ON SMARANDACHE SEMIGROUPS

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Abstract: The notion of completely regular element of a semigroup is applied to characterize Smarandache Semigroups. Examples are provided for justification.

Keywords: Semigroup, Regular element, Completely regular element, Divisibility, Idempotent element.

§ 1. Introduction

Smarandache notions on all algebraic and mathematical structures are interesting to the world of mathematics and researchers. The Smarandache notions in groups and the concept of Smaranadache Semigroups, which are a class of very innovative and conceptually a creative structure, have been introduced in the context of groups and a complete possible study has been taken in [11]. Padilla Raul intoduced the notion of Smarandache Semigroups in the year 1998 in the paper *Smarandache Algebraic Structures* [6].

In [5], the concept of regularity was first initiatied by J.V. Neumann for elements of rings. In general theory of semigroups, the regular semigroups were first studied by Thierrin [7] under the name demi-groupes inversifs. The completely regular semigroups were introduced by Clifford [2].

The notions of regular element, completely reagular element of a semigroup are very much useful to characterize Smarandache Semigroups. In this paper we present characterizations of Smarandache Semigroups. Besides, some more theorems on Smarandache Semigroups, examples are provided for justification. In section 2 we give some basic definitions from the theory of semigroups (See[3]) and definition of Smarandache Semigroup (See[11]). In section 3 we present our main characterization of Smarandache Semigroups and examples for justification.

§ 2. Preliminaries

Definition 2.1: ([3]) A semigroup is a nonempty set S in which for every ordered pair of elements $x, y \in S$, there is defined a new element called their product $xy \in S$, where for all $x, y, z \in S$ we have $(xy)z = x(yz)$.

Definition 2.2: ([3]) An element b of the semigroup S is called a right divisor of the element a of the semigroup if there exists in S an element x such that $xb = a$. b is called the left divisor of a if there exists in S an element y such that $by = a$.

If b is a right divisor of a, we say that a is divisible on the right by b. If b is a left divisor of a, we say that a is divible on the left by b.

Definition 2.3: ([3]) An element b of a semigroup S is called a right unit of the element a of the same semigroup, if $ab = a$.

Left unit is defined analogously. An element that is both a right and a left unit of some elemet is called two-sided unit of that element.

An element I which is its own two-sided unit is called an Idempotent : $I^2 = I$.

Definition 2.4: ([3]) An element a of a semigroup S is said to be regular, if we can find in S an element x such that $axa = a$.

A semigroup consisting entirely of regular elements is said to be Regular semigroup.

Definition 2.5: ([3]) An element a is said to be completely regular if we can find in S an element x such that $axa = a$; $ax = xa$.

A semigroup consisting entirely of completely regular elements is said to be completely regular.

Definition 2.6: ([3]) An element e of a semigroup S which is a left unit of the element $a \in S$ is called a Regualar left unit if it is divisible on the left by a.

e is called a regular right unit of a if it is a right unit of a and is divisible on the right by a.

e is called a regular two-sided unit of a if e is a two-sided unit of a and is divisible both on the left and on the right by a.

In [3], the following observations are known:

2.6.1. Concepts of regularity and complete regularity coincide for commutaive semigroup.

2.6.2. If $e \in S$ is a regular left unit of $a \in S$ there must exist an $x \in S$ such that $ea = a$, $ax = e$. The condition that e should be a right regular unit is $ae = a$, $xa = e$.

2.6.3. Every idempotent is completely regular. It is its own regular two-sided unit.

2.6.4. A regular left unit of an arbitrary element is always an idempotent.

2.6.5. No element in a semigroup S may have two regular two-sided units.

2.6.6. If an element has regular two-sided unit then it is completely regular.

§ 3. Proofs of the Theorems.

In this section we give characterizations of Smarandache Semigroups by proving the following theorems.

Theorem 3.1: A semigroup S is a Smarandache Semigroup if and only if S contains idempotents.

Proof : Let S be a Smarandache Semigroup then there is a proper subset $G \subset S$ such that G is a group under the operation defined on S. The identity element e of G is its own two-sided unit i.e., $e^2 = e$, in S. Hence, S contains idempotent.

Conversely, assume that the semigroup S contains idempotents. Let I be an arbitrary idempotent of the semigroup S. Write G_I for the set of all completely regular elements of S for which I is a regular two-sided unit. In view of (2.6.3), G_I is a nonempty subset of S as G_I contains I.

Now we show that G_I is a group under the operation on S. Let g_1, g_2 be any two elements in G_I. Since I is a regular two-sided unit of g_1 and g_2 we have for some u_1, u_2, v_1, v_2 in S.

$I = g_1u_1$, $I = g_2u_2$, $I = v_1g_1$, $I = v_2g_2$ from this we have

$(g_1g_2)(u_2u_1) = g_1(g_2u_2)u_1 = g_1Iu_1 = g_1u_1 = I$ next,

$(v_2v_1)(g_1g_2) = v_2(v_1g_1)g_2 = v_2Ig_2 = v_2g_2 = I.$

since, I is a two-sided unit of the element g_1g_2, I is a regular two-sided unit of g_1g_2. In view of (2.6.6), we have $g_1g_2 \in G_I$. Therefore G_I is a semigroup with unit I. Since I is clearly a two-sided unit for Iu_1I and

$I = II = g_1u_1I = g_1(Iu_1I)$,

$I = v_1g_1 = v_1IIg_1 = v_1g_1u_1Ig_1 = Iu_1Ig_1$,

it follows that I is a regular two-sided unit of the element Iu_1I. In view of (2.6.6), $Iu_1I \in G_I$ and further, Iu_1I is a two- sided inverse of g_1 with respect to I. From this we get the fact that every element in G_I has a two-sided inverse in G_I as g_1 is an arbitrary element of G_I with unit I. So, the proper subset $G_I \subset S$ is a group and hence S is a Smarandache Semigroup.

Theorem 3.2: A semigroup S is a Smarandache semigroup if and only if S contains completely regular elements.

Proof : Suppose that the semigroup S is a Smarandache semigroup then there is a proper subset $G \subset S$ which is group under the operation defined on S. Clearly, the identity element $e \in G$, which is a regular two-sided unit of any arbitrary element of the semigroup, is completely regular.

On the other hand if the semigroup S contains a completely regular element, say a, then a has an idempotent element I as its regular two-sided unit. In view of the theorem (3.1), the proper subset $G_I \subset S$ is a group. Hence, S is a Smarandache Semigroup.

Theorem 3.3: Let S be a Smarandache Semigroup. The set C of all completely regular elements of S can be expressed as the union of non-intersecting groups.

Proof : Let S be a Smarandache Semigroup, C be the set of all completely regular elements of S and H be the set of Idempotent elements of S.

In view of theorem(3.1) and theorem (3.2), $C \neq \phi$ and $H \neq \phi$. Let $c \in C$ then C has an idempotent I as its regualr two-sided unit. In view of theorem (3.1) $c \in G_I$ which is always a group. In view of (2.6.5), no element may have two regular two-sided units. It follows that the groups G_I, $I \in H$ are all mutually disjoint. Therefore, $C = \cup_{I \in H} G_I$

§ 4. Examples.

In this section we give examples for justification.

Example 4.1: Let $S = \{e, a, b, c\}$ be a semigroup under the operation defined by the following table.

	e	a	b	c
e	e	a	b	c
a	a	e	b	c
b	b	b	c	b
c	c	c	b	c

Table 1.

Clearly, the operation is commutative. Inview of (2.6.1), the completely regular elements of S are e, a, b, c as $eee = e$, $aaa = a$, $bbb = b$, $ccc = c$. Moreover the idempotent elements are e, c.

Now G_e= { e, a } as e is regular two-sided unit of e, a and G_c= { c, b } as c is regular two-sided uniit of c, b. Using the Table 1., we can easily see that G_e and G_c are groups. Further, $G_e \cap G_c = \phi$. Let $C = \{e, a, b, c\}$, we can easily see that $C = G_e \cup G_c$.

Example 4.2: Let $S = \{1, 2, 3, 4, 5, 6\}$ be a semigroup under the operation defined by $xy =$ the great common divisor of x, y for all $x, y \in S$. The composition table is as follows:

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	2	1	2
3	1	1	3	1	1	3
4	1	2	1	4	1	2
5	1	1	1	1	5	1
6	1	2	3	2	1	6

Table 2.

We can easily see that S is a commutative semigroup. The completely regular elements in S are $1, 2, 3, 4, 5, 6$ as $111 = 1$, $222 = 2$, $333 = 3$, $444 = 4$, $555 = 5$ and $666 = 6$. Write

$C = \{1, 2, 3, 4, 5, 6\}$ for the set of all completely regular elements of S and $H = \{1, 2, 3, 4, 5, 6\}$ for the set of all idempotent elements of S. Now, $G_1 = \{1\}$ as 1 is the only regular two-sided element of 1. Obviously, we have $G_2 = \{2\}$, $G_3 = \{3\}$, $G_4 = \{4\}$, $G_5 = \{5\}$, $G_6 = \{6\}$. Further $G_1, G_2, G_3, G_4, G_5, G_6$ are groups and they are mutually disjoint also $C = G_1 \cup G_2 \cup G_3 \cup G_4 \cup G_5 \cup G_6$.

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