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# ΜΙΑ ΝΕΑ ΓΕΝΙΚΗ ΚΑΤΗΓΟΡΙΑ «ΕΛΕΥΘΕΡΩΝ ΔΥΝΑΜΕΩΝ» ΠΕΔΙΩΝ BELTRAMI
# ΜΕΡΟΣ 1 : ΘΕΩΡΗΤΙΚΗ ΕΠΙΣΚΟΠΗΣΗ

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# A New Generic Class of Beltrami “Force-Free” Fields.
# Part-I: Theoretical considerations

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# A NEW GENERIC CLASS OF BELTRAMI "FORCE-FREE" FIELDS PART I : Theoretical Considerations

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# A New Generic Class of Beltrami "Force-Free" Fields. Part-I: Theoretical considerations

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***Abstract:*** *We report on a new general class of solutions of the Beltrami equation, with special characteristics. We also provide examples of solutions that also satisfy Maxwell equations. A subset of these solutions can be isolated which corresponds to "gauge" fields. A special projective geometry of vacuum fields is also revealed and discussed.*

## 1. Introduction

The notion of a force-free field comes directly from the work of 19th century mathematician Eugenio Beltrami (1835 - 1899) in hydrodynamics [1], [2]. In fact Beltrami made a very important contribution by a direct comparison between electrodynamics and hydrodynamics probably inspired by Lord Kelvin's vortex model of the atom. The central notion is that of a generalized eigenvalue of a rotation operator. As this is allowed to vary also with space and time it is preferable to call it the *eigen-vorticity* and we use this term for the rest of this paper. The special case of constant eigen-vorticity is often met in the literature under the name of a *Trkalian flow* from the Czech physicist and mathematician Viktor Trkal (1888 -1955) who has done similar work independently back at 1919 [3]. The subject is shown to be of significance in fluid mechanics, electromagnetism, magnetohydrodynamics and astrophysics while there is a proposition that the famous Vacuum Energy may be composed from such force-free fields [2].

In the following, we briefly introduce the subject of Beltrami fields and their general significance in section 2. In section 3, we present a general theorem for the construction of a class of solutions of the Beltrami equation. In section 4, we discuss the problem of the independent existence of such electric and magnetic fields either in vacuum or through appropriate sources and boundary conditions. In section 5, we further analyze the consequences of the previous abstract observations in the effort to develop a concrete algorithmic condition for constructing similar solutions of Maxwell equations. In section 6, we conclude on the possibility of a new class of devices that may lead to new applications in engineering electromagnetics as we intend to show in the second part of this article. This also provides strong indications that a new area of high energy physics is possible with macroscopic instruments effecting truly macroscopic results.

## 2. Geometry of Force-Free Fields

A Beltrami field is one that satisfies the equation

$$\nabla \times B = \lambda(\mathbf{r},t)B \qquad (1)$$

$\lambda(\mathbf{r},t)$ is the *eigen-vorticity,* which is a generalization of the eigenvalue of the rotation operator. The possibility of different orientations has been absorbed in the scalar coefficient. Such fields have been initially introduced in hydrodynamics to express vortex-like structures or in connection with the stability of solutions of the Euler equation. It has also been extensively used in the study of bi-anisotropic media especially by A. Lakhtakia [2], [4] and others. The later author has also shown that there exists a reformulation of Maxwell postulates in a purely covariant form in terms of two complementary complex Beltrami electromagnetic fields of the form

$$\mathbf{Q}_{\pm} = \frac{1}{2}\left[\mathbf{E} \pm \mathbf{i}Z\mathbf{B}\right]$$

where Z stand for the scalar vacuum impendance.

In the case of electrodynamics and particularly Magneto-hydrodynamics we usually start from the magnetic **B** field which can be derived through (1) and through Maxwell equations we can deduce the sources. On the other hand the second Maxwell equation demands that

$$\mathbf{B}\nabla\lambda(\mathbf{r},t)=0, \nabla B=0 \quad (2)$$

which restricts possible solutions of Beltrami equation (1). Not many general solutions of (1) are known in the literature (see [9] - [17]). A comprehensive review has been given by G. Marsh [5]. As is immediately obvious, application of *rot* operator into (1) turns it to a non-linear scalar and vector wave or Helmholtz equation, which has been solved in some cases for constant λ with appropriate boundary conditions. Trkalian fields with λ = ±2 are known in the mathematical literature as *Left Invariant Fields* with the simplest cases given by $\{x_i\mathbf{e}_i\}$. Non-constant eigen-vorticity is much more difficult and it has been known for decades in solar physics and astrophysics through the *Grad-Shafranov* equation [5] for which several special solutions have been studied extensively. Recent extensions of previous solutions have also been proposed [13] - [17] including the so-called *Kugelblitz* or Ball-like Force-Free fields [13], [14].

A nice approach on non-constant solutions is based on the decomposition of solenoidal (divergence-free) vector fields in $R^3$ in the form

$$B=\nabla\times(\varphi\mathbf{e}_3)+\nabla\times\nabla\times(\psi\mathbf{e}_3) \qquad (3)$$

where $\varphi$ and $\psi$ are called the *Debye-Hertz* potentials. This method has been employed by Benn and Kress in [10] to find solutions of equation (1) in terms of these potentials. One may also introduce *Hertz* potentials in a similar fashion. There is also an orthogonal basis of complex curl eigenfunction modes and the relevant *complex helical wave decomposition* first introduced by Lesieur [23] who defined helical waves through

$$\mathbf{V}^{\pm}(\mathbf{kx})=\left[\mathbf{b}(\mathbf{k})\mp\mathbf{ia}(\mathbf{k})\right]\exp\left(\mathbf{ikx}\right) \quad (4)$$

for which $\nabla\times\mathbf{v}^{\pm}=\pm|\mathbf{k}|\,\mathbf{v}^{\pm}$.

A deeper treatment by Kravchenko in [24] based on differential forms calculus, has resulted in an equivalent set of three Schrödinger equations with a non-constant complex eigen-vorticity. It is a notable fact that both Benn and Kress and especially Pantilie and Wood in [25] who have made a covariant treatment of (1) agree that the case of non-constant eigen-vorticity is inherently connected with curved metrics and the construction of self-dual metrics of the form

$$g=rh+r(dr+A)^2 \qquad (4)$$

In fact, there exists a linear morphism which maps solutions of the wave equation to solutions of a generalized Beltrami equation given as

$$*dA=\pm cA \qquad (5)$$

where * stands for the Hodge dual operator. Furthermore, Kassandrov and Trishin show in [26] that there exist spinorial generalizations of the Beltrami operator that occur in the theory of *Shear-Free Congruencies* in which every Maxwell-like field becomes *self-quantized.* There is also an associated work by H. Marmanis [28] who brought afore the old and forgotten subject of the analogy between hydrodynamics and electrodynamics in the effort to impose quantization conditions in turbulent flows. This direction of research was soon assimilated after 2000 in the new area called *Meta-fluid Dynamics* which is a form of Gauge field theory [29]. Similar in concept are the findings of Saygili

[27] in his attempt to construct a topologically massive abelian gauge field theory. In this construct abelian gauge potentials on Riemannian manifolds are Trkalian fields which define contact structures. The deep relation between contact structures and Beltrami fields has also arisen in the mathematical literature especially from the study of the "ABC" flows with unstable hyperbolic orbits by Arnold [28]. These works also have an intrinsically deep connection with some old and recent electromagnetic mass theories like the *geons* hypothesis first proposed by Wheeler and later elaborated by Melvin as well as with the primary attempts by George Reinich for a unified field theory.

Based on this evidence we find reasons to believe that the opposite route is also possible through which an understanding of both Maxwell and Yang-Mills fields could be reduced to an appropriate generalization of Navier-Stokes equations describing a self-quantizing, relativistic superfluid structure of a primordial, non-linear vacuum. Moreover, as far as a relativistic, covariant description of hydrodynamics exists it seems possible to include all Maxwell-like fields into a unified description where such forces will turn out to be "inertial" in nature. We provide evidence for such an opposite mapping in a forthcoming article through the introduction of generalized *Euler – Clebsch* vector potentials. In the next section we treat the problem of general solutions of (1) in the light of a new algebraic transform of vector fields. Applied to Maxwell fields this appears to be inherently related with an underlying projective geometry characteristic of the undisturbed vacuum.

### 3. A new class of Beltrami fields

In the following we will need to introduce some definitions in order to facilitate the exposition of the main results in both their generality and rigor. We first show that a set of special eigen-functions of the rotation operator can be constructed from arbitrary vector functions. In order to do that we will first need to construct a special transformation of vector fields which is defined by the following lemma.

**Field Equlibration Lemma 3.1 :** *Let* $\boldsymbol{F}$ *a real valued vector field in* $\boldsymbol{R}^3$ *and* $\boldsymbol{M}(\mathrm{n};\mathbf{x},\mathbf{y})$ *a set of matrices defining a 1-parameter group of continuous transformations* $\mathbf{F}^{(n)} = \mathbf{M}^{(n)}\mathbf{F}$ *such that* $\sum_{n=1}^{N}\mathbf{F}^{(n)} = \phi(\mathbf{R}\bullet\mathbf{d}), \mathbf{d} = [\mathbf{e}_{10}, \mathbf{e}_{20}, \mathbf{e}_{30}]$ *where* $\boldsymbol{R}(\mathbf{x})$*, a continuous "frame" transformation. Then,* $\mathbf{M}(n;\mathbf{x},\mathbf{y}) = \mathbf{r}(\mathbf{x}) \otimes \mathbf{k}(n;\mathbf{y})$ *is a trivial dyadic kernel with* $r_i = \sum_{j=1}^{3} Rij$ .

Proof: From the defining equation we have

$$\sum_{n=1}^{N}\mathbf{M}(n;\mathbf{x},\mathbf{y})\mathbf{F} = \sum_{n=1}^{N}[\sum_{j=1}^{3}M_{1j}F_j, \sum_{j=1}^{3}M_{2j}F_j, \sum_{j=1}^{3}M_{3j}F_j]$$

As the initial field elements get mixed we may equate terms by allowing the matrix elements to be factorised as $M_{ij} = r_i(\mathbf{x})K_{ij}(n;\mathbf{y})$ in which case we get

$$\sum_{j=1}^{3}k_j(n;\mathbf{y})F_j(\mathbf{x})[r_1(\mathbf{x})\mathbf{e}_{10}, r_2(\mathbf{x})\mathbf{e}_{20}, r_3(\mathbf{x})\mathbf{e}_{30}]$$

only if

$$\sum_{j=1}^{3}K_{1j}(n;\mathbf{y})F_j = \sum_{j=1}^{3}K_{2j}(n;\mathbf{y})F_j = \sum_{j=1}^{3}K_{3j}(n;\mathbf{y})F_j = \sum_{j=1}^{3}k_j(n;\mathbf{y})F_j$$

In order to have $[\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3] = \mathbf{R}[\mathbf{e}_{10}, \mathbf{e}_{20}, \mathbf{e}_{30}]^T$ we must also identify $r_i(\mathbf{x})$ with the sum over columns of the transformation matrix **R.**

Other non-trivial kernels for the same transform can be produced that cannot be expressed as dyadics by recognising that the main factorisation condition can be also satisfied in the more general form

$$\sum_{n=1}^{N}\sum_{j=1}^{3}K_{1j}(n;\mathbf{y})F_j = \sum_{n=1}^{N}\sum_{j=1}^{3}K_{2j}(n;\mathbf{y})F_j = \sum_{n=1}^{N}\sum_{j=1}^{3}K_{3j}(n;\mathbf{y})F_j$$

Assuming that $K_{1j}(n;\mathbf{y})$ forms a large random orbit of maximum period N, then all other elements in the group can be identified with arbitrary permutations of $K_{1j.}$

**Field Alignment Lemma 3.1 :** *Let* $\boldsymbol{F}$ *a real valued vector field in* $\boldsymbol{R}^3$, $\boldsymbol{M}(n;\mathbf{x},\mathbf{x})$ *a set of dyadics defining a 1-parameter group of continuous transformations* $\mathbf{F}^{(n)} = \mathbf{M}^{(n)}\mathbf{F}$, $\mathbf{J}^{(n)} = rot\mathbf{F}^{(n)}$ and $\mathbf{G} = \sum_{n=1}^{N}\mathbf{F}^{(n)}$. *Then* $\boldsymbol{G}$ *is an eigenfunction of rot operator with eigen-vorticity* $\Lambda = (J_1 + ... + J_N)(k_1F_1 + ... + k_NF_N)^{-1}$ *iff* $rot(\mathbf{M}^{(n)}\mathbf{F}) = \mathbf{M}^{(n)}rot\mathbf{F}$.

Proof: By definition

$$rot\mathbf{G} = \sum_{n=1}^{N}\mathbf{J}^{(n)} = \sum_{n=1}^{N}rot(\mathbf{M}^{(n)}\mathbf{F}) = \left(\sum_{n=1}^{N}\mathbf{M}^{(n)}\right)rot\mathbf{F} = \left(\sum_{n=1}^{N}\mathbf{M}^{(n)}\right)\mathbf{J}^{(0)} \qquad (6)$$

It is evident that if the transformations commute with the rotation operator then the set of $\{\mathbf{J}^{(n)}\}$ transforms the same way as the original field **F**. By Lemma 3.1, $\{\mathbf{M}(n;\mathbf{x})\}$ transform as

$$\mathbf{G} = (k_1(\mathbf{x})F_1 + k_2(\mathbf{x})F_2 + k_3(\mathbf{x})F_3)[\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3] = \phi(\mathbf{R} \bullet \mathbf{d}) \qquad (7)$$

In the above, $\{F_i\}$ are the elements of the initial vector field **F** and **d** stands for the "diagonal" vector at every point of space in the specific coordinate system used. By construction, if the rotations of $\{\mathbf{F}^{(n)}\}$ transform the same way then we must also have

$$rot\mathbf{G} = \sum_{n=1}^{N}\mathbf{J}^{(n)} = \varphi(\mathbf{Rd}) \qquad (8)$$

Then $\mathbf{G} \times rot\mathbf{G} = 0$ must hold true everywhere so that condition (1) is automatically satisfied. The eigenvorticity is simply taken as the local scaling factor $\varphi/\phi$. It is now possible to define solutions of this class for the Beltrami equation as follows.

**Theorem 3.1 :** *For every real valued vector field* $\boldsymbol{F}$ *in* $R^3$, $\mathbf{G} = \sum_{n=1}^{D}\mathbf{F}^{(n)}$ *is a Beltrami field iff* $\nabla\mathbf{F}^{(n)} = 0$.

Proof : Condition (2) is separately satisfied iff $\{\nabla\mathbf{F}^{(n)} = 0\}$. It then follows that

$$\nabla\phi = 0, \nabla\varphi = \sum\nabla(rot\mathbf{F}^{(n)}) = 0, \nabla\Lambda = 0$$

The above guarantees also that $\nabla\Lambda\mathbf{G} = 0$ so that **G** becomes a full solution of the Beltrami equation. We emphasize here the fact that $\nabla\mathbf{F}^{(0)} = 0$ does not automatically imply the solenoidal character of the rest of the initial field transformations due to the fact that the divergence does not necessarily commutes with the transformation operator. Another important property of these solutions in contrast with the field transformations is that the *"equilibrated"* character of **G** is coordinate free. This is a straightforward result of the fact that whatever the transformation matrix from one coordinate system to another it does not gets mixed with the multiplying scalar.

We now note that

$$rot(\phi\mathbf{Rd}) = \nabla\phi \times (\mathbf{Rd}) + \phi(\nabla \times \mathbf{Rd}) \qquad (9)$$

If we were to choose the initial field so that $\nabla\phi = \kappa(\mathbf{x})(\mathbf{Rd})$ then, (9) would simplify to the condition

$$\nabla \times \mathbf{Rd} = \varphi(\mathbf{Rd}) \qquad (10)$$

For arbitrary curvilinear coordinates (9) and (10) results in highly nonlinear equations that should be solved in order to define the classes of continuous groups of transformations of a single parameter (n) corresponding to the "moving frame" **Rd**. One may also see this frame motion associated with the so called *vielbeins* in relativistic theories. **R** could also stand for a more general rotation and boost matrix corresponding to the general Lorentz group. From this point of view it is much more preferable to interpret the above as a generic coordinate transformation $\{x_i\} \to \{u_i(x_i)\} : \mathbf{d} \to \mathbf{d}' = \mathbf{Rd}$ where we identify **R** with the jacobian of the transformation**.** In the new coordinate system we will always have

$$rot(\phi\mathbf{Rd}) \to rot'(\phi\mathbf{d}') = \left(\sqrt{g}\right)^{-1}(\nabla\phi \times \mathbf{d}' + \phi\mathbf{f}) \qquad (11)$$

The additional field **f** is obtained from the generic expression of the rotation in curvilinear systems as

$$f_i = \partial_j \mathbf{e}_k - \partial_k \mathbf{e}_j \qquad (12)$$

Both (9) and (11) are mostly useful for they allow associating the rotation operator with an exterior *algebra.* Most of all, its application into the complete set of Maxwell equations will reveal a very interesting underlying geometrical structure associated with a not so obvious *projective line bundle* geometry. These observations are utilized in the next section.

## 4. Geometry of the Electromagnetic Vacuum

We will now show that the result of the previous section extends even in the case where the arbitrary transformations do not commute with the *rot* operator! We first start with an important observation that can be summarized in the following Lemma

**Lemma 4.1 :** *A real vector field* $\phi(\mathbf{I} \bullet \mathbf{d}), \mathbf{d} = [\mathbf{e}_{10}, \mathbf{e}_{20}, \mathbf{e}_{30}]$ *is a gauge field iff there is a function* $\psi$ *such that* $\phi = \partial_1\psi = \partial_2\psi = \partial_3\psi$ .

Proof : Gauge fields are defined by the transformation of the vector potential $\mathbf{A} \to \mathbf{A} + \nabla\psi$ where the last term is irrotational. A trivially equilibrated field with $\mathbf{R} \equiv \mathbf{I}$ is always irrotational if it is a divergence of some scalar function. The only way this can happen is to have a complete symmetry such that all derivatives of $\psi$ are equal. This, due to (12) also implies the special case $\nabla\phi = (\partial_1^2\psi)\mathbf{d}$ so that $\nabla\phi \times \mathbf{d} = 0$ .

This also means that the dyadic transformation described in the previous section turns everything into a "gauge" field in an appropriately "shaped" space. In order to fully grasp its significance we have to analyze Maxwell equations in the light of the observations made in the previous section and in particular of equations (9) and (11). Specifically, given two abstract vector fields $\mathbf{a} = \nabla\phi_E, \mathbf{b} = \nabla\phi_B$ and using (11), Maxwell equations in vacuum can be written in arbitrary curvilinear coordinates as

$$\mathbf{a} \bullet \mathbf{d}' = \mathbf{b} \bullet \mathbf{d}' = 0 \qquad (12a)$$

$$\mathbf{a}\times\mathbf{d}' = -(\omega\sqrt{g}\phi_B\mathbf{d}' + \phi_E\mathbf{f}) \quad (12b)$$

$$\mathbf{b}\times\mathbf{d}' = \omega\sqrt{g}\phi_E\mathbf{d}' - \phi_B\mathbf{f} \qquad (12c)$$

This is a purely geometrical statement of which the abstract meaning is difficult to grasp without separating its content from the usual tensorial treatment. In fact, what we believe that happens is that there must exist two entirely different mathematical formulations that only happen to coincide in a 3-dimensional space. We will show that the second one which is now revealed can easily be generalized in any dimension in contrast with the usual pseudo-vector which demands the extension to higher rank tensors. This becomes obvious when we realize that any exterior product can be interpreted as a *projective space of rays*. Specifically, we start by recognizing the simple fact that the null space of any 3-dimensional exterior product represents a set of geometrical proportionalities over a line bundle that together constitute an affine space. To show this we simply analyze the components of an arbitrary exterior product with recourse to Fig. 1.

Let for example O be the cross section of a line bundle (*point ad infinitum*) and let the projective point $[x_i : y_i : 0]$ correspond to the elements of two abstract vectors such that $\mathbf{x}\times\mathbf{y} = 0$. By a simple rearrangement of terms we have

$$x_j y_k - x_k y_j = 0 \cong \frac{x_j}{x_k} = \pm\left(\frac{\mp y_j}{y_k}\right) \cong \frac{x_j}{y_j} = \pm\left(\frac{\mp x_k}{y_k}\right) \qquad (13)$$

Note here that the notion of a "line bundle" is generic and can be transferred to arbitrary geometries. We now recognize that the same structure can be transferred to an arbitrary local orthogonal frame given an atlas over an arbitrary manifold. Hence, any local values of two arbitrary vector fields **E** and **B** can be assigned to these local frames as in fig. 2 to represent the projective point $[E_i : B_i : 0] : \mathbf{E}\times\mathbf{B} = 0$**.** In the same spirit, the "force-free" condition $\mathbf{J}\times\mathbf{B} = 0$: $[J_i : B_i : 0]$ also represents a projective point. Thus, absence of radiation or absence of forces becomes isomorphic with the existence of a projective point. On the other hand, presence of radiation can be interpreted through equations (12a-c) as follows.

Both vectors **a, b** belong to a local tangent plane normal to **d'** in the transformed manifold as implied by (12a). Equations (12b-c) also denote the degree of <u>*loss of projectivity*</u> of the vacuum through a mixing of both characteristic scalars $\phi_E$ and $\phi_B$. Combining these two we derive

$$(\mathbf{a}\times\mathbf{d}')\bullet(\mathbf{b}\times\mathbf{d}') = \phi_E\phi_B(|\mathbf{f}|^2 - \omega g|\mathbf{d}'|^2) + \omega\sqrt{g}(\phi_B - \phi_E)(\mathbf{f}\bullet\mathbf{d}') \qquad (14)$$

$$(\mathbf{a}\times\mathbf{d}')\times(\mathbf{b}\times\mathbf{d}') = \omega(\phi_B^2 - \phi_E^2)(\mathbf{d}'\times\mathbf{f}) \qquad (14)$$

From the last expression (14) we immediately see that a sufficient condition for the existence of Beltrami fields is given by $\phi_E = \pm\phi_B$ and/or $\partial_j\mathbf{e}_k - \partial_k\mathbf{e}_j \propto \kappa\mathbf{e}_i$. It is possible in principle to incorporate "source" or current terms in the above as long as they can be expressed in a similar form, that is $\mathbf{J}^{(m)} = \mathbf{M}_E(m;\mathbf{x})\mathbf{F}$.

A preliminary examination reveals that given the freedom we have to choose the two basic functional components of the dyadics, it seems reasonable that for every set of vacuum solutions we should be able to define an appropriate set of functional equations defining the dyadic in such a way that there will always be a maximal period N at which equilibration in the sense of Lemma 3.1 and the condition (14) takes place.

We also see that a crucial difference between trivial gauge fields and the rest lies in the existence of a frame transformation. This implies a deep relationship between gauge field theories and the way a field may manifest itself out of a vacuum as a result of a transformation more general than the one implied by simple observer motion (relativity). In order to further examine this issue, we need to clarify any relationship between the vector potential and the equilibrated fields that may form vacuum solutions of Maxwell equations. In case $\mathbf{M_B}$ is a dyadic, then by Lemma 3.1, **A** is an equilibrated field itself and it will be aligned with the electric and magnetic components as well. Simple examples of such fields exist in the literature [5]. It is natural to ask then, for an appropriate reformulation of the electromagnetic tensor in the generic form

$$F_{\mu\nu} = M_{\mu\nu} A^{\nu} \qquad (15)$$

The above approach could also suggest the real, physical existence of the vector potential as already suggested by certain phenomena like the Aharonov-Bohm effect. It is possible to show that an appropriate dyadic can always be constructed by a simple inversion so that an arbitrary set of arbitrary fields $\{\mathbf{E},\mathbf{B}\}$ can always be written as a functional transform of its associated vector potential. By definition we have

$$(\mathbf{r}(\mathbf{x}) \otimes \mathbf{k}_{E,B}(\mathbf{x}))\mathbf{A} = \{\mathbf{E},\mathbf{B}\} \qquad (16)$$

By the symmetry of the dyadic the above can be inverted as

$$(\mathbf{r} \otimes \mathbf{A})\mathbf{k}_{E,B} = \{\mathbf{E},\mathbf{B}\}$$

so that

$$\mathbf{k}_{E,B} = (\mathbf{r} \otimes \mathbf{k})^{-1}\{\mathbf{E},\mathbf{B}\} \qquad (17)$$

Then an algorithm for the construction of generic Beltrami fields satisfying condition (13b) can be stated as follows. We start by taking a set of TE and TM eigenfunctions $\{\mathbf{E}^{(n)},\mathbf{B}^{(n)}\}$ for a certain set of boundary conditions. We then define the associated set of dyadics $\{\mathbf{k}^{(n)}_{E,B}\}$. By an appropriate "clipping" procedure we try to isolate a specific subset of eigenfunctions $\{\mathbf{E}^{m)},\mathbf{B}^{(m)}\}$ for which

$$\sum_{m=m0}^{N}\sum_{i=1}^{3} k^{(m)}_{E,i} E^{(m)}_{i} = \pm \sum_{m=m0}^{N}\sum_{i=1}^{3} k^{(m)}_{B,i} B^{(m)}_{i} \qquad (18)$$

Such a "clipping" procedure corresponds to a successive breaking of the symmetry of an initial set of boundary conditions. Current sources could also be transformed accordingly.

## 5. Application in Maxwell Fields

We will give below some simple examples of constructions that result into Beltrami-like electric and magnetic fields. We first show on very general grounds that if one of these components separately satisfies the Beltrami equation, then so does the other. Indeed if both **E** and **B** are Beltrami fields then the problem is automatically reduced into (1) and (2). Given the electric and magnetic eigen-vorticities $\alpha$ and $\beta$ respectively we have

$$\nabla\mathbf{E} = \nabla\mathbf{B} = 0$$
$$a\mathbf{E} = -\omega\mathbf{B}, b\mathbf{B} = \omega\mathbf{E}$$

From the last two we easily deduce that

$$ab + \omega^2 = 0 \qquad (19)$$

Starting from the original equations and assuming that **B** is a Beltrami field with eigen-vorticity $\beta$, we derive the relations

$$rot\mathbf{E} = -(\frac{\omega}{\beta})rot\mathbf{B} = -(\frac{\omega^2}{\beta})\mathbf{E}$$

which again justifies (19).

The simplest example of a coordinate transformation capable of producing locally a Beltrami-like pair of fields can be given with the aid of three linearly polarized plane waves specially rotated. Let then

$$\mathbf{E}_1 = E_0[\sin(k(z-z'),0,0], \mathbf{B}_1 = B_0[0,\cos(k(z-z')),0]$$
$$\mathbf{E}_2 = E_0[0,\sin(k(x-x')),0], \mathbf{B}_2 = B_0[0,0,\cos(k(x-x'))]$$
$$\mathbf{E}_3 = E_0[0,0\sin(k(y-y'))], \mathbf{B}_3 = B_0[\cos(k(y-y')),0,0]$$

(we assume the same time dependence given as $\exp(\pm\mathbf{i}\omega t)$ ). These are emitted from three different sources at (0,0,z'), (x',0,0) and, (0,y',0). With the aid of Fig. 2, by simple geometrical reasoning we see that the total fields are locally parallelized $(\mathbf{E} = \kappa\mathbf{B})$ at a periodic set of points at which (13) holds true. This in turn only depends on the relative positions of the source (emission) points. For $x' = y' = z'$ we have a Beltrami field.

The above example represents also a simple scheme for a *spatial helicity modulation* which we will explore further in the second part of this presentation. We note that the helicity of a simple plane wave is zero. The local helicity density of an arbitrary field is usually given by the quantity $\mathbf{A} \bullet \mathbf{B}$ which becomes maximal for parallel electric and magnetic components [5]. A helicity modulator could then be constructed by varying the relative phase or the positions of the sources.

We note that the permutation above corresponds to a special case of rotations. In general, there will be different choices for the transformation parameters such that when combined with the symmetry of the two last Maxwell equations will result in a type of interference fields

$$\mathbf{E}' = \sum_{n=1}^{N} \mathbf{E}(\mathbf{R}^{(n)}(a,...)\mathbf{x}), \mathbf{B}' = \sum_{n=1}^{N} \mathbf{B}(\mathbf{R}^{(n)}(b,...)\mathbf{x})$$

such that

$$E'_1(X') \equiv E'_2(X') \equiv E'_3(X')$$
$$B'_1(X') \equiv B'_2(X') \equiv B'_3(X')$$

This type of field equilibration guarantees that (13) will be satisfied everywhere. For a given set of boundary conditions, a pair of Maxwell fields can always be written as a combination of TE and TM eigenfunctions reflecting the symmetry of the sources. The above construction implies that given a certain set of symmetries, we may always construct an associated pair of separate Beltrami-like electric and magnetic components by just rotating and scaling multiple copies of the initial sources. Feasibility of such a construction should be checked separately for each type of field as some constructions may be unrealistic from an engineering viewpoint.

## 6. Conclusions

The interpretation of the vacuum as a hidden projective space is a non-trivial result when applied to electrodynamics for the simple reason that till now we have no justification whatsoever for the particular reason that nature has chosen the particular differential operators for the laws of the electromagnetic field. Moreover, even the expression that attributes any “choice” to nature –which is not a subject-is obviously fallacious! Any such underlying geometry would permit us to enlarge our understanding on the inner workings of nature by submitting its contents into the much deeper necessities of a purely geometrical origin.

Let us now remind that the essence behind relativistic treatments of electromagnetism and other forces can be summarized effectively in the equivalence of all frames, of which the geometrical content is simply that any point of space is equally valid for being the zero of the axis O of any local frame. The analysis of the exterior algebra involved in electromagnetism from the point of view of projective affine spaces is by itself suggestive of an additional and complementary axiom that could be stated as "Every point of an empty space is equally valid as a point *ad infinitum*". Radiation in this context would appear as a deviation from a perfect "projectivity" of empty space. This also suggests an additional and complementary equivalence principle rather different than ordinary covariance. In a sense, it could also be interpreted as saying that every point of our "real" space can be seeing as a special projection from a much larger universe!

A special geometric construction that could show this is inevitably associated with the fact that "time" as we understand it, does not appear as a completely separate component of the electric and magnetic components but as a multiplicative factor wherever separation of variables is possible. In a sense this is similar with the way the additional coordinate of a *projective plane* or a *Riemann Sphere* in a (n+1)-dimensional projective space appears as a divisor over the rest of the coordinates of the n-dimensional subspace. Hence, our "physical" time would be no more than an abstract "inverse time" over the projective plane or the projective sphere of a 4-dimensional Riemannian manifold. The fact that our universe has been made so as to obey the Lorentz group can be given a meaning in a purely geometrical content as the result of a fundamental selection rule acting upon the projective plane or the projective sphere in such a way that any orbits will have the topology of a hyperbolic tiling (e.g. *Teichmuller Theory*) [32], [33].

In the second part of this presentation we will deal with more complex constructions based on special types of magnetization that can imitate some of the existing solutions and we will try to extend them using the general scheme adopted here. Our effort is towards macroscopic devices which could prove their ability to exert a direct macroscopic influence on the assumed non-linear vacuum field structure operating in a high power regime. Specifically we intend to develop methods for direct Helicity Modulation, both spatial and temporal. As any such influence on the helicity content of vacuum fields appears equivalent to metric distortions of a kind, such devices could prove the electromagnetic analogue of macroscopic *Warp Engines.*

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**FIGURES**

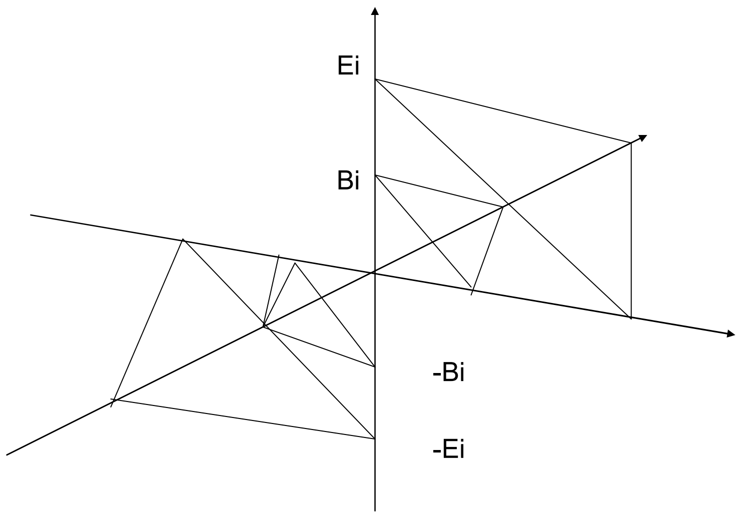


**Fig. 1**

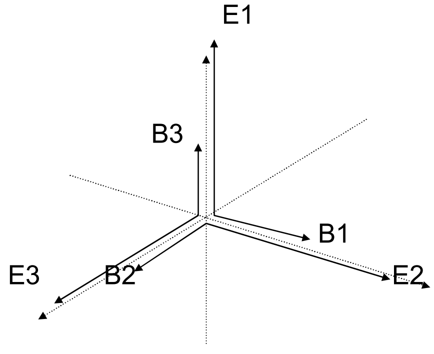


**Fig. 2**