A Causal Net Approach to Relativistic Quantum Mechanics

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Abstract

In this paper we discuss a causal network approach to describing relativistic quantum mechanics where each vertex on a causal net represents a possible point event or particle observation. By constructing the simplest causal net based on Reichenbach-like conjunctive forks in proper time we can exactly derive the 1+1 dimension Dirac equation for a relativistic fermion and correctly model quantum mechanical statistics. Symmetries of the net provide various quantum mechanical effects such as quantum uncertainty and wavefunction, phase, spin, negative energy states and the effect of a potential. The causal net can be embedded in 3+1 dimensional space-time and is consistent with the conventional Dirac equation. In the low velocity limit the causal net approximates to the Schrödinger equation and Pauli equation for a fermion in an electromagnetic field. Extending to different momentum states the net is compatible with the Feynman path integral approach to quantum mechanics that allows calculation of well known quantum phenomena such as diffraction.

1.0 Introduction.

Causality, or the concept of "cause and effect", has witnessed renewed interest in the physics community as a fundamental principle in recent years. Although Newton's laws were clearly written in causal terms the indeterminacy of quantum mechanics lead to confusion of the role of causality in describing quantum systems. Several attempts have been made to fuse relativity and quantum indeterminism by Belnap and others [1] where they assume that a theory of branching space-time can be built on the primitives of a set of "possible point events" and the causal relations between them. Also recently, Rafael Sorkin and collaborators have explored a causal set approach to quantum gravity [2] using a "Poisson sprinkling" of events in space-time in a Lorentz invariant manner.

In this paper we will outline a discrete causal network approach to describing relativistic quantum mechanics. The fundamental equation of relativistic quantum mechanics is the Dirac equation for a fermion [3]. The Dirac equation is based on the concept of continuous space and time and an exact lattice based formulation has never been achieved. In the famous book *Quantum Mechanics and Path Integrals* [4] Richard Feynman presented a discrete space-time derivation of the 1+1 dimension Dirac equation for a free particle – the "Feynman chequerboard" – since a particle is viewed in the calculation as "zig-zagging" diagonally forwards through space-time in a similar manner to a bishop in chess. At each turn in the path the particle picks up a small phase term and by summing over all paths the Dirac propagator is found. Although a very clever way of discretising the Dirac equation the treatment is an approximation and assumes that the particle moves unrealistically at instantaneously at the speed of light and strangely the right answer is only found in the limiting low velocity case. Many people have pondered over the problem ever since and it has lead to other variants of discrete or lattice based quantum mechanics [5,6,7,8,9]. Notably Kauffman reworked the original chequerboard and suggested "bit string" physics to represent quantum mechanics. Also Kac and others used a discrete underlying diffusion process (again at the speed of light) to derive a continuous time differential equation, reminiscent to the 1+1 dimension Dirac equation, if mass or time is treated as an imaginary number.

In this paper we discuss a causal network discretisation approach which exactly derives the full 4-vector Dirac equation and provides all the common fermion features, such as spin, negative energy states, action of a potential and summation of paths. The most basic causal net describes a plane wave solution with the space axis aligned along the direction of momentum. This 1+1 dimension net can be embedded in 3+1 dimension space-time using the Pauli matrices and is consistent with the full Dirac equation. By combining causal nets with different momentum states a Feynman path integral approach can be used to describe other quantum phenomena such as diffraction.

1.1 Introduction to Common Cause Principles.

The application of probability theory to causality and its relation to the direction of time was developed by Hans Reichenbach. His *principle of common cause* (PCC) [10] was summarized as follows: "If coincidences of the two events A and B occur more frequently than would correspond to their independent occurrence, that is, if these events satisfy $P(A.B) > P(A)P(B)$ then there exists a common cause C for these events that the fork ACB is conjunctive." That is the probability of A and B occurring together is greater than the product of the individual probabilities of A and B. A conjunctive fork ACB (see Fig. 1) between events is open on one side where C is earlier in time than A or B. This asymmetry Reichenbach argued provides a definition of the flow of the direction of time in terms of microstatistics. Essentially a common cause is expected when coincidences or correlations between events occur repeatedly with greater frequency than complete statistical independence $P(A.B) = P(A)P(B)$. The principle of common cause can provide a definition of simultaneity. If A and B are simultaneous there cannot be a causal linkage between them except through the earlier event C.

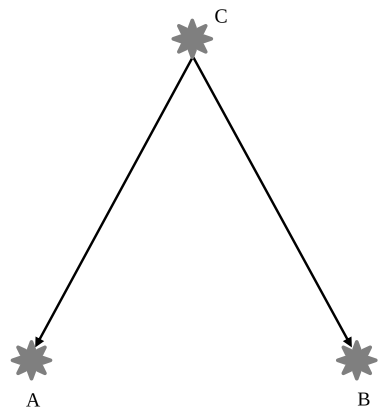

Figure 1: Reichenbach "conjunctive" fork linking events A and B with common cause C. C is earlier in time than simultaneous events A and B.

Reichenbach's principle of common cause is very general and must be obviously slightly modified in a relativistic framework. The relativistic connectivity of possible events must be considered and if from Einstein's relativity the speed of light is the maximum signal speed then the light cone structure must determine the causal structure. Penrose developed a *law of conditional independence* [11] relating the probabilities in regions of space-time that have overlapping past light cones. This is very similar to Reichenbach's PCC but considers the prior relativistic space-time regions C that have causal influence on say events A and B and provide a causal correlation between A and B. The work of Malament is also interesting in providing a relativistic definition of simultaneity between space-time events based on the light cone structure. Malament defined a *standard simultaneity condition* [12] where actual simultaneous events lie on a hyperplane orthogonal to the particle world-line and he defined several symmetries (translation along world line, scale expansion, reflection and spatial rotations) for valid hypersurfaces of simultaneity.

It is widely accepted that quantum mechanics cannot be developed from a basic application of Reichenbach's principle of common cause with a *single* conjunctive fork since general quantum mechanical statistics violate the principle. The principle of common cause involving a single conjunctive fork can even actually be used to formulate the well known Bell's theorem [13,14], which motivated the famous experiments by Alain Aspect [15] to exclude the possibilities of certain types of hidden variables. Here we shall consider a modified framework where a complete causal network of possible events is comprised of conjunctive forks such that each possible event has *two* effective local common causes or screening factors. Adjacent possible events on the net that are simultaneous are thus considered to share a common cause. It appears that, at least in the simple case we consider, this allows our common cause principle – based on the simultaneity of neighbouring *possible* events – to be applied consistently with quantum statistics.

1.2 Space-Time Causal Nets.

We shall adopt a relational view of time as an ordered series of closely spaced "events". Now if we consider time as a series of closely spaced events then from this perspective a classical particle trajectory could appear as a statistically correlated series of events in space-time (for example, a series of actual observations). If the correlation is perfect then one may loosely say that an event at one point in space "causes" the event at the next point, providing a Newtonian trajectory (Fig. 2.i).

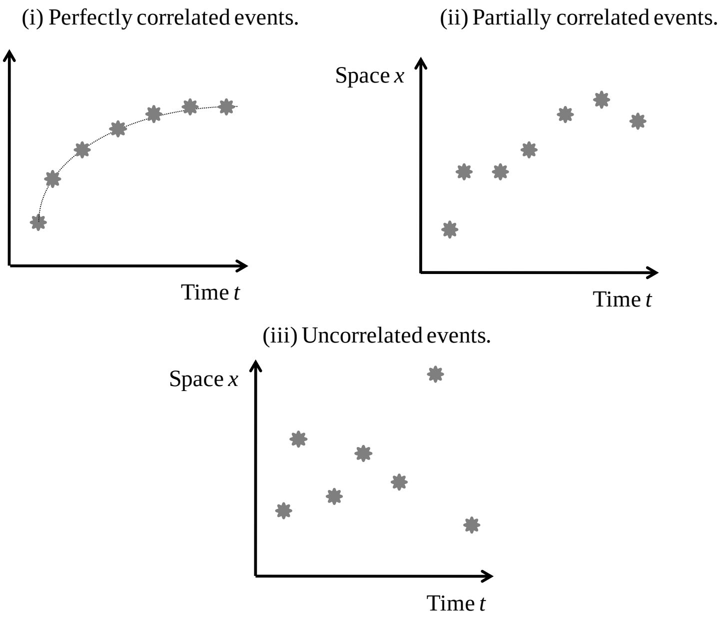

Figure 2: Correlated and uncorrelated events in space-time.

However, if the correlation of events is imperfect, but greater than that resulting from statistical independence, then adjacent events in space are implied to have a common cause originating at a previous time. A trajectory becomes probabilistic in nature and we would have to involve a statistical interpretation (see Fig. 2.ii). We might suppose that the event at an earlier time could be considered to provide a common cause analogous to the common cause discussed by Reichenbach (Section 1.1) [10].

Reichenbach argued in quite general terms that a satisfactory definition of time could only be obtained on the basis of a principle of common cause [10]. By this he meant that if two events A and B occur repeatedly with greater frequency than complete statistical independence predicts (ie. $P(\mathrm{A.B}) > P(\mathrm{A})P(\mathrm{B})$) then there exists an event C at previous time such that the fork ACB is conjunctive, or has one side open (see Fig. 1). A network of such conjunctive forks constitutes a causal net in which time is ordered and events may be considered simultaneous only when they share a common cause.

In discussing the correlation between events we have introduced the notion of probability. We can apply the conventional Lapacian definition of probabilities based on the observed frequency of events divided by the possible number of events. Jaynes [16] demonstrated that probability theory, the mathematics of probabilities, could be essentially derived from logic and common sense. In any case, to further develop our causal net we need to apply probability theory.

To construct the causal net for a particle motion in space-time, we consider a 1 dimensional space aligned with the direction of particle motion, and embedded in 3 dimensional space. In this 1 dimensional space the simplest causal net that satisfies our definition of simultaneity is a 1+1 dimensional “diamond” lattice with causal links connecting the lattice points as in Figure 3.

Each causal connection is defined by a connecting arrow giving a definite lineal order and an associated probability. Each vertex on the causal net represents a possible event – meaning a possible observation of the particle – and has two incoming and two outgoing causal connections so that each event has two effective possible common causes. Starting at a vertex and following an outgoing arrow at random at each subsequent vertex describes a “causal chain” as a series of possible events. The causal net thus describes the connectivity between all causal connections and can statistically model all future possibilities.

Measurement or observation at a vertex or a region of the net provides, through Bayesian statistics, a re-evaluation of these probabilities after a measurement. For example, if we possess no knowledge of where an actual event might occur on the net but observe an event at a particular location then, due to the connectivity of the net, we can say that certain causal chains or paths could have lead to the event and that it was more likely that the event was preceded by an event in a cone or region of previous possible events. This is illustrated in Figure 3 where an event at A is more likely to have been caused by an event at B than C and D is an impossibility due to zero connectivity between the paths. Bayes theorem provides a way of translating this common sense concept into a formal probabilistic context since $P(A|B) > P(A|C) > P(A|D)$.

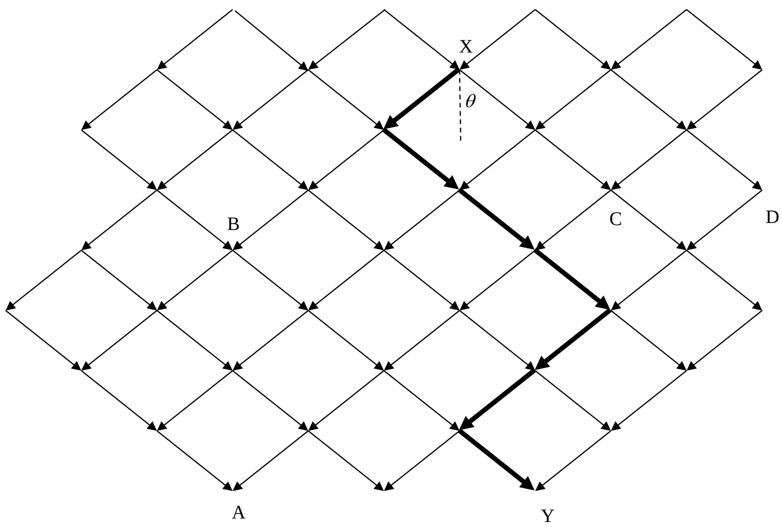

Figure 3: Causal net showing causal chain from X to Y.

1.3 Relativistic Causal Nets for a Free Particle.

First we will consider the simple case of a particle randomly diffusing on the causal net shown in Figure 3. In this model time and space can be discretely “counted” by attaching an integer to each of the vertex points but there is no underlying continuous space-time. To relate to conventional mechanics we interpolate this set of integers by a set of real number coordinates. Expecting that space and time have different dimensions we need to introduce a constant c with dimensions [space/time]. The net is then made up of elementary triangles labelled with $(\Delta x, c\Delta t, c\Delta\tau)$ as shown in Figure 4. We have not yet added any specific interpretation to these quantities. To guarantee invariance of causality on the net we impose c as the speed of light [17]. Since, from geometry, $\frac{\Delta x}{c\Delta t} = sin\theta \leq 1$, we then identify Δx and Δt as relativistic space-time intervals in an observer frame S and $\Delta\tau$ as the particle proper time interval in its rest frame S'. The net geometry guarantees the invariant space-time interval

$$(c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2.$$

(1)

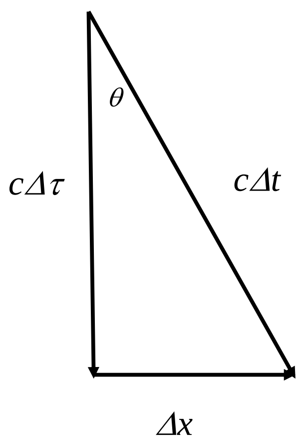

Figure 4: The elementary space-time “triangle” for the causal net.

Having abandoned the concept of absolute and continuous space-time we need to define the observed velocity in terms of finite differences. The definition we shall adopt is $v = \Delta x/\Delta t$ which we equate to the expectation of the velocity on the causal net. The two time intervals are then related by $\Delta\tau = \Delta t/\gamma$ where γ is the Lorentz factor

$$\gamma = 1/\sqrt{1 - v^2/c^2}$$

specifying the net angles

$$\cos\theta = \frac{1}{\gamma}\,, \qquad sin\theta = \frac{v}{c}$$

We now specialise to the case of the motion of a free particle. Clearly Eq. (1) and thus the net can be scaled by a factor. If we identify this with the particle rest mass m then rearranging Eq. (1) we then have the relativistic dispersion relation $E^2 = p^2c^2 + m^2c^4$ where E is the particle energy $E = \gamma mc^2$ and $p = \gamma mv$ the momentum.

By construction we require the lattice to describe only physically admissible motions of the particle. Experience shows that real particle trajectories obey a principle of least action – that is the integral $\int p\, dx$ is stationary. This is a restatement of Maupertuis principle which is a weak form of the well known principle of least action and is experimentally verified in classical mechanics. On our net if the action $\sum p\Delta x$ differed for different trajectories then this would rule some trajectories as physically inadmissible. Therefore we conclude that $\sum p\Delta x$ is the same on the lattice for all paths between two points which means that $p\Delta x$ is a constant η for a valid causal net.

1.4 Free Particle Motion on the Causal Net.

To impose our imperfect correlation of events we shall assume that there is an indeterminism or randomness to the particle motion at each net vertex. We shall make the assumption that this indeterminism is governed by Eq. (1) on the causal net. Thus a particle in its own rest frame S' over interval $\Delta\tau$ moving at a speed |v| in frame S can move to a position $\pm\Delta x$ in time Δt. This produces a random trajectory in space-time (see Fig. 3).

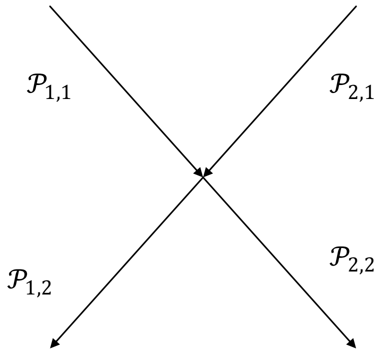

Figure 5: A vertex (1,2) on the causal net with associated probabilities.

Initially we shall consider "classical" or non quantum probabilities in construction of the causal net. Consider an individual vertex on the net and label the incoming probabilities on row 1 $\mathcal{P}_{1,1}$ and $\mathcal{P}_{2,1}$ and outgoing probabilities on row 2 $\mathcal{P}_{1,2}$ and $\mathcal{P}_{2,2}$ (Fig. 5). Probability is conserved at the vertex and the total probability at a vertex is given by $\Omega_{1,2} = \mathcal{P}_{1,1} + \mathcal{P}_{2,1}$. If the average velocity measured on the lattice is uniform then $\mathcal{P}_{1,1} = \mathcal{P}_{2,2}$ and $\mathcal{P}_{1,2} = \mathcal{P}_{2,1}$. This implies that the probabilities "cross" at each vertex without actually interfering although the probabilities are coupled. We shall see that this corresponds to the equilibrium case of a free particle. If we consider normalised branching probabilities at the vertex defined as $\hat{\mathcal{P}}_{1,1} + \hat{\mathcal{P}}_{2,1} = 1$ then since expected velocity at the vertex is defined to be v we have

$$\mathbb{E}[v] = \gamma \frac{\Delta x}{\Delta t}\left[\hat{\mathcal{P}}_{1,1} - \hat{\mathcal{P}}_{2,1}\right] = v. \tag{2}$$

The branching probabilities are then given by

$$\hat{\mathcal{P}}_{1,1} = \frac{E + mc^2}{2E}\ , \qquad \hat{\mathcal{P}}_{2,1} = \frac{E - mc^2}{2E}. \tag{3}$$

From this we can see that in the low velocity limit $|\mathrm{v}| \to 0$ then $\hat{\mathcal{P}}_{1,1} \to 1$ and $\hat{\mathcal{P}}_{2,1} \to 0$ and in the high velocity limit $|\mathrm{v}| \to c$ then $\hat{\mathcal{P}}_{1,1} \to \hat{\mathcal{P}}_{2,1} \to 1/2$. The branching ratio Γ can be written as a function of γ or the net angle θ.

$$\Gamma = \frac{\hat{\mathcal{P}}_{1,1}}{\hat{\mathcal{P}}_{2,1}} = \frac{E + mc^2}{E - mc^2} = \frac{\gamma + 1}{\gamma - 1} = \frac{1 + \cos\theta}{1 - \cos\theta} \tag{4}$$

Using the branching probability (Eq. 4) we can write a non trivial matrix equation linking the probabilities

$$\boldsymbol{\mathcal{P}} = \begin{pmatrix} \mathcal{P}_{1,1} \\ \mathcal{P}_{2,1} \end{pmatrix} = \begin{pmatrix} 0 & \Gamma \\ 1/\Gamma & 0 \end{pmatrix} \begin{pmatrix} \mathcal{P}_{1,1} \\ \mathcal{P}_{2,1} \end{pmatrix}. \tag{5}$$

1.5 Relativistic Quantum Mechanics on the Causal Net.

We shall now see how the causal net is compatible with the quantum mechanics of the Dirac equation for a free particle. We notice that identifying the net constant η with Planck's constant h provides the de Broglie relation $\lambda\, p = h$ [18] with $\Delta x = \lambda/2$, and a Heisenberg like relation $\Delta p \Delta x \sim h/2$ [19,20]. The discrete nature of the net automatically entails a de Broglie relation and an uncertainty principle.

Now a very general way of forming the probabilities $\mathcal{P}_{i,j}$ for the branch (i,j) is through a vector dot product $\mathcal{P}_{i,j} = \boldsymbol{\phi}_{\mathrm{i,j}} .\, \boldsymbol{\phi}_{\mathrm{i,j}}$ with $\boldsymbol{\phi}_{\mathrm{i,j}} = \sqrt{\mathcal{P}_{i,j}}\ (\mathrm{a}(\tau), b(\tau))$ with real components that depend on the proper time τ at the net vertices. The probability is invariant in the rest frame of the particle and equivalent to a gauge relationship that conserves probability in S' so using the relativistic invariance $-mc^2\Delta\tau = p\Delta x - E\Delta t$ we can write the vector as

$$\boldsymbol{\phi}_{\mathrm{i,j}} = \sqrt{\mathcal{P}_{i,j}} \begin{pmatrix} \mathrm{a}(\tau) \\ \mathrm{b}(\tau) \end{pmatrix} = \sqrt{\mathcal{P}_{i,j}} \begin{pmatrix} \cos(mc^2\tau/\hbar) \\ \sin(mc^2\tau/\hbar) \end{pmatrix}$$

$$= \sqrt{\mathcal{P}_{i,j}} \begin{pmatrix} \cos\left((px - Et)/\hbar\right) \\ \sin\left((px - Et)/\hbar\right) \end{pmatrix}.$$

However, a more conventional and compact way of forming the probabilities is through introducing complex numbers, rather than vectors, to carry the phase information. So instead by combining complex probability amplitudes we can write $\mathcal{P}_{i,j} = \phi_{\mathrm{i,j}} .\, \phi^*_{\mathrm{i,j}}$ with

$$\phi_{\mathrm{i,j}} = \sqrt{\mathcal{P}_{i,j}}\ \mathrm{e}^{-\frac{\mathrm{i}mc^2\tau}{\hbar}}, \tag{6}$$

which depends on the proper time τ at the net vertices. Notably the phase is independent of position x for a particular τ. Again using the relativistic invariance we can write the probability amplitude as

$$\phi_{\mathrm{i,j}} = \sqrt{\mathcal{P}_{i,j}}\ \mathrm{e}^{-\frac{\mathrm{i}mc^2\tau}{\hbar}} = \sqrt{\mathcal{P}_{i,j}}\mathrm{e}^{\frac{\mathrm{i}(px - Et)}{\hbar}}, \tag{7}$$

Where x and t are defined at the discrete net vertices and for the moment we consider only positive roots. We can then rewrite Eq. (5) as

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_{1,1} \\ \phi_{2,1} \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\Gamma} \\ 1/\sqrt{\Gamma} & 0 \end{pmatrix} \begin{pmatrix} \phi_{1,1} \\ \phi_{2,1} \end{pmatrix}, \tag{8}$$

which can be alternatively expressed for Eq. (7) in terms of a *unique* transfer matrix $\boldsymbol{M}$

$$\boldsymbol{\phi} = \begin{pmatrix} \phi_{1,1} \\ \phi_{2,1} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} \phi_{1,1} \\ \phi_{2,1} \end{pmatrix}, \tag{9}$$

defined as

$$\boldsymbol{M} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} 1/\gamma & v/c \\ v/c & -1/\gamma \end{pmatrix} = \frac{1}{E}\begin{pmatrix} mc^2 & pc \\ pc & -mc^2 \end{pmatrix} = \frac{H_D}{E}. \tag{10}$$

Here we recognise H_D as the Dirac Hamiltonian for a free particle [3, 21] with defined momentum p. To connect with the complete quantum mechanics we note that Eq. (10) can be put in the conventional form [21] by assuming that space-time is locally differentiable at the vertex, allowing us to use the usual momentum operator $\hat{\boldsymbol{p}}$ to replace the momentum eigenvalues p writing

$$\begin{pmatrix} mc^2 & c\hat{\boldsymbol{p}} \\ c\hat{\boldsymbol{p}} & -mc^2 \end{pmatrix}\boldsymbol{\Psi} = \mathrm{E}\boldsymbol{\Psi} = i\hbar\frac{\partial\boldsymbol{\Psi}}{\partial t},$$

(11)

where we have replaced the probability amplitudes $\boldsymbol{\phi}$ with the familiar 2 component Dirac spinor $\boldsymbol{\Psi}$ for the free particle [22]

$$\boldsymbol{\Psi}(\mathrm{x,t}) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \boldsymbol{\phi} = \begin{pmatrix} \phi_{1,1} \\ \phi_{2,1} \end{pmatrix} = A\begin{pmatrix} 1 \\ \dfrac{pc}{E+mc^2} \end{pmatrix}\mathrm{e}^{\frac{\mathrm{i}(px-Et)}{\hbar}}$$

(12)

and A is an appropriate normalisation constant.

1.6 Causal Net Quantum Symmetries and Spin.

Note that the unique matrix $\boldsymbol{M}$ above is a unitary, orthogonal matrix which provides an SU(2) group transformation corresponding to an improper rotation – that is a rotation $\boldsymbol{R}(\theta)$ followed by an inversion $\boldsymbol{\beta}$ so $\boldsymbol{M} = \boldsymbol{\beta}\boldsymbol{R}(\theta)$. The matrix provides the transformations for the probabilities $\boldsymbol{\mathcal{P}} = \mathbf{M}^2\boldsymbol{\mathcal{P}} = \mathbf{I}\boldsymbol{\mathcal{P}}$ and probability amplitudes $= \boldsymbol{M}\boldsymbol{\Psi}$. Importantly because $\mathbf{M}^3 = \mathbf{M}$ there automatically exists only two levels of symmetry at the vertex and the causal net provides simultaneously both the probabilities and the underlying probability amplitudes. Since it is an improper rotation the symmetry determines a preferred axis which provides helicity along the axis of movement. If we revisit Eq. (7) and consider both possible positive and negative roots we can see that even and odd solutions that provide helicity $\lambda = \pm 1$ with positive energy $\varepsilon = +1$ are given by

$$\boldsymbol{\phi}_{\substack{\varepsilon=+1 \\ \lambda=+1}} = \begin{pmatrix} \sqrt{\mathcal{P}_{1,1}} \\ \sqrt{\mathcal{P}_{2,1}} \end{pmatrix} \qquad \boldsymbol{\phi}_{\substack{\varepsilon=+1 \\ \lambda=-1}} = \begin{pmatrix} \sqrt{\mathcal{P}_{1,1}} \\ -\sqrt{\mathcal{P}_{2,1}} \end{pmatrix}$$

(13)

corresponding to transfer matrices $\boldsymbol{M}_{\varepsilon=+1,\lambda=\pm 1} = \boldsymbol{\beta}\boldsymbol{R}(\pm\theta)$. Note that in the above and following discussion we have omitted the phase factor $\mathrm{e}^{-\mathrm{i}mc^2\tau/\hbar}$ since this cancels in both sides of Eq. (9).

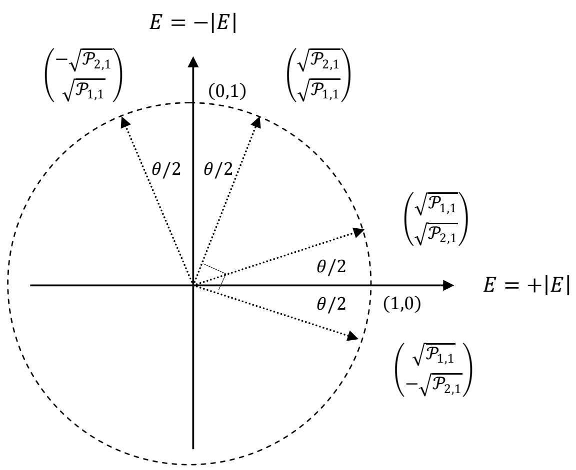

Figure 6: Representation of 1+1 dimension causal net solutions (Eq. 13 and Eq. 14) to the Dirac equation.

1.7 Negative Energy States.

Until now we have considered only the positive energy states, but negative energy solutions arise from the negative solution of the relativistic dispersion relation $E = \varepsilon\sqrt{p^2c^2 + m^2c^4} = \varepsilon|E|$ = $\epsilon\gamma mc^2$ ($\varepsilon = \pm 1$). This results in a reversal of the branching probabilities in Eq. (3) and two additional possible even and odd spinor solutions

$$\boldsymbol{\phi}_{\substack{\varepsilon=-1\\ \lambda=+1}} = \begin{pmatrix} -\sqrt{\mathcal{P}_{2,1}} \\ \sqrt{\mathcal{P}_{1,1}} \end{pmatrix} \qquad \boldsymbol{\phi}_{\substack{\varepsilon=-1\\ \lambda=-1}} = \begin{pmatrix} \sqrt{\mathcal{P}_{2,1}} \\ \sqrt{\mathcal{P}_{1,1}} \end{pmatrix} \tag{14}$$

for transfer matrices $\boldsymbol{M}_{\varepsilon=-1,\lambda=\pm 1} = -\boldsymbol{\beta R}(\pm\theta)$. These states provide inverted branching ratios in Eq. (4) so in our model negative energy states correspond to particles moving in the opposite spatial direction or with negative velocity. This implies that either the negative energy solutions are invalid in our framework or the net should be perhaps redefined as speed rather than velocity with $\mathbb{E}[v] = \pm|v|$ instead of Eq. (2). These negative energy solutions however, have a phase that evolves in the opposite sense with proper time in Eq. (6).

1.8 The Foldy–Wouthuysen Transformation.

The causal net is also consistent with the extraordinarily simple Foldy–Wouthuysen representation [23] of the Dirac equation where the positive and negative energy states are decoupled through a rotation of θ (the lattice angle) $\boldsymbol{R}(\theta)$ of the Dirac Hamiltonian. For example, one Foldy–Wouthuysen state is given by the rotation through $\theta/2$ of the Dirac state Eq. (12)

$$\boldsymbol{\phi}_{\substack{FW\\ \varepsilon=+1}} = \boldsymbol{R}(\theta/2)\boldsymbol{\phi}_{\substack{Dirac\\ \varepsilon=+1\\ \lambda=+1}} = \begin{pmatrix} \sqrt{\mathcal{P}_{1,1}} & \sqrt{\mathcal{P}_{2,1}} \\ -\sqrt{\mathcal{P}_{2,1}} & \sqrt{\mathcal{P}_{1,1}} \end{pmatrix}\begin{pmatrix} \sqrt{\mathcal{P}_{1,1}} \\ \sqrt{\mathcal{P}_{2,1}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{15}$$

This rotated state and the previous Dirac states (Eq. 12 and Eq. 13) can be represented more clearly as vectors in the Figure 6. The Foldy–Wouthuysen states correspond with our framework where the Hamiltonian and velocity operator satisfy their classical analogues. Importantly, in this representation, establishing an exact particle position is impossible (there is only a mean position operator) and a particle is viewed as spread out over a finite region of about a wavelength which is consistent with our causal net picture since we cannot localise a particle between two adjacent net vertices without an averaging over vertices being performed in a measurement.

1.9 The Complete 4-Vector Dirac Equation in 1+1 Dimension.

If we include the negative energy states then, by combining all 4 net solutions above (Eq. 13 and Eq. 14), we can write 4 orthogonal 4-vectors which for helicity $\lambda = \pm 1$ and $\varepsilon = \pm 1$
are

$$\boldsymbol{\Psi}_{p,\varepsilon,\lambda=+1}(\mathrm{x}, \mathrm{t}) = \mathrm{A}\sqrt{\frac{E + mc^2}{2E}}\begin{pmatrix} 1 \\ 0 \\ \dfrac{cp}{E + mc^2} \\ 0 \end{pmatrix}\mathrm{e}^{\frac{\mathrm{i}(px - Et)}{\hbar}}$$

$$E = \varepsilon|E|, \lambda = +1$$

$$\boldsymbol{\Psi}_{p,\varepsilon,\lambda=-1}(\mathrm{x},\mathrm{t}) = \mathrm{A}\sqrt{\frac{E+mc^2}{2E}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-cp}{E+mc^2} \end{pmatrix}\mathrm{e}^{\frac{\mathrm{i}(px-Et)}{\hbar}}$$

$$E = \varepsilon|E|, \lambda = -1 \qquad (16)$$

where A is again an appropriate normalization constant. Using the 4 possible transfer matrices $\boldsymbol{M}_{\varepsilon=\pm1,\lambda=\pm1}$ then the l+1 dimension 4-matrix Dirac equation Eq. (15) is

$$\begin{pmatrix} mc^2 & c\boldsymbol{\beta}\widehat{\boldsymbol{p}} \\ c\boldsymbol{\beta}\widehat{\boldsymbol{p}} & -mc^2 \end{pmatrix}\boldsymbol{\Psi} = \mathrm{E}\boldsymbol{\Psi} = i\hbar\frac{\partial\boldsymbol{\Psi}}{\partial t}. \qquad (17)$$

Now this Dirac equation and the spinor wavefunction Eq (16) correspond to exactly the conventional 3+1 dimension Dirac spinor for the special case of the particle moving along the x-axis and with a well defined spin (helicity) aligned parallel and antiparallel with the x-axis [21].

1.10 The 3+1 Dimension Dirac Equation.

To extend to the general 3+1 dimension case we must consider transformations of the causal net that leave it invariant under variation of direction of velocity **v**. Using polar coordinates then for momentum $\boldsymbol{p} = |\boldsymbol{p}|(sin\vartheta cos\varphi, sin\vartheta sin\varphi, cos\vartheta)$ and we can expect that the wavefunction components become dependent on the coordinates (ϑ,φ) so $\sqrt{\mathcal{P}_{i,j}}$ becomes $\sqrt{\mathcal{P}_{i,j}}\boldsymbol{\chi}(\vartheta,\varphi)$. Using a reduced vector notation $\left(cos\vartheta, sin\vartheta e^{i\varphi}\right)$ and following Dirac's convention [3] we can replace the 1 dimensional momentum operator $\widehat{\boldsymbol{p}}$ with the 3 dimensional momentum operator Eq. (17)

$$\widehat{\boldsymbol{\sigma}}.\widehat{\boldsymbol{p}} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = |\boldsymbol{p}|\begin{pmatrix} cos\vartheta & sin\vartheta e^{-i\varphi} \\ sin\vartheta e^{i\varphi} & -cos\vartheta \end{pmatrix} \qquad (18)$$

formed from Pauli matrices $\boldsymbol{\sigma}_k$

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (19)$$

Eq. (18) is an improper rotation and by definition the provides the relation $(\widehat{\boldsymbol{\sigma}}.\widehat{\boldsymbol{p}})\boldsymbol{\chi}_{\pm} = \pm|\boldsymbol{p}|\boldsymbol{\chi}_{\pm}$ with two eigenvectors

$$\boldsymbol{\chi}_{+} = \begin{pmatrix} cos\frac{\vartheta}{2} \\ sin\frac{\vartheta}{2}e^{i\varphi} \end{pmatrix} \quad , \quad \boldsymbol{\chi}_{-} = \begin{pmatrix} -sin\frac{\vartheta}{2}e^{-i\varphi} \\ cos\frac{\vartheta}{2} \end{pmatrix} \qquad (20)$$

The general solutions for the wavefunction then become (from Eq. 12 and Eq. 13) four 4-component orthogonal vectors corresponding to up and down spin $S = \pm1/2$ with positive and negative energies $\varepsilon = \pm1$. Omitting the phase factors $\mathrm{e}^{-\mathrm{imc}^2\tau/\hbar}$ and normalisation constant these are

$$\boldsymbol{\Psi}_{\substack{\varepsilon=+1 \\ S=+1/2}} = \begin{pmatrix} \sqrt{\mathcal{P}_{1,1}}\boldsymbol{\chi}_{+} \\ \sqrt{\mathcal{P}_{2,1}}\boldsymbol{\chi}_{+} \end{pmatrix} \qquad \boldsymbol{\Psi}_{\substack{\varepsilon=+1 \\ S=-1/2}} = \begin{pmatrix} \sqrt{\mathcal{P}_{1,1}}\boldsymbol{\chi}_{-} \\ -\sqrt{\mathcal{P}_{2,1}}\boldsymbol{\chi}_{-} \end{pmatrix}$$

$$\boldsymbol{\Psi}_{\substack{\varepsilon=-1\\S=+1/2}} = \begin{pmatrix} -\sqrt{\mathcal{P}_{2,1}}\boldsymbol{\chi}_+ \\ \sqrt{\mathcal{P}_{1,1}}\boldsymbol{\chi}_+ \end{pmatrix} \qquad \boldsymbol{\Psi}_{\substack{\varepsilon=-1\\S=-1/2}} = \begin{pmatrix} \sqrt{\mathcal{P}_{2,1}}\boldsymbol{\chi}_- \\ \sqrt{\mathcal{P}_{1,1}}\boldsymbol{\chi}_- \end{pmatrix}$$

(21)

These are the general solutions to the conventional 3+1 dimension Dirac equation

$$\begin{pmatrix} mc^2 & c\widehat{\boldsymbol{\sigma}}.\widehat{\boldsymbol{p}} \\ c\widehat{\boldsymbol{\sigma}}.\widehat{\boldsymbol{p}} & -mc^2 \end{pmatrix}\boldsymbol{\Psi} = \mathrm{E}\boldsymbol{\Psi} = i\hbar\frac{\partial\boldsymbol{\Psi}}{\partial t},$$

(22)

which can also be written in the well known form

$$\left[c\widehat{\boldsymbol{\alpha}}.\widehat{\boldsymbol{p}} + mc^2\widehat{\boldsymbol{\beta}}\right]\boldsymbol{\Psi} = \mathrm{E}\boldsymbol{\Psi} = i\hbar\frac{\partial\boldsymbol{\Psi}}{\partial t},$$

(23)

using the matrices $\widehat{\boldsymbol{\alpha}}$ and $\widehat{\boldsymbol{\beta}}$ defined as

$$\boldsymbol{\alpha}_k = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{\sigma}_k \\ \boldsymbol{\sigma}_k & \boldsymbol{0} \end{pmatrix}, \qquad \widehat{\boldsymbol{\beta}} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I} \end{pmatrix}.$$

(24)

Thus for a 3 dimensional space we require all 3 Pauli matrices to construct the vector $\widehat{\boldsymbol{\sigma}}$ and the dimensionality of space defines the Pauli matrices. If we rotate Eq. (22) by $(-\vartheta/2)$ then we see that the net is constructed along the momentum direction $\left(cos\vartheta, sin\vartheta e^{i\varphi}\right)$ and that the causal net transfer matrix in Eq. (10) is invariant. Velocity or momentum of the particle at any point in space-time only has one direction even though it is embedded in 3 dimensional space and a plane wave solution to the Dirac equation has a unique velocity direction. The connectivity of the plane wave causal net itself only requires one space axis. The specification of the Pauli scheme is not unique and we could use a different matrix system or permutation and combining with Eq. (9) this allows 12 different permutations.

1.11 Effect of Potential on Causal Net and the Pauli Equation.

The case we have examined is that of a free particle but we could include a potential V on the causal net since E can be replaced by $E–V$ in the construction of the lattice and the branching ratios. Returning to the 1+1 dimension case, between two media with different potentials the net is compressed or stretched in space in the potential region with a form similar to Snell's law $cos\theta_2/cos\theta_1 = E/(E-V)$. We can write Eq. (10) conveniently as

$$\boldsymbol{M} = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ \sin\theta_2 & -\cos\theta_2 \end{pmatrix} = \frac{1}{E-V}\begin{pmatrix} mc^2 & pc \\ pc & -mc^2 \end{pmatrix} = \frac{H_D}{E-V}.$$

(25)

Recall that an incremental change in potential is equivalent to a force acting on the particle so our net can encompass the full mechanics of the particle. The probability current is conserved at a potential barrier if we consider the relativistic change in probability across the barrier arising from a Lorentz contraction/expansion.

Taking these ideas a bit further we can consider that introducing a phase in Eq. (6) is equivalent to a global gauge invariance represented by a unitary group U(1). The Lagrangian for our net can be written in the form $\boldsymbol{\phi}^+\boldsymbol{M}\boldsymbol{\phi} - \boldsymbol{\phi}^+\boldsymbol{\phi} = 0$. What happens if we consider invariance under a local gauge transformation? For example, if we consider an SU(2) matrix, such as a rotation $\boldsymbol{U} = \boldsymbol{R}(\zeta)$, then in general $\boldsymbol{M}(\boldsymbol{U}\boldsymbol{\phi}) \neq (\boldsymbol{U}\boldsymbol{\phi})$ so to retain invariance we must add an additional term to the

Lagrangian and the Dirac equation which corresponds to an electromagnetic potential term. We can see that a small change ζ in net angle relates to a vector potential providing an apparent force. To illustrate, consider the special case of a transformation where the proper time interval $\Delta\tau$ is unchanged by a potential. The triangle in Figure 3 is deformed by an amount δt in time and δx in space

$$(c\Delta\tau)^2 = (c(\Delta t - \delta t))^2 - (\Delta x - \delta x)^2. \tag{26}$$

If we write $eA_0 = \gamma mc\delta t/\Delta t$ and $eA_1 = \gamma mc\delta x/\Delta t$ then we have the dispersion relation for an electron of charge e in an electromagnetic field (A_0, A_1)

$$(E - eA_0)^2 = (pc - eA_1)^2 + m^2c^4, \tag{27}$$

and the corresponding transfer matrix $\boldsymbol{M}$ is given by

$$\boldsymbol{M} = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ \sin\theta_2 & -\cos\theta_2 \end{pmatrix} = \frac{1}{E - eA_0}\begin{pmatrix} mc^2 & pc - eA_1 \\ pc - eA_1 & -mc^2 \end{pmatrix} \tag{28}$$

If as Eq. (22) we embed the causal net in a continuous 3 dimensional space we can replace p with the 3 dimensional momentum operator $\hat{\boldsymbol{\sigma}}.\hat{\boldsymbol{p}}$ and use the full vector form for the field $(\boldsymbol{A}, A_0)$ and write this as

$$\begin{pmatrix} mc^2 & c\hat{\boldsymbol{\sigma}}.\hat{\boldsymbol{p}} - e\boldsymbol{A} \\ c\hat{\boldsymbol{\sigma}}.\hat{\boldsymbol{p}} - e\boldsymbol{A} & -mc^2 \end{pmatrix}\boldsymbol{\Psi} = (E - eA_0)\boldsymbol{\Psi} \tag{29}$$

Consider the case of non-relativistic motion in a weak field so that $E' = (E - eA_0)$ and $|E' - eA_0| \ll mc^2$. We can neglect smaller component of the spinor and have, following [29], the Pauli equation for a non-relativistic spin-1/2 particle

$$\left[\frac{(\hat{\boldsymbol{p}} - e\boldsymbol{A})^2}{2m} + eA_0 - \frac{e\hbar}{2mc}(\hat{\boldsymbol{\sigma}}.\boldsymbol{H})\right]\psi_1 = E'\psi_1 \tag{30}$$

where $\boldsymbol{H} = curl\,\boldsymbol{A}$.

1.12 Momentum States and the Feynman Path Integral.

The causal net, denoted ℵ, we have considered is for a free particle with one specific momentum state. It is interesting to consider the more general case of a range of momentum states with each momentum state occupying an individual net $\aleph(p)$. This can be visualised in Figure 7 as a stacked "deck" of infinite causal nets. Importantly, due to the different net sizes (Eq. 1) the vertices of each net do not overlap.

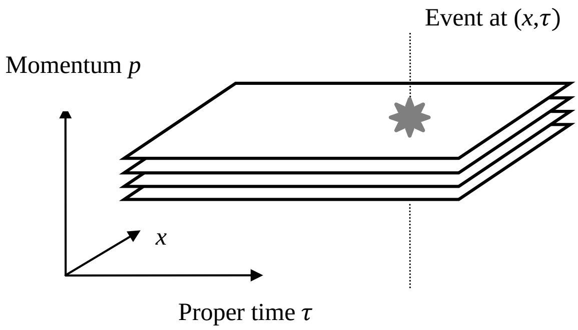

Figure 7: A stacked "deck" of causal nets for different momentum states.

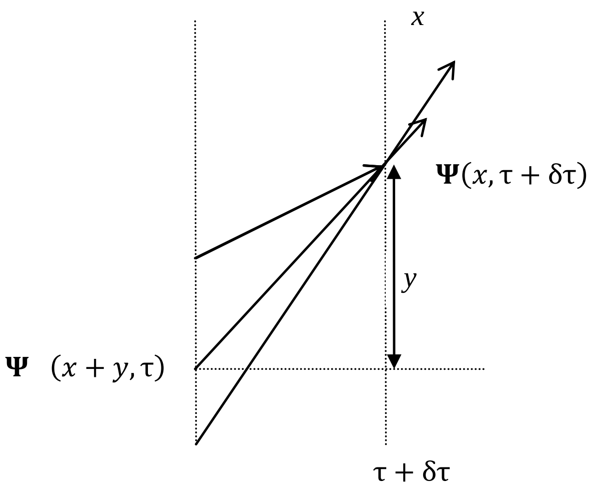

Figure 8: Causally connected paths traversing a single space-time point.

For a point in space-time (in practice this could be defined as an infinitesimal region of measurement) we can consider which events can causally act on a point at (x,t) from the past. To retain our definition of simultaneity as being defined by events that occur at the same proper time then it is a more consistent description to consider a point in proper time at $(x, \tau + \delta\tau)$ and the prior events that are causally connected from a earlier slice of proper time at τ which are given by *different* space points x from each momentum net $\aleph(p)$ (Fig. 8). This is equivalent to considering which possible particle paths pass through the point. To illustrate we shall consider only positive energy states but we can easily extend to include negative energy states. If we sum the different spinor components contributing to the overall probability amplitude at (x, τ) and include the change in phase over interval $\delta\tau$ from Eq. (6) we have

$$\mathbf{\Psi}(\mathrm{x}, \tau + \delta\tau) = \sum_{\substack{\text{causally} \\ \text{connected} \\ \text{events}}} \mathbf{\Psi}(\mathrm{x} + \mathrm{y}, \tau) \mathrm{e}^{-\frac{\mathrm{imc}^2 \delta\tau}{\hbar}}$$

(31)

here y is the relative space coordinate (Fig. 8). For one casual net with a free particle with a single momentum state Eq. (31) is trivial since velocity and probability are uniform across the net with only the phase varying with τ.

$$\mathbf{\Psi}(\mathrm{x},\tau+\delta\tau)=\mathbf{M}\,\mathbf{\Psi}(\mathrm{x},\tau+\delta\tau)=\ \mathbf{\Psi}(\mathrm{x}+\mathrm{y},\tau)\mathrm{e}^{-\frac{\mathrm{imc}^2\delta\tau}{\hbar}} \tag{32}$$

providing a simple delta function propagator for proper time interval $\delta\mathrm{t}'$

$$\mathrm{K}(\mathrm{x},\mathrm{x}+\mathrm{y};\ \tau+\delta\tau,\tau)=\ \delta(\mathrm{y})\mathrm{e}^{-\frac{\mathrm{imc}^2\delta\tau}{\hbar}} \tag{33}$$

However if there is a continuum of momentum nets by geometry the sum in Eq. (31) selects a single probability amplitude contribution from each net with momentum $\mathrm{p}=\mathrm{my}/\delta\tau$ for a given relative position y. Writing the relativistic infinitesimal action $\mathrm{S}(\delta\tau)=-\mathrm{mc}^2\delta\tau$ we can write Eq. (31) as an integration

$$\mathbf{\Psi}\quad(\mathrm{x},\tau+\delta\tau)=\int_{-\infty}^{\infty}\mathbf{\Psi}_{\mathrm{p}}(\mathrm{x}+\mathrm{y},\tau)\mathrm{e}^{\frac{\mathrm{iS}(\delta\tau)}{\hbar}}\,\mathrm{dy} \tag{34}$$

where $\mathbf{\Psi}_{\mathrm{p}}$ denotes the spinor with momentum $\mathrm{p}=\mathrm{my}/\delta\tau$. If we consider the non-relativistic limit where one spinor component ψ_1 dominates and $\tau\to t$ we can use the semi-classical action

$$\mathrm{S}_{\mathrm{cl}}=\int_{\mathrm{t}}^{\mathrm{T}}\frac{\mathrm{m}}{2}\left(\frac{\mathrm{dx}}{\mathrm{dt}}\right)^2\,\mathrm{dt} \tag{35}$$

and write this infinitesimal path integral in the limit of large time interval T to give the conventional Feynman path integral [4]

$$\psi_1(\mathrm{x},\mathrm{t}+\mathrm{T})=\int_{-\infty}^{\infty}\mathrm{K}_0(\mathrm{x},\mathrm{y};\mathrm{T})\,\psi_1(\mathrm{y},\mathrm{t})\,\mathrm{dy} \tag{36}$$

where K_0 is the free particle propagator for the Schrodinger equation

$$\mathrm{K}_0(\mathrm{x},\mathrm{y};\mathrm{T})=\left(\frac{\mathrm{m}}{2\pi\mathrm{i}\hbar\mathrm{T}}\right)^{1/2}\,\mathrm{e}^{\mathrm{imy}^2/2\hbar\mathrm{T}} \tag{37}$$

The causal net model is thus consistent with the quantum mechanical summation of paths and solutions to various problems such as slit diffraction using Feynman integrals as detailed in [4].

1.13 Non-Euclidean Space-Time and General Relativity.

Lastly, it is worth considering the case of non-Euclidean or curved space-time. Is our causal net model consistent with Einstein's general relativity? In curved space-time our elementary triangles (Fig. 3) comprising our causal net will become distorted and we can no longer apply Pythagoras' theorem to evaluate the space-time interval. If we embed our causal net in 4 dimensional space-time, then in the language of general relativity the space-time interval is given by the metric $g_{\mu\upsilon}$ so $ds^2=g_{\mu\upsilon}dx^\mu dx^\nu$. Previously, we have considered the special case of the Minkowski metric $\eta_{\mu\nu}$ for flat space-time. General relativity considers Riemann spaces that have quadratic metric equations and are characterised as locally flat. This means that the first derivatives of the metric tensor are zero so for a small displacement in space Δx from a point x using Taylor expansion we have the metric

$$g_{\mu\nu}(x+\Delta x) = \eta_{\mu\nu} + \frac{1}{2} g_{\mu\nu,\rho\sigma} \Delta x^{\rho} \Delta x^{\sigma}. \tag{38}$$

So the change in the metric depends only on the second derivatives of the metric and is related to the *curvature* of the space-time. On the basis of Eq. (28) we might assume that variation to the scalar Minkowski action S (Eq. 34) would produce a correction S_g which, to be a scalar under general coordinate transformations, can only include second order derivatives of the metric. Mathematically, the simplest curvature scalar is the well known Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ formed from the Ricci curvature tensor $R_{\mu\nu}$. We can postulate that the *simplest* additional action might be of the form

$$S_g = B \int R \ d^4x. \tag{39}$$

If we set the constant $B = -\varepsilon_0 / 16\pi G c^4$, where ε_0 is a "density" and G Newton's gravitational constant then we have Einstein's *unimodular* gravity. If we further impose the density as $\varepsilon_0 = -\sqrt{-g}$ where $g = \det(g_{\mu\nu})$ then we recover the Einstein–Hilbert action

$$S_g = -\frac{1}{16\pi G c^4} \int R \ \sqrt{-g} \ d^4x. \tag{40}$$

Now to balance the Einstein–Hilbert action, so that the overall action is stationary, we must add an additional term or Lagrangian. This, in general relativity, describes a matter field and is *defined* by setting $\delta S = 0$ to provide the characteristic stress-energy tensor and the famous Einstein field equations. Thus the causal net model, although a microscopic theory, would appear to be consistent with the macroscopic theory of general relativity. The elementary triangles of our causal net model must be distorted to "tile" curved space-time between causally connected possible events.

1.14 Discussion.

The causal net is a net of probabilities and a measurement provides through Bayesian statistics a re-evaluation of these probabilities after a measurement – equivalent to "collapse of the wavefunction". The causal net provides a net of possibilities in space-time describing the position and momentum of the particle. To illustrate, assume as Section 1.4 the probability of finding a particle at vertex labelled (*i,j*) on a measurement is given by $\Omega_{i,j}$. Thus a statistical measurement will provide through Bayesian analysis a change in all the probabilities on the net for the current time and all previous times. The mysterious "collapse of the wavefunction" just corresponds to a statistical measurement and the change or refinement in all the retrospective probabilities. This is easily seen for the case of a single particle. Consider two different space locations at the same proper time. If for example the probability of finding the particle at $\Omega_{1,2}$ is the same as $\Omega_{2,2}$ before the measurement and we perform a measurement at vertex (1,2) and the answer is "yes we have found a particle" then $\Omega_{1,2} = 1$ and $\Omega_{2,2} = 0$. Instantaneously, from Bayes theorem and basic probability theory, the probabilities earlier in time will change: $\Omega_{1,1}$ will increase and the probability $\Omega_{2,1}$ will decrease.

The above approach has reduced the general quantum mechanical measurement process to one of causal statistics of an ensemble of events. Determination or measurement of the actual state of a vertex comprises a statistical measurement which is fully consistent with all the probabilities on the net. If we consider a single particle similar to the Aspect experiments [15] then when the particle is measured in a particular location there is no true "collapse of the wavefunction" but merely a statistical observation yes or no. From this observation Bayesian analysis can be applied to compute the retrospective or historic probabilities and statistics of earlier vertices but these are all perfectly consistent with probability theory. So-called "action at a distance" is just a natural consequence of measuring and statistically determining the state of the particle of the system.

In Section 1.5 we saw that $p\Delta x$ is a constant η for a valid lattice and, for the particle to obey the simplest form of the principle of least action, that the sum $\sum p\Delta x$ is the same for all paths on the causal net. Setting this constant η to be Planck's constant h is an arbitrary decision but provides the de Broglie relation $\lambda\, p = h$ with lattice constant $\Delta x = \lambda/2$. The discretisation of the net then provides a Heisenberg like relation $\Delta p\Delta x \sim h/2$.

We find that on the causal net the transition from quantum to classical behaviour occurs for massive objects. For massive objects since $\Delta x = h/2p$ from the uncertainty relations when p is large – true for high mass or virtually any velocity – then the net size Δx is small and uncertainty in x is small relative to the size of the object. The object can be well localized or resolved on the net and is effectively non quantum although of course it can be relativistic.

It is worth emphasising that the full conventional Dirac equation arises only when we impose probability amplitudes on the causal net. In contrast the quantisation is automatically provided by the net and does not rely on the full formalism of quantum mechanics in terms of the wavefunction. To illustrate this consider the relativistic particle in the (1 dimensional) box problem. For a potential well of depth V and width L bound states exist for a "forbidden zone" given by imaginary momentum states in the potential region. Now from the causal net model we can consider there to be an integer number n of net vertices to be contained in the well. This provides the quantization condition $n = L\Delta x$ giving quantised momentum states $p = 2nL/h$. This corresponds to the solution of the 1+1 dimensional Dirac equation [22] and reduces to the Schrödinger particle in the box problem in the non relativistic limit. Thus for the particle in say its n = 2 state then there are 2 space events on the net that can occur at the same time with equal probability. In conventional quantum mechanics the bound state problem is treated as a superposition of two plane waves with opposite momentum states and a reflective boundary condition applied at the edge of the potential well. In our model we could treat the problem as a superposition of two causal nets although as we have seen this is not necessary to realise that any bound state is quantised into maxima and minima since this is given by the discrete nature of the net, as described above.

Since we have demonstrated that a "stack" of causal nets provides the Feynman path integral (Section 1.12) we can use this well known technique to solve other quantum problems such as diffraction from slits. If a single net is denoted by $\aleph$ then we must consider the set of $\aleph(p)$ nets of all possible p momentum states applicable to the problem. Following Feynman, as detailed in [4], for a single slit all possible trajectories of the particle which can travel from the source and through the slit(s) must be considered. The sum over paths provides well known Fresnel integrals which provide the observed diffraction phenomena at the detector. In the language of our causal nets we must consider all causal chains that causally link the source and the detector that are not limited by the slit apparatus. We do not know which causal path was followed to reach the detector but upon final measurement we could, if necessary, compute using Bayesian analysis the range of possible paths the particle could have followed.

We could consider the net and the set of all nets exist only as a mathematical description (as say Pythagoras' theorem) or a computing device for possibilities in the same way that a particle on the net is probabilistic. Only when an event or measurement occurs then the region of space-time has reality and an existence and until then it remains a possibility represented by probability. As with the Copenhagen interpretation we cannot say anything about the particles trajectory or local reality when it is not observed. However, in our interpretation, between observations particles have their own local reality and "exist" in their own stationary reference frame since probability is conserved and the particle is stationary. The precise position the particle occupies in space-time in other inertial frames is unknown. Determining the location of the particle in the observer frame temporarily collapses the causal net for that particular reference frame. This agrees with the wave-particle duality of nature – that particles move on the probabilistic causal net and appear as waves when they are not observed. However, when we measure them we have an actual event and localise and find a particle. Special relativity showed that there is no universal time but we have replaced it with each particle having its own invariant proper time or "personal" time – a kind of "universal personal time". Obviously this is an essentially realist interpretation of space-time – that both space and time exist outside the human mind.

Interesting the causal net using the principle of the common cause has other possible implications relating to the flow of time and irreversibility. Reichenbach [10] argued that second law of thermodynamics could be derived from application of the principle to statistical thermodynamics. On the flow of time he writes: "Neither the laws of [classical] mechanics nor mechanical observables give us the direction of time, unless such a direction has been defined previously by reference to some irreversible process. For instance, if the velocity of a body is regarded as observable, its direction must be ascertained by comparison with some temporally directed process such as the time of psychological experience, which is derived from the irreversible processes of the human organism." In our causal net theory of quantum mechanics actual statistical measurement of vertex states define such irreversible processes which can provide a direction to time.

From the basis of simple casual connections between elementary events we have constructed a model where the Dirac equation and the fermion particles it describes are seemingly emergent properties. Does the causal net automatically imply a quasiparticle such as a fermion? In an inertial frame an observer will view an ordered series of events in space-time as an entity behaving as either a wave or a particle, depending on how the experimental measuring setup is conceived, and thus exhibits wave-particle duality exactly as proposed by de Broglie. Geometrical quantities of the causal net correspond to measurable physical qualities: mass (scaling factor), momentum and energy (net angle and geometry) and acceleration and forces (change of net angle). The global gauge symmetry of the net provides quantum phase and the other degenerate solutions arising from the symmetries of the net provide the Dirac spin and negative energy states. In addition the discretisation of the net provides an analogy with the Heisenberg uncertainty principle and a "stack" of nets provides, consistent with the Feynman path integral approach, quantum phenomenon such as diffraction. None of these emergent aspects of our net would have been apparent from our simple starting point of an equilibrium distribution of ordered events.

In many ways this statistical causal net model is a non-local hidden variable theory – the hidden variable could be considered to be the actual net. Bell's theorem applies to local hidden variable theories and since our theory is a non-local hidden variable theory which exactly derives the Dirac equation it should always pass Bell's tests and give the exact quantum mechanical result. The causal net model is perhaps also consistent with the famous Einstein, Podolsky and Rosen (EPR) paper [24] that the "description of reality as given by a wave function is not complete" since the wavefunction is not complete without the connectivity of the causal net. The analysis of the superposition of nets in Section 1.12 could be possibly extended multiple particles. Consider an EPR type spin experiment with parallel settings with the experimental apparatus shown schematically in Figure 9 below. Two spin ½ particles are simultaneously emitted in opposite directions from a source in a singlet (or antiparallel) state. The particles are later detected along three different chosen spatial axes using magnetic fields in the left and right wings of the apparatus.

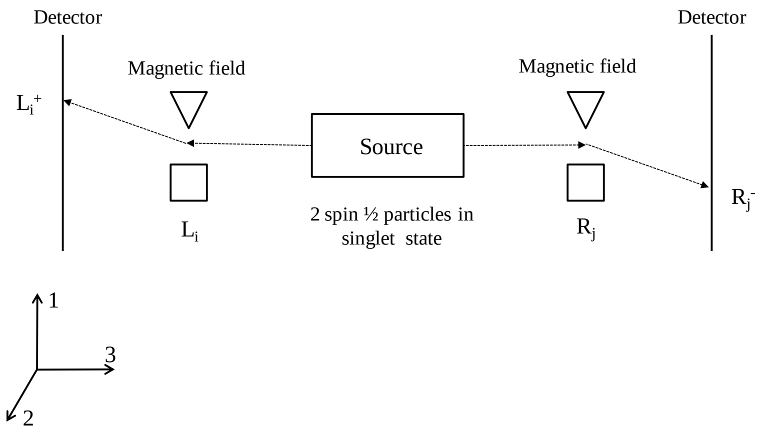

Figure 9: EPR experimental setup (from [14]).

Following the notation of Grasshof [14], if the measurement apparatus is set to measure the spin direction in the left and right wings $L_i(R_j)$ $i \in \{1,2,3\}$ $j \in \{1,2,3\}$ and $L_i^a(R_j^b)$ symbolizes the spin event type $a \in \{+,-\}$, $b \in \{+,-\}$ then the experiment measures $P(L_i^a.R_j^b|L_i.R_j)$. However for two particles there exists a set of possible pairs of causal nets $\aleph(L_l.R_m)$ comprising all possible experimental setups *l* and *m* for each particle. If we choose to measure the combination $\aleph(L_i.R_j)$ then we select the relevant pair of nets. This selection is mutually exclusive of all the other possible pairs of nets existing in the set. For perfect correlation *i* = *j* and we can see that $P(L_i^+|L_i.R_j) = 1/2$ and due to the initial antiparallel spin arrangement the other net must provide $P(R_i^-|L_i.R_j.L_i^+) = 1$. For imperfect correlation we must consider probabilities such as

$$P(L_i^-.R_j^+|L_i.R_j) = P(R_j^+|L_i.R_j.L_i^-)P(L_i^-|L_i.R_j)$$
$$= f(\varphi_{ij})/2.$$

Due to mutual exclusivity it does not matter the order we measure L_i or R_j since determination of one state implies the other. The function $f(\varphi_{ij})$ where φ_{ij} is the angle between *i* and *j* can be considered to be a projection of a causal net corresponding to projecting R_i onto a different spatial axis *j* to give R_j and has a form $f(\varphi_{ij}) = cos^2\varphi_{ij}$. This result corresponds to quantum mechanical statistics.

Thus importantly, it would seem that there exists a full set of possible causal nets between physical observations. This set of causal nets is associated with the entire wavefunction of the system. Combining many simple causal nets together we can form a sort of “super” causal net that accommodates not only all possible position and momentum states but also various possible experimental arrangements and corresponds to the entire wavefunction. On a physical observation or measurement one member (or subset in the case of several particles) of the set of all possible nets $\aleph$ is selected through mutual exclusivity based on the experimental arrangement and detection of the particle(s). This is somewhat similar to the theory of de Broglie [25] where a kind of “many worlds” quantum state exists in the microscopic world but in the macroscopic world only one member of the ensemble is statistically realised when a particular measurement is specified. In our model the mechanism for selection of one member of the set of causal nets and measurement of a particle on the net is through *mutual exclusivity* – that is the particle can exist at a particular location on the net and only one net fits the experimental conditions. The existence of one net per fermion state is consistent with the Pauli exclusion principle.

Finally, we shall finish with a brief comment by John Bell himself in 1986: “because of the EPR experiments ... I want to say there is a real causal sequence which is defined in the aether.” Bell did not imply that the original notion of the aether should be resurrected but “behind the apparent Lorentz invariance of the [quantum] phenomena, there is a deeper [causal] level which is not Lorentz invariant.”

1.15 References.

1. Belnap N. (1992) *Branching Space-Time*, Synthese, 92, p. 385.
2. Sorkin R. D. (1997) *Forks in the Road, on the way to Quantum Gravity*, Int. J. Theor. Phys., 36, p. 2759.
3. Dirac P.A.M. (1928) *The Quantum Theory of the Electron*, Proc. Royal Soc. A117.
4. Feynman R.P. and Hibbs A.R. (1965) *Quantum Mechanics and Path Integrals* (McGraw-Hill).
5. Kauffman L. H. and Noyes H. P. (1996) *Discrete Physics and the Dirac Equation,* SLAC–PUB–7115.
6. Gaveau B., Jacobson T., Kac M., Schulman L.S. (1984) *Relativistic Extension of the Analogy between Quantum Mechanics and Brownian Motion*, Phys. Rev. Lett. 53, p. 419.
7. Coddens G. (1995) *A Remark on the Dirac Equation*, Found. Physics Lett., 8, p. 3.
8. Bialynicki-Birula I. (1994) *Weyl, Dirac, and Maxwell Equations on a Lattice as Unitary Cellular Automata*, Phys. Rev. D, 49, p. 12.
9. Schotte K.D., Iwabuchi S. and Truong T.T. (1985) *Ice Models and a Lattice Version of the Dirac Equation*, Z. Phys. B 60, p. 255.
10. Reichenbach H. (1956) *The Direction of Time* (University California Press).
11. Penrose O. and Percival I. C. (1962) *The Direction of Time*, Proc. Phys. Soc., 79, p. 605.
12. Malament D. (1977) *Causal Theories of Time and the Conventionality of Simultaneity*, Nous, 11, p. 293.
13. Bell J. (1966) *On the Problem of Hidden Variables in Quantum Mechanics,* Rev. Mod. Phys., 38, p. 447-52.

14. Grasshof G., Portmann S. and Wuthrich A. (2005) *Minimal Assumption Derivation of a Bell-Type Inequality*, Brit. J. Phil. Sci., 56, p. 663.
15. Aspect A., Dalibard J., Roger G. (1982) *Experimental Test of Bell's Inequalities using Time-Varying Analyzers*, Phys. Rev. Lett. 49, p. 1804-1807.
16. Jaynes E.T. (2003) *Probability Theory: The Logic of Science*, 2nd Ed. (Cambridge University Press).
17. Rindler W. (1982) *Introduction to Special Relativity* (Oxford).
18. De Broglie L. Ann. Phys. (Paris) 3, p. 22 (1925) and Nobel lecture (1929) *The Wave Nature of the Electron*.
19. Heisenberg W. (1927) *Uber den Anschaulichen Inhalt der Quantentheoretischen Kinematik und Mechanik*, Zeitschrift fur Physik, 43, p. 172-198.
20. Schurmann T. and Hoffmann I. (2009) *A Closer Look at the Uncertainty relation of Position and Momentum*, Found. Phys. 39, p. 958.
21. Davydov A.S. (1965) *Quantum Mechanics* (Pergamon Press).
22. Coulter B. and Adler C. (1971) *The Relativistic One Dimensional Square Potential*, AJP 39, p. 305.
23. Foldy L. and Wouthuysen S. (1950) *On the Dirac Theory of Spin ½ Particles and its Non-Relativistic Limit*, Phys. Rev., 78, p. 1.
24. Einstein A., Podolsky B., Rosen K. (1935) *Can Quantum-Mechanical Description of Physical Reality be Considered Complete*, Phys. Rev. 47, p. 777.
25. Broglie, L. (1956) *Tentative d'Interprétation Causale et Non-linéaire de la Méchanique Ondulatoire* (Gauthier-Villars).