A new proof Daniel Pedoe and Oene Bottema inequalities

Profesor Marian Dincă, Şcoala Generală nr. 1
Comuna Pantelimon,Județul Ilfov,România

Abstract:*In the paper given a new proof the two inequalities using unitary method.*

Let a, b, c and a_1, b_1, c_1 the sides of the triangles ABC and $A_1B_1C_1$;
F and F_1 the area of the triangles, such that :
$a^2(b_1^2+c_1^2-a_1^2)+b^2(c_1^2+a_1^2-b_1^2)+c^2(a_1^2+b_1^2-c_1^2) \geq 16FF_1$(*Daniel Pedoe*) (1)

$a_1R_1+b_1R_2+c_1R_3 \geq \left[\frac{a^2(b_1^2+c_1^2-a_1^2)+b^2(c_1^2+a_1^2-b_1^2)+c^2(a_1^2+b_1^2-c_1^2)}{2}+8FF_1\right]^{\frac{1}{2}}$ (*Oene Bottema*) (2)

Where R_1, R_2, R_3 the distance from point P to the vertice of triangle ABC
Proof: Let P the point to plane of the triangle ABC : $\sphericalangle BPC=\alpha$,$\sphericalangle CPA=\beta$, $\sphericalangle APB=\gamma$
For $P\in IntABC$ rezult $\alpha+\beta+\gamma=2\pi$

$$\sum_{ciclic} a^2(b_1^2+c_1^2-a_1^2)=\sum_{ciclic}(R_2^2+R_3^2-2R_2R_3\cos\alpha)(b_1^2+c_1^2-a_1^2)=$$
$$=\sum_{ciclic}(R_2^2+R_3^2)(b_1^2+c_1^2-a_1^2)+\sum_{ciclic}(-2R_2R_3\cos\alpha)(2b_1c_1\cos A_1)=2\sum_{ciclic}R_1^2a_1^2-$$
$$-4\sum_{ciclic}(R_2b_1)(R_3c_1)\cos\alpha\cos A_1 \qquad (3)$$

$16FF_1$ $=8(R_2R_3\sin\alpha+R_3R_1\sin\beta+R_1R_2\sin\gamma)F_1$ $=4R_2R_3\sin\alpha\bullet b_1c_1\sin A_1$ $+4R_3R_1\sin\beta\bullet$ $\bullet c_1a_1\sin B_1$ $+4R_1R_2\sin\gamma\bullet a_1b_1\sin C_1$, (4) and obtain the inequality:

$$2\sum_{ciclic}(R_1a_1)^2\geq 4\sum_{ciclic}(R_2b_1)(R_3c_1)\cos\alpha\cos A_1+4\sum_{ciclic}(R_2b_1)(R_3c_1)\sin\alpha\sin A_1=$$
$$=4\sum_{ciclic}(R_2b_1)(R_3c_1)(\cos\alpha\cos A_1+\sin\alpha\sin A_1)=4\sum_{ciclic}(R_2b_1)(R_3c_1)\cos(\alpha-A_1).$$

Let $R_1a_1=x$, $R_2b_1=y$,$R_3c_1=z$, and $(\alpha-A_1)+(\beta-B_1)+(\gamma-C_1)=2\pi-\pi=\pi$
Following Wolstenholme inequality:
$\sum_{ciclic}x^2\geq 2\sum_{ciclic}yz\cos A$, for $A+B+C=\pi$,and proved the inequality (1)
For inequality (2) using identity (3)and (4) obtain:

$$\left(\sum_{ciclic}R_1a_1\right)^2=\sum_{ciclic}(R_1a_1)^2+2\sum_{ciclic}(R_2b_1)(R_3c_1)\geq\frac{2\sum_{ciclic}(R_1a_1)^2-4\sum_{ciclic}(R_2b_1)(R_3c_1)\cos\alpha\cos A_1}{2}+$$
$$+2\sum_{ciclic}(R_2b_1)(R_3c_1)\sin\alpha\sin A_1$$ and obtain:

$$2\sum_{ciclic}(R_2b_1)(R_3c_1)\geq-2\sum_{ciclic}(R_2b_1)(R_3c_1)\cos\alpha\cos A_1+2\sum_{ciclic}(R_2b_1)(R_3c_1)\sin\alpha\sin A_1$$

obtain:$2\sum_{ciclic}(R_2b_1)(R_3c_1)(1+\cos(\alpha-A_1))$ ≥ 0 it's true because 1+cosx$\geq$ 0 for any x
and proved the inequality (2)

References:
[1]O.Bottema,R.Z.Dordevic,R.R.Janic,D.S.Mitrinovic,P.M.Vasic:Geometric inequalities, Groningen,1969
[2]G.Bennett:Multiple triangle inequalities,Publikacije Elektrotehnic Fakulteta,seria Mat.Fiz. ,nr.577-598,(1977)p.39-44