An Application of Sondat's Theorem

Regarding the Orthohomological Triangles

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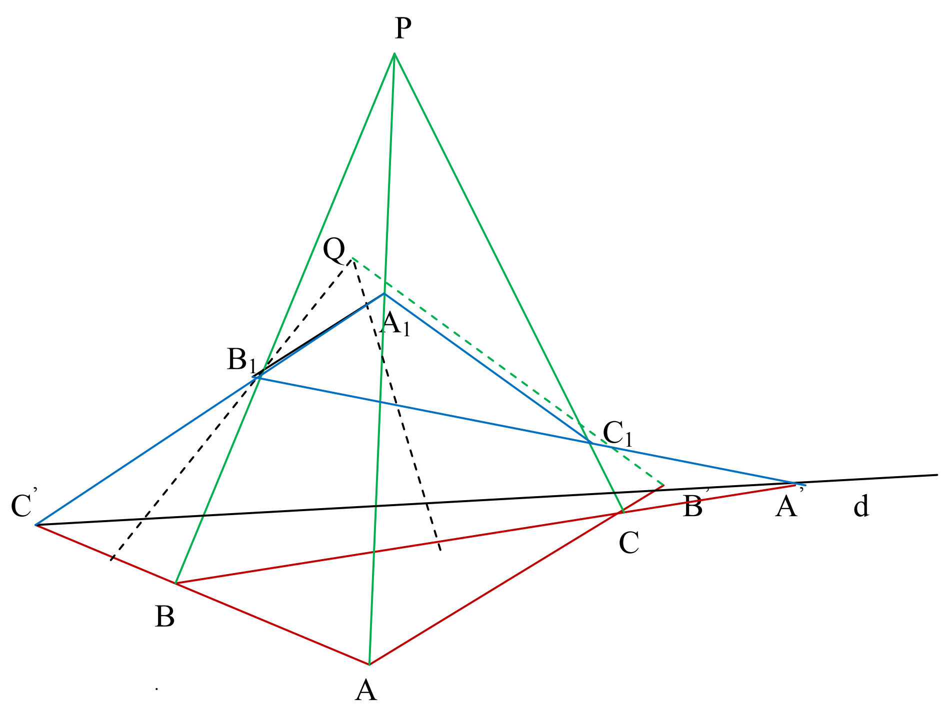
In this article we prove the Sodat's theorem regarding the orthohomological triangle and then we use this theorem and Smarandache-Patrascu's theorem in order to obtain another theorem regarding the orthohomological triangles.

Theorem (P. Sondat)

Consider the orthohomological triangles ABC, $A_1B_1C_1$. We note Q, Q_1 their orthological centers, P the homology center and d their homological axes. The points P, Q, Q_1 belong to a line that is perpendicular on d



Proof.

Let Q the orthologic center of the ABC the $A_1B_1C_1$ (the intersection of the perpendiculars constructed from $A_1,\ B_1,\ C_1$ respectively on $BC,\ CA,\ AB$), and Q_1 the other orthologic center of the given triangle.

We note $\{B'\} = CA \cap C_1A_1$, $\{A'\} = BC \cap B_1C_1$, $\{C'\} = AB \cap A_1B_1$.

We will prove that $PQ \perp d$ which is equivalent to

$$B'P^2 - B'Q^2 = C'P^2 - C'Q^2 \qquad (1)$$

We have that

$$\overrightarrow{PA_1} = \alpha\overrightarrow{A_1A},\ \overrightarrow{PB_1} = \beta\overrightarrow{B_1B},\ \overrightarrow{PC_1} = \gamma\overrightarrow{C_1C}$$

From Menelaus' theorem applied in the triangle PAC relative to the transversals $B',\ C_1,\ A_1$ we obtain that

$$\frac{B'C}{B'A} = \frac{\alpha}{\gamma} \qquad (2)$$

The Stewart's theorem applied in the triangle PAB' implies that

$$PA^2 \cdot CB' + PB'^2 \cdot AC - PC^2 \cdot AB' = AC \cdot CB' \cdot AB' \qquad (3)$$

Taking into account (2), we obtain:

$$\gamma PC^2 - \alpha PA^2 = (\gamma - \alpha)PB'^2 - \alpha B'A^2 + \gamma B'C^2 \qquad (4)$$

Similarly, we obtain:

$$\gamma QC^2 - \alpha QA^2 = (\gamma - \alpha)QB'^2 + \gamma B'C^2 - \alpha B'A^2 \qquad (5)$$

Subtracting the relations (4) and (5) and using the notations:

$$PA^2 - QA^2 = u,\ PB^2 - QB^2 = v,\ PC^2 - QC^2 = t$$

we obtain:

$$PB'^2 - QB'^2 = \frac{\gamma t - \alpha u}{\gamma - \alpha} \qquad (6)$$

The Menelaus' theorem applied in the triangle PAB for the transversal $C',\ B,\ A_1$ gives

$$\frac{C'B}{C'A} = \frac{\alpha}{\beta} \qquad (7)$$

From the Stewart's theorem applied in the triangle $PC'A$ and the relation (7) we obtain:

$$\alpha PA^2 - \beta PB^2 = (\alpha - \beta)C'P^2 + \alpha C'A^2 - \beta C'B^2 \qquad (8)$$

Similarly, we obtain:

$$\alpha QA^2 - \beta QB^2 = (\alpha - \beta)C'Q^2 + \alpha C'A^2 - \beta C'B^2 \qquad (9)$$

From (8) and (9) it results

$$C'P^2 - C'Q^2 = \frac{\alpha u - \beta v}{\alpha - \beta} \qquad (10)$$

The relation (1) is equivalent to:

$$\alpha\beta(u-v)+\beta\gamma(v-t)+\gamma\alpha(t-u)=0 \quad (11)$$

To prove relation (11) we will apply first the Stewart theorem in the triangle CAP, and we obtain:

$$CA^2\cdot PA_1+PC^2\cdot A_1A-CA_1^2\cdot PA=PA_1\cdot A_1A\cdot PA \quad (12)$$

Taking into account the previous notations, we obtain:

$$\alpha CA^2+PC^2-CA_1^2(1+\alpha)=PA_1^2+\alpha A_1A^2 \quad (13)$$

Similarly, we find:

$$\alpha BA^2+PB^2-BA_1^2(1+\alpha)=PA_1^2+\alpha A_1A^2 \quad (14)$$

From the relations (13) and (14) we obtain:

$$\alpha BA^2-\alpha CA^2+PB^2-PC^2-(1+\alpha)\left(BA_1^2-CA_1^2\right)=0 \quad (15)$$

Because $A_1Q\perp BC$, we have that $BA_1^2-CA_1^2=QB^2-QC^2$, which substituted in relation (15) gives:

$$BA^2-CA^2+QC^2-QB^2=\frac{t-v}{\alpha} \quad (16)$$

Similarly, we obtain the relations:

$$CB^2-AB^2+QA^2-QC^2=\frac{u-t}{\beta} \quad (17)$$

$$AC^2-BC^2+QB^2-QA^2=\frac{v-u}{\gamma} \quad (18)$$

By adding the relations (16), (17) and (18) side by side, we obtain

$$\frac{t-v}{\alpha}+\frac{u-t}{\beta}+\frac{v-u}{\gamma}=0 \quad (19)$$

The relations (19) and (11) are equivalent, and therefore, $PQ\perp d$, which proves the Sondat's theorem.

In [2] it was proved the **Smarandache-Patrascu Theorem**:
If the triangles ABC and $A_1B_1C_1$ inscribed into the triangle ABC are orthohomological, where Q is the center of orthology (i.e. the point of intersection of the perpendiculars in A_1 on BC, in B_1 on AC, and in C_1 on AB), and Q_1 is the second center of orthology of the triangles ABC and $A_1B_1C_1$, and $A_2B_2C_2$ is the pedal triangle of Q_1, then the triangles ABC and $A_2B_2C_2$ are orthohomological.

Now we prove another theorem:

Theorem (Pătraşcu-Smarandache)

Consider the triangle ABC and the inscribed orthohomological triangle $A_1B_1C_1$, with Q, Q_1 their centers of orthology, P the homology center and d their homology axes. If $A_2B_2C_2$ is the pedal triangle of Q_1, P_1 is the homology center of triangles ABC and $A_2B_2C_2$, and d_1 their homology axes, then the points P, Q, Q_1, P_1 are collinear and the lines d and d_1 are parallel.

Proof.

Applying the Sondat's theorem to the ortho-homological triangle ABC and $A_1B_1C_1$, it results that the points P, Q, Q_1 are collinear and their line is perpendicular on d. The same theorem applied to triangles ABC and $A_2B_2C_2$ shows the collinearity of the points P_1, Q, Q_1, and the conclusion that their line is perpendicular on d_1.

From these conclusions we obtain that the points P, Q, Q_1, P_1 are collinear and the parallelism of the lines d and d_1.

References

[1] Cătălin Barbu, Teoreme Fundamentale din Geometria Triunghiului, Editura Unique, Bacău, Romania, 2008.

[2] Florentin Smarandache, Multispace & Multistructure Neutro-sophic Transdisciplinarity (100 Collected Papers of Sciences), Vol. IV. North-European Scientific Publishers, Hanko, Finland, 2010.