Planck constant estimation using constant period relativistic symmetric oscillator

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The electromagnetic wave quantum-energy depends only on its frequency, not on the emitting system's radiation power. The proportionality constant between the frequency and the quantum-energy of the electromagnetic wave, the Planck's constant is in the essence of quantum mechanics. This constant is known experimentally but till now there was no clue for calculating its value on a theoretical basis. In the present work a methodology for calculating a lower bound for Planck's constant is presented, based on simple principles. In order to get a reasonable good lower bound it is necessary to have a model of a relativistic oscillator whose period is independent of its energy and which efficiently radiates electromagnetic energy. It is highly desired that the mathematics involved is simple enough to enable good insight into the results. Such a model can also be used for other investigations, and therefore, in this work a potential that conserves the vibration period of symmetric oscillators at relativistic velocities is found and analyzed. The electrically charged system of constant period is used to calculate a lower bound H_m of the Planck's constant h. The value of H_m is smaller than h by a factor very close to $\sqrt{3}$. The explanation of this factor also explains the value of Planck's constant. From this value the fine structure constant value is calculated and a new interpretation of this constant obtained.

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I. INTRODUCTION

At the beginning of the 20th century when the quantization of electromagnetic (EM) energy was discovered, there were no theoretical tools to estimate the value of Planck's constant; a constant that relates the energy of EM quanta to the frequency of the electromagnetic waves. Later it was believed that this is a basic constant so there was no point to investigate its value theoretically. There is a need for an improved model to reasonable estimate Planck's constant. This model will replace the harmonic oscillator model. The simple harmonic oscillator (SHO) is frequently used to explain different aspects of physical phenomena. The simple analytical solutions of the perti-

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nent differential equations, both in the classical and quantum mechanics frames, make this system special .The one dimensional (1D) SHO has in addition a unique property namely, at non relativistic velocities it has a constant period, independently of its energy. When a 1D SHO is electrically charged, its radiation frequency is the same as the oscillations frequency both according to classical electromagnetism and quantum mechanical theories. A two dimensional (2D) symmetric SHO exhibits more interesting properties, while retaining the simplicity of a single frequency of oscillation and radiation. It is well known that at high energies, when relativistic effects are important the oscillation period is not constant [1]. It is energy dependent. The potential for a 1D oscillator that ensures the constant period, independently of energy, at relativistic velocities has been calculated [2]. However the exact potential was not described by an analytic expression.

The first objective of the present work is to find the potential that ensures constant period in the circular mode of vibration (Symmetric Constant Period Potential - SCPP) at relativistic velocities. In this work, the potential is expressed in closed-form. At relativistic velocities, 2D electrically charged system is expected to radiate its energy more efficiently than the 1D system. For the general elliptic modes of vibration there is a need to fit a potential for each value of eccentricity. In this paper a general form for an approximate potential with one parameter is presented. The value of this parameter may be found and can be checked for accuracy in the future. I hope that the present work provides the mathematical tools that will help to enhance our understanding of relativistic classical mechanics and relativistic quantum mechanics.

The motivation for this study is the attempt to understand the EM quantum-energy, i.e., the photon energy. This understanding starts with the simple observation that according to classical wave theory, In order that EM radiation will have a defined frequency within some accuracy, it needs to be radiating at least during one cycle. Planck's interpretation of black body radiation [3] includes the model where the radiation is produced by elementary systems, presumably elementary harmonic oscillators, with quanta of energy independent of the oscillators' energy, despite the fact that the radiation power increases with the oscillators' energy growth. The frequency is one of the basic properties of the photon; a property that determines its energy. The combined result is that at any frequency *the EM quantum-energy must be larger than the amount that any elementary system of constant period*[1] *can produce in one cycle*. At very high system's energies, new channels, in addition to the EM radiation, are opened for the release of energy. Such processes of energy release will limit the EM radiation of an oscillator. A high energy oscillator is naturally in the relativistic velocities domain. In order to get a lower bound for the energy quantum we need an elementary system with constant period that radiates energy efficiently. For this purpose a symmetric constant period oscillator (SCPO) is selected. A model of a particle having mass m and elementary electric charge of an electron e is used as the moving component of the SCPO. A competitive energy release process is pair creation. Above the required energy of $2mc^2$ this process may be dominant. *Only under the energy of $2mc^2$ we can be certain that an isolated charged SCPO will emit its available energy as EM waves.*

[1] For a radiating system of constant period, the independence of its radiated EM quantum-energy from the system's energy follows from the fact that the EM quantum-energy only depends upon its frequency, which is constant.

The radiation power of the charged SCPO at energy equals $2mc^2$ will be calculated. Calculating the energy radiated in one period of oscillation at that power gives a lower bound for the EM quantum $E_m = H_m\nu$. Here ν is the radiation frequency and H_m is a calculated factor. Comparing this result with the experimental value of Plank's constant yields a surprisingly simple relation $h = \sqrt{3}H_m$. The error in this relation is 0.014%. Although there is no theoretical explanation for this factor of $\sqrt{3}$, it may be explained in the future. Considering this relation as an expression for h in terms of H_m allows the calculation of the fine structure constant (within the same relative error). This constant is inversely proportional to the square of the reference normalized acceleration derived from the SCPO model at energy of $2mc^2$.

In the macroscopic world the EM quantum-energy is described as a minute amount of energy. For an elementary system of molecular size, this is actually a huge amount of energy that cannot be radiated during a single period of oscillation. Actually, according to classical electrodynamics laws in usual cases there is a need for thousands of periods of oscillations in order to produce one quantum. The quantum mechanical equivalent is the average waiting time (or decay time) until a photon is created, the same time that this number of oscillations takes. The present work gives an intuitive explanation for the magnitude of the EM quantum.

II. THE RELATIVISTIC SYMMETRIC CONSTANT PERIOD POTENTIAL

For circular motion the potential that ensures constant period T should compensate for the centrifugal potential (related to the centrifugal force); that is the same potential with opposite sign. Such a relativistic centrifugal potential is already known [4]. The derivation for this potential is repeated here in order to have intermediate results.

In a uniform circular motion the tangential velocity v and radius of rotation r are related to the period T and angular velocity ω according to

$$T = \frac{2\pi}{\omega} = \frac{2\pi r}{\mathrm{v}}. \quad (1)$$

This relation is used to express the velocity of a particle rotating with angular velocity ω as a function of the radius

$$\beta(r) = \frac{\mathrm{v}}{c} = \frac{\omega r}{c} \quad (r < \frac{c}{\omega}). \quad (2)$$

In the circular motion the relativistic proper (or intrinsic) centripetal acceleration $\mathbf{a}$ is

$$\mathbf{a} = -\frac{\gamma^2 \mathrm{v}^2}{r}\hat{\mathbf{r}} = -c^2\frac{\gamma^2\beta^2}{r}\hat{\mathbf{r}}, \quad (3)$$

where $\hat{\mathbf{r}}$ is a radial unit vector and $\gamma^2 = 1/\left(1-\beta^2\right)$. Inserting $\beta(r)$ yields the proper acceleration as a function of r

$$\mathbf{a} = -c^2 \frac{\left(\omega r / c\right)^2}{r\left[1-\left(\omega r / c\right)^2\right]}\hat{\mathbf{r}}. \qquad (4)$$

The force that generates this proper acceleration at any r, for a given period T is according to Newton's second law:

$$\mathbf{F} = m\mathbf{a} = -\frac{m\omega^2 r}{1-\left(\omega r / c\right)^2}\hat{\mathbf{r}}. \qquad (5)$$

The potential is found by integration

$$\begin{aligned} V &= -\int \mathbf{F} \cdot d\mathbf{r} = -0.5mc^2 \ln\left(1-q^2\right) \\ &= mc^2 \ln\left(\frac{1}{\sqrt{1-q^2}}\right) \quad \left(q<1\right). \end{aligned} \qquad (6)$$

Here a dimensionless variable $q = \omega r / c = 2\pi r / \left(cT\right)$ is used in order to simplify expressions.

This constant-period potential is a generalization of the SHO into the relativistic regime in two or three dimensions. There is no advantage of working in 3 dimensions, therefore in the present work we will restrict to the two dimensional framework. The constancy of the period for any amplitude (and therefore for any energy), is only guaranteed for circular motion.

A. Properties of the symmetric potential of constant period

A graph of the potential, in units of the rest mass energy, as a function of normalized distance is shown in figure 1.

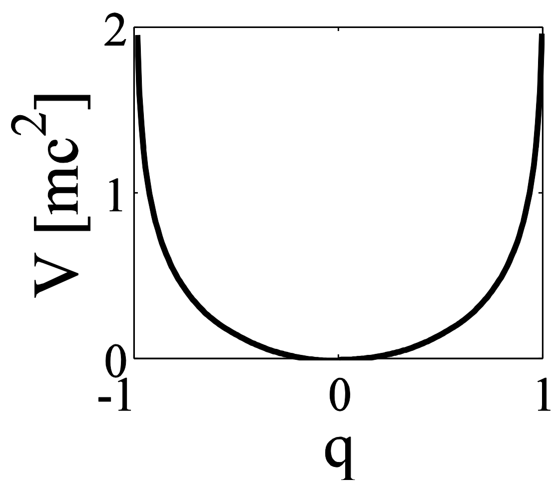

FIG. 1. A cut of the SCPP through the origin in units of rest mass energy. $q = 2\pi r/(cT)$

1. Central region

As r tends to zero, the potential tends to behave as the usual symmetric harmonic oscillator

$V \xrightarrow[r\to 0]{} \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}m\omega^2\left(x^2+y^2\right).$

2. Edge region

There is a potential singularity at $2\pi r = cT$. It posts a barrier that prevents circulation of a particle on an orbit of circumference longer than cT.

B. Comparison with the one dimensional case

The 1D oscillator of constant period exact potential [2] is not given analytically. Analytical approximations are available; none of them are of a logarithmic form. The derived force of one of the approximations is however similar to the force of SCPP:

$$F_{linear} \simeq -m\omega^2 \frac{x}{\left[1-\left(\frac{4x}{cT}\right)^2\right]^{\frac{5}{4}}}. \tag{7}$$

C. Approximate relativistic constant period potential for elliptic motion

This force given by eq. (7) and the force derived in the present work can be written in terms of a common ellipse notation using the eccentricity parameter ε and the ellipse related circumference of unit major axis $L(\varepsilon)=4\mathrm{E}(\varepsilon^2)$. The function E is the complete elliptic integral of the second kind [8]. It takes the form

$$\mathbf{F}_{ellipse}=-\frac{m\omega^2\left(x\mathbf{x}+y\mathbf{y}\right)}{\left[1-\left(\frac{L(\varepsilon)}{cT}\right)^2\left(x^2+\frac{y^2}{1-\varepsilon^2}\right)\right]^{n(\varepsilon)}}. \tag{8}$$

Here $\hat{\mathbf{x}}$ and $\mathbf{y}$ are orthogonal unit vectors in the xy plane. For a circular orbit $\left[\varepsilon,L,n\right]\ =\ \left[0,\pi,1\right]$ and for a linear trajectory $\left[\varepsilon,L,n\right]\ =\left[1,4,5/4\right]$.

It is expected that for a general ellipse case the value of the exponent n can be expressed as a function of the eccentricity ε. At non relativistic velocities, proper initial conditions are enough to make the particle enter into a desired elliptical path and to oscillate in the right period T. At the extreme relativistic case the particle will stick to the elliptical singularity line and accomplish a round trip at a period of $L(\varepsilon)/c=T$. A proper choice of the exponent $n(\varepsilon)$ will give a good fit for the intermediate velocities.

III. OSCILLATOR PROPERTIES

When a particle is oscillating under the influence of the SCPP, it follows a trajectory in the XY plane according to its initial conditions.

A. The nonrelativistic limit

In the nonrelativistic limit ($E\ll mc^2$) all orbits of vibration will stay in the central region of the potential and will have the same period of oscillations independent of energy or eccentricity.

B. The super-relativistic limit

In the super-relativistic limit ($E\gg mc^2$) the trajectory will be very close to the line of singularity. For circular motion's initial condition $\left[\mathbf{r}_0,\mathbf{v}_0\right]$: 1) $\mathbf{r}_0\perp\mathbf{v}_0$, 2) $\omega\left|\mathbf{r}_0\right|=\left|\mathbf{v}_0\right|$, the orbits are of constant period. In the next sections we will refer only to such orbits. Deviation of the Initial conditions from these two rules leads to a distorted trajectory that is usually not periodic (not a closed curve). Though for special cases the trajectory is periodic. The period is than energy dependent. See appendix for calculation and examples.

IV. ENERGY VERSUS VELOCITY AND RADIATION LIMIT

A relativistic particle rotating in the SCPP possess total available (not including rest mass) energy of

$$E = mc^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) - \tfrac{1}{2}mc^2\ln(1-\beta^2). \tag{9}$$

Here we use the relation found in eq. (2), $q = \beta$. In the next section the radiation of electrically charged SCPO will be considered. Theoretically, there is no obvious limit for the EM power it can radiate. The limit for the radiation appears when a competitor energy release process exists. Such a process is a creation of particles pairs. That is when the energy of the SCPO reaches the level of $2mc^2$. To be precise we assume that the particle oscillating under the influence of the SCPP is an electron, and the creation is of electron-positron pair. Using equation (9) the velocity of the SCPO at that critical energy of $2mc^2$ is determined according to

$$\left[\left(1-\beta_m^2\right)^{-1/2} - 1\right] - \tfrac{1}{2}\ln(1-\beta_m^2) = 2. \tag{10}$$

The resulted value is $\beta_m = 0.8915558$.

A. Maximum radiated energy at one cycle

The power radiated by accelerating particle, with an elementary electric charge, e , is according to Larmor formula [5]

$$P = \frac{e^2 a^2}{6\pi\varepsilon_0 c^3}, \tag{11}$$

where a is the particle's acceleration.

The extension to relativistic velocities [6] is made by substituting the proper acceleration for a [7]. The revolving particle's proper acceleration is given in eq. (3) where the radius of revolution is connected to the velocity according to eq. (2). It is useful to normalize the acceleration of the particle by the angular velocity ω and the speed of light to get a dimensionless quantity. The definition and expression for the present case according to equations (2, 3) are

$$\underline{a} \overset{\text{def}}{=} \frac{a}{c\omega} = \frac{c^2\beta^2}{c\omega\left(\frac{c}{\omega}\beta\right)(1-\beta^2)} = \frac{\beta}{(1-\beta^2)}. \tag{12}$$

Combining equations (11, 3, and 12), the radiation power of the system takes the form

$$P=\frac{e^2}{6\pi\varepsilon_0 c^3}\left(\frac{\gamma^2\,\mathrm{v}^2}{r}\right)^2 \underset{r=\frac{\omega}{c}\beta}{=} \frac{(2\pi)^2 e^2 \underline{a}^2}{6\pi\varepsilon_0 c}\nu^2, \tag{13}$$

where $\nu = 1/T$ is the oscillation frequency.

The energy E_1 radiated during one oscillation is just this power multiplied by the period time T .

The result is (using $T\nu = 1$)

$$E_1 = PT = \frac{2\pi e^2 \underline{a}^2}{3\varepsilon_0 c}\,\nu. \tag{14}$$

In order to find the maximum radiated energy per period that the charged SCPO can emit one should substitute the limiting velocity value for pair creation β_m to get normalized acceleration $\underline{a}_m$. The maximum radiated energy is then

$$E_{1,m} = \frac{2\pi e^2 \underline{a}_m^2}{3\varepsilon_0 c}\,\nu = H_m \nu. \tag{15}$$

We introduced the constant H_m defined by the equation above. The value of this constant is

$H_m = 3.82609679 \cdot 10^{-34}$ J s

The energy $E_{1,m}$ is a lower bound to the EM quantum-energy: $E = h\nu > H_m\nu$.

H_m is therefore a lower bound for the Planck constant

$$h > H_m. \tag{16}$$

The value of H_m is very close to the value of Plank's constant, this is an interesting issue to look at more closely. This is done in section V.

B. Comparison with the one dimensional case

Using the approximate potential, related to the force given by eq. (7), of the one dimensional oscillator of constant period (1DCPO) and its equations of motion [2], it is straight forward to calculate its radiation when electrically charged. When the oscillator's energy equals $2mc^2$ the radiated energy per cycle is about 2.6 times less than the energy radiated by the SCPO at same conditions. This result is not very intuitive. At non relativistic velocities the efficiencies (defined as the radiated energy per cycle and per system's energy) of the two systems (1DCPO and

SCPO) are identical. The radiation of a relativistic linearly accelerating charge gets a factor of γ^6 [7]. This provides the one dimensional case with an extra γ^2 factor compared to the circular case [see equation (13)]. The radiation is, therefore enhanced by relativistic effects. The fact is that a 1DCPO at high energies exhibits a saw-tooth profile of position versus time; so that along most of its trajectory there is almost no acceleration. High acceleration is obtained only at the turning points where the velocity (and therefore γ) is not high. From this comparison one can deduce that dimensionality is important concerning radiation efficiency, at relativistic velocities.

V. THE MATCHING FACTOR AND INTERPRETATIONS

Numerically, to a good approximation, one can write

$$H_m = \frac{h}{\sqrt{3}}. \tag{17}$$

The accuracy of the relation is 0.014%.

Such a relation is puzzling: On one hand it can be accidental, on the other hand, as $\sqrt{3}$ is for example, the diagonal of a unit cube; it may be a sign that there is a hidden physical relation.

A. Values of fundamental constants

Some interesting conclusions could be drawn for the fine structure constant α, if we considering H_m not only as the lower bound for h, but also fix the relation:

$$h = \sqrt{3} H_m,$$

and leave it as an investigation issue to explain it. Then, using eq. (15), we find:

$$h = \frac{2\pi e^2 \underline{a}_m{}^2}{\sqrt{3}\varepsilon_0 c}, \tag{18}$$

and being

$$\alpha = \frac{e^2}{2h\varepsilon_0 c}, \tag{19}$$

when inserting the expression for h one gets

$$\alpha = \frac{\sqrt{3}}{4\pi \underline{a}_m^{\,2}}. \tag{20}$$

Here $\underline{a}_m$ is the normalized acceleration calculated according to equation (12) using the limiting velocity β_m of the SCPO; with energy sufficient for pair creation [see value bellow equation (10)]. It is usual to consider the inverse value, for which one gets:

$$\alpha^{-1} = 137.055138.$$

The value of α is in agreement with the experimental value, within the same error, as in the value of h calculated according to eq. (18), of 0.014%. The relative errors are of the order of α^2 which is acceptable if a good physical explanation to the factor of $\sqrt{3}$ is available.

B. Interpretation of alpha

The fine structure constant is thought by many physicists to be even more fundamental quantity than the Planck's constant, in quantum mechanics in general and especially in quantum electrodynamics. This dimensionless quantity is first used in the calculation of relativistic corrections to atomic energy levels together with the spin-orbit coupling [9]. From the result of the present work it is obvious that α origin lies in the relativity theory, i.e., the relativistic connection between mass and energy. Its actual value is determined by the physics of radiation as a result of acceleration, according to the laws of EM theory. To illustrate the meaning of eq. (20) and the normalized acceleration that appears there let's make a use of it.

The rate of photons production A by an elementary oscillating charge according to classical electromagnetism is just the power emitted by the charge divided by the EM quantum-energy of the same frequency. Introducing a normalized acceleration according to the definition in equation (12), A becomes

$$A = \frac{\langle P \rangle}{h\nu} = \frac{e^2 \langle a^2 \rangle}{6\pi\varepsilon_0 c^3}\frac{1}{h\nu} = \frac{4\pi}{3}\alpha \langle \underline{a}^2 \rangle \nu. \tag{21}$$

The angle brackets denote time average.

Inserting the value of α according to eq. (20) into eq. (21) one can write the photon production rate as

$$A = \frac{1}{\sqrt{3}} \left\langle \left(\frac{\underline{a}}{\underline{a}_m} \right)^2 \right\rangle \nu. \tag{22}$$

The quantity $\underline{a}_m$ appears in equation (22) as a reference value for the average normalized acceleration $\langle \underline{a} \rangle$ of elementary radiating systems. The ratio of $\langle \underline{a} \rangle$ to $\underline{a}_m$ is a measure for how fast the system emits photons on a scale of 0 to 1.

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APPENDIX

The force derived from the SCPP that acts on the particle according to Newton's laws in the inertial coordinate system moving instantaneously with the particle is

$$m\mathbf{a}' = \mathbf{F}(\mathbf{r}). \quad \text{(A1)}$$

Bold letters designate vectors.

The acceleration is divided into two components: one is parallel to the particle velocity $\mathbf{v}$, and the other is perpendicular to it

$$\begin{aligned} \mathbf{a}'_{\parallel} &= \frac{\mathbf{a}' \cdot \mathbf{v}}{\mathbf{v}^2}\mathbf{v} \\ \mathbf{a}'_{\perp} &= \mathbf{a}' - \mathbf{a}'_{\parallel}. \end{aligned} \quad \text{(A2)}$$

The acceleration transformation to the coordinate system attached to the center of rotation with fixed orientation is according to special relativity rules [10]

$$\begin{aligned} \mathbf{a}_{\parallel} &= \gamma^{-3}\mathbf{a}'_{\parallel} \\ \mathbf{a}_{\perp} &= \gamma^{-2}\mathbf{a}'_{\perp}. \end{aligned} \quad \text{(A3)}$$

The velocity and position are then calculated in Cartesian coordinates, using $\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp} = [a_x, a_y]$,

$$\begin{aligned} \frac{d\,\mathrm{v}_i}{dt} &= a_i \quad i = x, y \\ \frac{d\,\mathrm{r}_i}{dt} &= \mathrm{v}_i. \end{aligned} \quad \text{(A4)}$$

We preformed numerical computer integration of the set of equations A1-A4 for different initial conditions. All computations were done for $m=1$, and $T=1$ and the results are analyzed in the following sections.

A. Super-relativistic case

Calculation was made for initial velocity $\left[\beta_x,\beta_y\right]_0=\left[0.995,0\right]$, with different initial locations. When the initial location does not cause a circular motion the particle usually bounces between different points on the enclosing circular barrier of potential singularity (See section II.A.2) along lines that are almost straight, without closing a loop. In certain cases a polygonal shape is formed. Two examples, one for a triangle and a second for a square are shown in figure 2 together with the circle resulting from proper initial conditions. The periods of these 3 trajectories are different and they are summarized in table I.

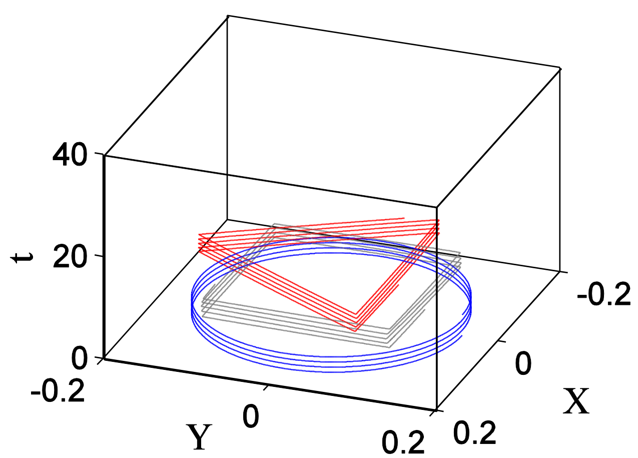

FIG. 2. Calculated periodic trajectories in the super-relativistic ($\beta=0.995$) region on the *xy* plane. The third axis is the time axis. Initial conditions are given in table I.

TABLE I. Periodic trajectories initial conditions and resulted periods and shapes.

Initial conditions			outcome	
No.	$\beta_{x,0}$	$2\pi y_0/\beta_{x,0}$	Period	shape
1	0.995	1	1	circle
2		0.7138	0.905	square
3		0.5053	0.831	triangle
4	0.75	1.1933	0.98	Rounded triangle

B. Intermediate to high energies

At moderate relativistic velocities, the non circular trajectories are rounded. From about $\beta = 0.7$ and up, by fitting the value of the initial perpendicular radius, we find at least one periodic trajectory in addition to the circular one. Such a periodic rounded triangle is shown in figure 3. The initial conditions and the period for this case are summarized in the last row of table I.

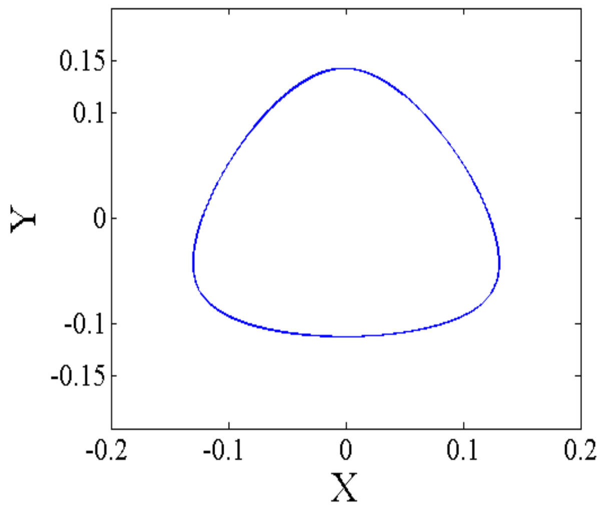

FIG. 3. A calculated periodic trajectory in the intermediate energy region ($\beta = 0.75$), on the xy plane.

C. Low to intermediate energies

At very low nonrelativistic energies the usual elliptic trajectories are obtained, all with the same period and different shapes according to their initial conditions. As the energy increases the elliptical trajectories start to show precession effects, typical to relativistic orbits. . This effect increases with increasing energy until it is no longer possible, in the general case to have a defined period (unless initial conditions ensure a circular orbit). One example of such a trajectory is shown in Figure 4 together with a circle resulting from the same value for β_{x0} .

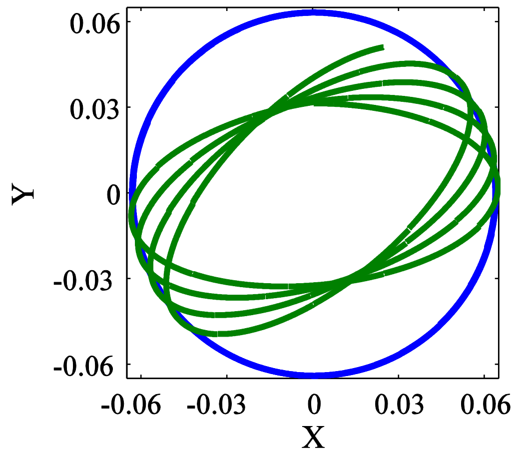

FIG. 4 Calculated trajectories on the xy plane for cases where $\beta_0 = [0.4, 0]$.

1) $\mathbf{r}_0 = [0, 0.2/(2\pi)]$ Results in a precessing ellipse. 2) $\mathbf{r}_0 = [0, 0.4/(2\pi)]$ Results in a circle.

D. Linear trajectories

In addition to the mentioned trajectories, there are linear trajectories resulting from the initial conditions $\mathbf{r}_0 = 0$ or $\mathbf{r}_0 \parallel \boldsymbol{\beta}_0$. The period of such trajectories changes with energy, and in the super-relativistic limit we get:

$$T \xrightarrow[\beta_0 \to 1]{} \frac{2}{\pi} T_0 .$$

Here T_0 is the low energy period of oscillation.

The graph of the period as a function of the initial velocity, for the case $\mathbf{r}_0 = 0$ is shown in figure 5.

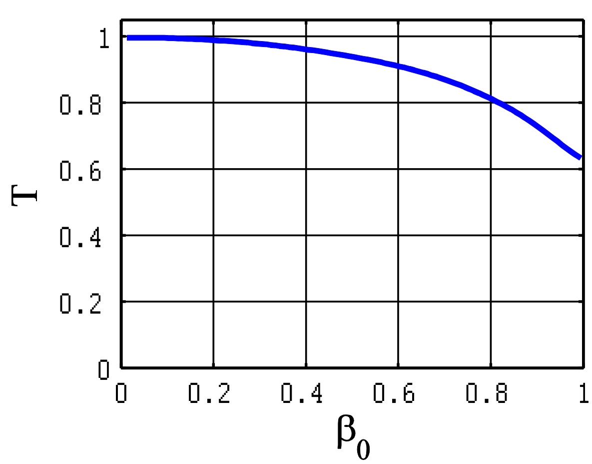

FIG. 5. Linear oscillations' period versus initial velocity (initial position at the origin).

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