 ppb [9].) Particularly, note that the following four constants

$$1/g_{12}N \quad , \quad g_{13}/N \quad , \quad 1/g_{12} \quad , \quad \text{and } g_{13} \quad ,$$

which occur in Eq. (46), also arise independently in some form in Eqs. (13)–(16) and Eqs. (25)–(28). In this way these four constants provide a nexus between the fine structure constant and the mixing model.

Accordingly, the above mixing model employs no arbitrary parameters, but rather employs constants of independent mathematical interest, though at this stage no physical explanation can be given for the remarkable behavior of Eq. (36) in generating these useful physical constants.

VII. MODEL PREDICTIONS

The assignments of Eqs. (43)–(44) produce mixing matrices that are close to those determined by experiment. *Omitting phase*, the above mixing model predicts the following magnitudes for the quark mixing matrix elements

$$V_{g_{12}=\frac{1}{10},g_{13}=\frac{1}{30000}} = \begin{array}{c} \begin{array}{ccc} \quad d \quad & \quad s \quad & \quad b \quad \end{array} \\ \begin{bmatrix} 0.974674 & 0.223606 & 0.003333 \\ 0.223550 & 0.973817 & 0.041308 \\ 0.005991 & 0.041007 & 0.999141 \end{bmatrix} \end{array} \begin{array}{c} \\ u \\ c \\ t \end{array} \tag{47}$$

while for leptonic mixing it predicts

$$U_{g_{12}=\frac{1}{10},g_{13}=\frac{1}{30000}} = \begin{array}{c} \begin{array}{ccc} \quad \nu_1 \quad & \quad \nu_2 \quad & \quad \nu_3 \quad \end{array} \\ \begin{bmatrix} 0.836660 & 0.547723 & 0.000238 \\ 0.387157 & 0.591700 & 0.707107 \\ 0.387439 & 0.591516 & 0.707107 \end{bmatrix} \end{array} \begin{array}{c} \\ \nu_e \\ \nu_\tau \\ \nu_\mu \end{array} \tag{48}$$

again with $\nu_\mu - \nu_\tau$ interchange. These matrices employ mixing angles of

$$\theta_{23} = 2.367442^\circ \quad , \qquad \theta_{13} = 0.190986^\circ \quad , \qquad \theta_{12} = 12.920966^\circ \quad ,$$
$$\varphi_{23} = 135^\circ \quad , \qquad \varphi_{13} = 0.013665^\circ \quad , \qquad \varphi_{12} = 33.210911^\circ \quad ,$$

with only φ_{23} exceeding convention by 90°.

For quark mixing, the model predicts the following sines squared

$$\sin^2\theta_{12} = g_{12} \times \sin^2\varphi_{23} = \frac{1}{20} \quad ,$$
$$\sin^2\theta_{23} = 1.706342 \times 10^{-3} \quad ,$$
$$\sin^2\theta_{13} = \frac{g_{13}}{N} = \frac{1}{90000} \quad .$$

For leptonic mixing the model predicts the following sines squared

$$\sin^2 \varphi_{12} = N g_{12} = \frac{3}{10} \quad ,$$
$$\sin^2 \varphi_{23} = \frac{1}{2} \quad ,$$
$$\sin^2 \varphi_{13} = 5.687808 \times 10^{-8} \quad ,$$

which can be compared against these 2008 corresponding experimental values [7]

$$\sin^2 \varphi_{12} = 0.304^{+0.022}_{-0.016} \quad ,$$
$$\sin^2 \varphi_{23} = 0.50^{+0.07}_{-0.06} \quad ,$$
$$\sin^2 \varphi_{13} = 0.01^{+0.016}_{-0.011} \quad ,$$

from which they differ by less than one standard deviation.

Finally, for quark mixing, the model also predicts the CKM matrix elements

$$|V_{us}| = 2.236 \times 10^{-1} \quad , \quad |V_{ub}| = 3.33 \times 10^{-3} \quad , \quad |V_{cb}| = 4.13 \times 10^{-2} \quad ,$$

each of which is unaffected by phase, and each of which derives primarily from a different mixing angle. Given that the model described by this article was first introduced in 2007 [5], it is logical to begin by comparing these predictions against 2006 CKM mixing data [8], which was then current.

As it turns out, the model differs from these 2006 data

$$|V_{us}| = \left(2.272^{+0.01}_{-0.01}\right) \times 10^{-1}, \quad |V_{ub}| = \left(3.96^{+0.09}_{-0.09}\right) \times 10^{-3}, \quad |V_{cb}| = \left(4.221^{+0.010}_{-0.080}\right) \times 10^{-2}$$

by 3.6, 7.0, and 1.1 standard deviations, respectively, a large discrepancy.

However, this same source also offers 2008 data [2]

$$|V_{us}| = \left(2.257^{+0.01}_{-0.01}\right) \times 10^{-1}, \quad |V_{ub}| = \left(3.59^{+0.16}_{-0.16}\right) \times 10^{-3}, \quad |V_{cb}| = \left(4.15^{+0.10}_{-0.11}\right) \times 10^{-2} \quad ,$$

from which the model differs by just 2.1, 1.6, and 0.2 standard deviations, respectively. It is particularly striking that the discrepancy regarding $|V_{us}|$ is reduced from 3.6 to 2.1 standard deviations, and that for $|V_{ub}|$ from 7.0 to 1.6, though for this second case the reduction occurs in part because of a widening of its error bar from ± 0.09 to ± 0.16.

Finally, it should be noted that if, as before

$$g_{12} = \frac{1}{10} \quad , \tag{49}$$

but g_{13} is now allowed to occupy the interval

$$\frac{1}{100000} < g_{13} < \frac{1}{10000} \quad , \tag{50}$$

then for any such g_{13} the model predicts

$$\frac{|V_{cb}|}{|V_{ub}|} = 12.4 \quad . \tag{51}$$

In other words, the ratio $|V_{cb}|/|V_{ub}|$ is practically unaffected by the exact g_{13} chosen to fit the data and must therefore be fit almost solely by g_{12}. It follows that g_{12} must by itself bear the burden of fitting no less than *three* distinct experimental values: θ_{12}, φ_{12}, *and* $|V_{cb}|/|V_{ub}|$.

As it turns out, the above 2006 experimental values produce the ratio

$$\frac{|V_{cb}|}{|V_{ub}|} = \frac{4.221 \times 10^{-2}}{3.96 \times 10^{-3}} = 10.7 \quad , \tag{52}$$

while the corresponding 2008 values produce

$$\frac{|V_{cb}|}{|V_{ub}|} = \frac{4.15 \times 10^{-2}}{3.59 \times 10^{-3}} = 11.6 \quad , \tag{53}$$

where this later value is significantly more consistent with Eq. (51).

VIII. SUMMARY AND CONCLUSION

The mixing model described here exploits a non-traditional version of the CKM quark mixing matrix, a version in which the second and third rows of this matrix appear interchanged. It also exploits the fact that when a 3×3 rotation matrix whose elements are squared is subtracted from its transpose, the matrix produced has non-diagonal elements that possess a common absolute value. For the above non-traditional CKM matrix this value equals one-third the corresponding value for the leptonic matrix, a fact this article takes as its starting point for model-building. A framework of three such constraints is then built, in which the quark and lepton mixing matrices can be specified with just two free parameters. A realistic mixing model is then specified using values for these two parameters that derive from a key solution to a simple equation, Eq. (36), where this solution also accurately reproduces the fine structure constant. This resultant model is, therefore, largely mathematical in origin; but given that it is entirely free from parameters adjusted to fit the mixing data, and that it correctly forecast that the 2006 value for $|V_{ub}|$ was significantly in error, its further study is nevertheless justified. In particular, a logical next step is to examine the physical implications of both the above framework and its derivative mixing model.

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