Possibility of gravitational wave from generalized Maxwell equations

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(Dated: July 5, 2010)

The generalized Maxwell equations in vacuum are basically the same with Dirac's extended Maxwell equations, although intrinsic charges and currents are defined by the time differential and gradation of scalar fields, respectively. Consequently, the electromagnetic stress-energy tensors make important conservation laws. Then, we found scalar fields acting like the gravitational wave interacting with the electromagnetic wave. Interestingly, those gravitational waves due to the scalar fields push out the electromagnetic waves. Moreover, there is a possibility of the existence of the materials, from which we feel no gravitational forces although the electromagnetic waves are kicked out by those gravitational waves. We also discussed about the relation with weight.

PACS numbers:

I. INTRODUCTION

Maxwell equations were proposed many years ago,[1] and Dirac extended those to have the magnetic charges and currents.[2] On the other hand, it was proposed that there is a possibility of the existence of longitudinal waves because the gravitational force works perpendicular to the wave front.[3] Moreover, Einstein field equation was proposed, in which the gravitational wave was proposed as a wave of distortion of space with light speed.[4] In recently, a possibility of the existence of scalar fields caused by broken Lorentz condition begins to be discussed in both the classical electromagnetic dynamics and quantum gravity.[5,6]

In this paper, according to Ref. [5], we derive the generalized Maxwell equations in vacuum. Then, we explain that those are basically the same with Dirac's extended Maxwell equations except that intrinsic charges and currents are defined by the scalar fields made by the potential. Consequently, we find longitudinal waves acting like the gravitational wave interacting with the electromagnetic wave. We note that we only discuss the gravitational wave in the classical physics.

II. BASIC EQUATION

A. Definition

We define a quaternion as

$$a + bi + cj + dk, \tag{1}$$

where '$a \sim d$' are real numbers and '$i \sim k$' are so-called Hamilton's unit vectors with following relations,[7]

$$\begin{gathered} i^2 = j^2 = k^2 = ijk = -1, \\ ij = k,\ jk = i,\ ki = j, \\ ij = -ji,\ jk = -kj,\ ki = -ik. \end{gathered} \tag{2}$$

Then, we expand this quaternion with imaginary numbers,

$$\mathbb{Q} \;=\; (a + \mathrm{i}b) + (c + \mathrm{i}d)i + (e + \mathrm{i}f)j + (g + \mathrm{i}h)k, \tag{3}$$

where '$a \sim h$' are real numbers. We also define its complex conjugate as

$$\mathbb{Q}^* \;=\; (a - \mathrm{i}b) + (c - \mathrm{i}d)i + (e - \mathrm{i}f)j + (g - \mathrm{i}h)k. \tag{4}$$

In order to simplify equations, we define a vector as

$$\mathbf{A} \;=\; \mathrm{A_x}i + \mathrm{A_y}j + \mathrm{A_z}k = \begin{pmatrix} \mathrm{A_x} \\ \mathrm{A_y} \\ \mathrm{A_z} \end{pmatrix}. \tag{5}$$

For example, we prepare two quaternions as

$$\mathbb{A} \;=\; a + \mathbf{A} = a + \mathrm{A_x}i + \mathrm{A_y}j + \mathrm{A_z}k, \tag{6}$$

$$\mathbb{B} \;=\; b + \mathbf{B} = b + \mathrm{B_x}i + \mathrm{B_y}j + \mathrm{B_z}k, \tag{7}$$

where a, b, $\mathrm{A_i}$, and $\mathrm{B_i}$ are all complex numbers. Then, those multiplication and interproduct become

$$\begin{aligned} \mathbb{A}\mathbb{B} &= (a + \mathbf{A})\,(b + \mathbf{B}) \\ &= (a + \mathrm{A_x}i + \mathrm{A_y}j + \mathrm{A_z}k) \\ &\quad\ (b + \mathrm{B_x}i + \mathrm{B_y}j + \mathrm{B_z}k) \\ &= \sum_{\text{all}} \left\{ \begin{matrix} ab & a\mathrm{B_x}i & a\mathrm{B_y}j & a\mathrm{B_z}k \\ \mathrm{A_x}bi & \mathrm{A_x}\mathrm{B_x}ii & \mathrm{A_x}\mathrm{B_y}ij & \mathrm{A_x}\mathrm{B_z}ik \\ \mathrm{A_y}bj & \mathrm{A_y}\mathrm{B_x}ji & \mathrm{A_y}\mathrm{B_y}jj & \mathrm{A_y}\mathrm{B_z}jk \\ \mathrm{A_z}bk & \mathrm{A_z}\mathrm{B_x}ki & \mathrm{A_z}\mathrm{B_y}kj & \mathrm{A_z}\mathrm{B_z}kk \end{matrix} \right\} \\ &= \sum_{\text{all}} \left\{ \begin{matrix} ab & a\mathrm{B_x}i & a\mathrm{B_y}j & a\mathrm{B_z}k \\ \mathrm{A_x}bi & -\mathrm{A_x}\mathrm{B_x} & \mathrm{A_x}\mathrm{B_y}k & -\mathrm{A_x}\mathrm{B_z}j \\ \mathrm{A_y}bj & -\mathrm{A_y}\mathrm{B_x}k & -\mathrm{A_y}\mathrm{B_y} & \mathrm{A_y}\mathrm{B_z}i \\ \mathrm{A_z}bk & \mathrm{A_z}\mathrm{B_x}j & -\mathrm{A_z}\mathrm{B_y}i & -\mathrm{A_z}\mathrm{B_z} \end{matrix} \right\} \\ &= ab + a\mathbf{B} - \mathbf{A}\cdot\mathbf{B} + \mathbf{A}b + \mathbf{A}\times\mathbf{B}, \end{aligned} \tag{8}$$

$$\begin{aligned} \mathbb{A}\cdot\mathbb{B} &= (a + \mathbf{A})\cdot(b + \mathbf{B}) \\ &= ab + \mathrm{A_x}\mathrm{B_x}ii + \mathrm{A_y}\mathrm{B_y}jj + \mathrm{A_z}\mathrm{B_z}kk \\ &= ab - \mathbf{A}\cdot\mathbf{B}. \end{aligned} \tag{9}$$

B. Unit

For the convenience, we change variables with following relations,

$$\mathbf{B}_{\text{new}} = c\mathbf{B},\ t_{\text{new}} = ct, \tag{10}$$

where $\mathbf{B}$ is the magnetic induction (Vs/m^2), and c is light speed (m/s). Then the well known Maxwell equations in vacuum become

$$\begin{aligned} \nabla\cdot\mathbf{B} &= 0, && (11)\\ \nabla\cdot\mathbf{E} &= 0, && (12)\\ \nabla\times\mathbf{B} &= \dot{\mathbf{E}}, && (13)\\ -\nabla\times\mathbf{E} &= \dot{\mathbf{B}}, && (14)\end{aligned}$$

where $\mathbf{E}$ is the electric field (V/m). We use this unit system in following sections.

C. Derivation

According to Ref. [5], we derive the generalized Maxwell equations in vacuum. We define a potential and an operator as

$$\begin{aligned} \mathbb{V} &= \mathrm{i}\phi+\psi+\mathbf{A}+\mathrm{i}\mathbf{C}, && (15)\\ \Box &= \mathrm{i}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\\ &= \mathrm{i}\frac{\partial}{\partial t}+\nabla. && (16)\end{aligned}$$

Then, we calculate a following equation,

$$\left[\Box\left[\Box\mathbb{V}\right]^*\right]^* = 0, \quad (17)$$

where the right hand side's zero means that there are no external charges and currents. So,

$$\begin{aligned} \Box\mathbb{V} &= \Box\left[\mathrm{i}\phi+\psi+\mathbf{A}+\mathrm{i}\mathbf{C}\right]\\ &= \mathrm{i}\left[\mathrm{i}\dot{\phi}+\dot{\psi}\right]+\mathrm{i}\left[\dot{\mathbf{A}}+\mathrm{i}\dot{\mathbf{C}}\right]-\nabla\cdot\left[\mathbf{A}+\mathrm{i}\mathbf{C}\right]\\ &+ \nabla\left[\mathrm{i}\phi+\psi\right]+\nabla\times\left[\mathbf{A}+\mathrm{i}\mathbf{C}\right]\\ &= -\mathrm{i}\left\{-\dot{\psi}+\nabla\cdot\mathbf{C}\right\}+\left\{-\dot{\phi}-\nabla\cdot\mathbf{A}\right\}\\ &+ \left\{-\dot{\mathbf{C}}+\nabla\psi+\nabla\times\mathbf{A}\right\}-\mathrm{i}\left\{-\dot{\mathbf{A}}-\nabla\phi-\nabla\times\mathbf{C}\right\}\\ &= -\mathrm{i}\eta+\xi+\mathbf{B}-\mathrm{i}\mathbf{E}, && (18)\end{aligned}$$

where

$$\begin{aligned} \eta &= -\dot{\psi}+\nabla\cdot\mathbf{C}, && (19)\\ \xi &= -\dot{\phi}-\nabla\cdot\mathbf{A}, && (20)\\ \mathbf{B} &= -\dot{\mathbf{C}}+\nabla\psi+\nabla\times\mathbf{A}, && (21)\\ \mathbf{E} &= -\dot{\mathbf{A}}-\nabla\phi-\nabla\times\mathbf{C}. && (22)\end{aligned}$$

Then,

$$\begin{aligned} \Box\left[\Box\mathbb{V}\right]^* &= \Box\left[\mathrm{i}\eta+\xi+\mathbf{B}+\mathrm{i}\mathbf{E}\right]\\ &= -\mathrm{i}\left\{-\dot{\xi}+\nabla\cdot\mathbf{E}\right\}+\left\{-\dot{\eta}-\nabla\cdot\mathbf{B}\right\}\\ &+ \left\{-\dot{\mathbf{E}}+\nabla\xi+\nabla\times\mathbf{B}\right\}-\mathrm{i}\left\{-\dot{\mathbf{B}}-\nabla\eta-\nabla\times\mathbf{E}\right\}\\ &= 0. && (23)\end{aligned}$$

Therefore, the generalized Maxwell equations in vacuum,

$$\begin{aligned} -\nabla\cdot\mathbf{B} &= \dot{\eta}, && (24)\\ \nabla\cdot\mathbf{E} &= \dot{\xi}, && (25)\\ \nabla\times\mathbf{B} &= \dot{\mathbf{E}}-\nabla\xi, && (26)\\ -\nabla\times\mathbf{E} &= \dot{\mathbf{B}}+\nabla\eta, && (27)\end{aligned}$$

are found. Consequently, we substitute (19), (20), (21), and (22) in (23), and we get a following equation,

$$\begin{aligned} \left[\Box\left[\Box\mathbb{V}\right]^*\right]^* &= \mathrm{i}\left\{\ddot{\phi}-\Delta\phi\right\}+\left\{\ddot{\psi}-\Delta\psi\right\}\\ &+ \left\{\ddot{\mathbf{A}}-\Delta\mathbf{A}\right\}+\mathrm{i}\left\{\ddot{\mathbf{C}}-\Delta\mathbf{C}\right\}\\ &= 0. && (28)\end{aligned}$$

So,

$$\begin{aligned} \ddot{\phi} &= \Delta\phi, && (29)\\ \ddot{\psi} &= \Delta\psi, && (30)\\ \ddot{\mathbf{A}} &= \Delta\mathbf{A}, && (31)\\ \ddot{\mathbf{C}} &= \Delta\mathbf{C}. && (32)\end{aligned}$$

If we define following intrinsic charges and currents,

$$\begin{aligned} -\rho_{\mathrm{m}} &= \dot{\eta}, && (33)\\ \rho_{\mathrm{e}} &= \dot{\xi}, && (34)\\ \mathbf{j_m} &= \nabla\eta, && (35)\\ -\mathbf{j_e} &= \nabla\xi, && (36)\end{aligned}$$

then the intrinsic charge conservation laws are obtained as

$$\begin{aligned} \nabla\cdot\mathbf{j_m} &= \Delta\eta=\ddot{\eta}=-\dot{\rho}_{\mathrm{m}}, && (37)\\ -\nabla\cdot\mathbf{j_e} &= \Delta\xi=\ddot{\xi}=\dot{\rho}_{\mathrm{e}}. && (38)\end{aligned}$$

Thus, we get the same type of equations with Dirac's extended Maxwell equations,[2]

$$\begin{aligned} \nabla\cdot\mathbf{B} &= \rho_{\mathrm{m}}, && (39)\\ \nabla\cdot\mathbf{E} &= \rho_{\mathrm{e}}, && (40)\\ \nabla\times\mathbf{B} &= \dot{\mathbf{E}}+\mathbf{j_e}, && (41)\\ -\nabla\times\mathbf{E} &= \dot{\mathbf{B}}+\mathbf{j_m}, && (42)\end{aligned}$$

In the tensor form, the generalized Maxwell equations in vacuum become

$$\begin{aligned} \partial_\nu \mathrm{F_e}^{\mu\nu} &= 0, && (43)\\ \partial_\nu \mathrm{F_m}^{\mu\nu} &= 0, && (44)\end{aligned}$$

where

$$\begin{aligned} \mathrm{F_e}^{\mu\nu} &= \begin{pmatrix} \mathrm{F_e}^{00} & \mathrm{F_e}^{01} & \mathrm{F_e}^{02} & \mathrm{F_e}^{03}\\ \mathrm{F_e}^{10} & \mathrm{F_e}^{11} & \mathrm{F_e}^{12} & \mathrm{F_e}^{13}\\ \mathrm{F_e}^{20} & \mathrm{F_e}^{21} & \mathrm{F_e}^{22} & \mathrm{F_e}^{23}\\ \mathrm{F_e}^{30} & \mathrm{F_e}^{31} & \mathrm{F_e}^{32} & \mathrm{F_e}^{33}\end{pmatrix}\\ &= \begin{pmatrix} \xi & -\mathrm{E_x} & -\mathrm{E_y} & -\mathrm{E_z}\\ \mathrm{E_x} & -\xi & -\mathrm{B_z} & \mathrm{B_y}\\ \mathrm{E_y} & \mathrm{B_z} & -\xi & -\mathrm{B_x}\\ \mathrm{E_z} & -\mathrm{B_y} & \mathrm{B_x} & -\xi\end{pmatrix}, && (45)\end{aligned}$$

$$\mathrm{F_m}^{\mu\nu} = \begin{pmatrix} \eta & \mathrm{B_x} & \mathrm{B_y} & \mathrm{B_z} \\ -\mathrm{B_x} & -\eta & -\mathrm{E_z} & \mathrm{E_y} \\ -\mathrm{B_y} & \mathrm{E_z} & -\eta & -\mathrm{E_x} \\ -\mathrm{B_z} & -\mathrm{E_y} & \mathrm{E_x} & -\eta \end{pmatrix}. \qquad (46)$$

Then, the electromagnetic stress-energy tensors become

$$\mathrm{T_e^{\mu\nu}} = \varepsilon_0 c^2 \xi \mathrm{F_e}^{\mu\nu}, \qquad (47)$$
$$\mathrm{T_m^{\mu\nu}} = \varepsilon_0 c^2 \eta \mathrm{F_m}^{\mu\nu}. \qquad (48)$$

where ε_0 is the electric permeability (As/Vm). Since we assume that there are no external fields, from $\partial_\nu \mathrm{T}^{\mu\nu} = 0$, the energy and force conservation laws,

$$0 = \frac{\partial}{\partial t}\frac{1}{2}\xi^2 + \mathbf{j_e}\cdot\mathbf{E}, \qquad (49)$$
$$0 = \rho_\mathrm{e}\mathbf{E} + \mathbf{j_e}\times\mathbf{B} - \nabla\frac{1}{2}\xi^2, \qquad (50)$$
$$0 = \frac{\partial}{\partial t}\frac{1}{2}\eta^2 + \mathbf{j_m}\cdot\mathbf{B}, \qquad (51)$$
$$0 = \rho_\mathrm{m}\mathbf{B} - \mathbf{j_m}\times\mathbf{E} - \nabla\frac{1}{2}\eta^2, \qquad (52)$$

are found. So, the energy conservation law for all fields becomes

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\mathbf{E}^2 + \mathbf{B}^2 + \xi^2 + \eta^2\right] = \nabla\cdot(\mathbf{B}\times\mathbf{E} + \xi\mathbf{E} - \eta\mathbf{B}). \qquad (53)$$

Lorentz condition,

$$\begin{aligned} \Box\cdot\mathbb{V} &= \left[\mathrm{i}\frac{\partial}{\partial t} + \nabla\right]\cdot[\mathrm{i}\phi + \psi + \mathbf{A} + \mathrm{i}\mathbf{C}] \\ &= \left\{-\dot{\phi} - \nabla\cdot\mathbf{A}\right\} - \mathrm{i}\left\{-\dot{\psi} + \nabla\cdot\mathbf{C}\right\} \\ &= \xi - \mathrm{i}\eta = 0, \end{aligned} \qquad (54)$$

is kept in the case of $\xi = \eta = 0$, which leads the well known Maxwell equations in vacuum, (11), (12), (13), and (14).

III. ANALYSIS

A. Definition

There are four potentials as

ϕ : electroscalar potential,
ψ : magnetoscalar potential,
$\mathbf{A}$: electrovector potential,
$\mathbf{C}$: magnetovector potential,

and four fields as

ξ : electroscalar field,
η : magnetoscalar field,
$\mathbf{E}$: electrovector field,
$\mathbf{B}$: magnetovector field.

Then, from these fields, we can make four waves with light speed as

$\mathfrak{p}$: electroscalar wave,
$\mathfrak{g}$: magnetoscalar wave,
$\mathcal{P}$: electrovector wave,
$\mathcal{G}$: magnetovector wave.

If we think about a forced oscillation of the $\mathbf{E}$ field with starting no potentials. From (22), we get a relation,

$$\mathbf{E} = -\dot{\mathbf{A}}, \qquad (55)$$

and get an well known transverse wave as the $\mathcal{P}$ wave, which is so called the electromagnetic wave. On the other hand, from a forced oscillation of the $\mathbf{B}$ field, we get another relation,

$$\mathbf{B} = -\dot{\mathbf{C}}. \qquad (56)$$

From this equation, we get again the electromagnetic wave as the $\mathcal{G}$ wave, which has the same relation among the $\mathbf{E}$, $\mathbf{B}$, and advancing direction with the $\mathcal{P}$ wave.

To make the scalar waves, we have to break Lorentz condition. We need to prepare only the $\mathbf{E}$ or $\mathbf{B}$ field by the way of, for example, crossing two electromagnetic waves. When only the $\mathbf{E}$ field is realized, from (22), the following relation,

$$\mathbf{E} = -\nabla\phi, \qquad (57)$$

makes a longitudinal wave as the $\mathfrak{p}$ wave, which is related with the plasma physics. On the other hand, when only the $\mathbf{B}$ field is realized, the following relation,

$$\mathbf{B} = \nabla\psi, \qquad (58)$$

makes a longitudinal wave as the uncommon $\mathfrak{g}$ wave.

B. Interaction between the electromagnetic and scalar waves

We prepare the $\mathfrak{p}$ wave as

$$\xi = \cos(\omega(t+z)), \qquad (59)$$
$$\rho_\mathrm{e} = \dot{\xi} = -\omega\sin(\omega(t+z)), \qquad (60)$$
$$\mathbf{j_e} = -\nabla\xi = \omega\sin(\omega(t+z))\mathbf{e_z} = -\rho_\mathrm{e}\mathbf{e_z}. \qquad (61)$$

We also prepare the electromagnetic wave having the same advancing direction with the $\mathfrak{p}$ wave as

$$\mathbf{E} = \cos(\omega'(t+z)))\mathbf{e_y} = |\mathbf{E}|\mathbf{e_y}, \qquad (62)$$
$$\mathbf{B} = \cos(\omega'(t+z)))\mathbf{e_x} = |\mathbf{E}|\mathbf{e_x}. \qquad (63)$$

In this case, the cross terms between those waves in (50) disappear as

$$\rho_\mathrm{e}\mathbf{E} + \mathbf{j_e}\times\mathbf{B} = \rho_\mathrm{e}|\mathbf{E}|\mathbf{e_y} - \rho_\mathrm{e}\mathbf{e_z}\times|\mathbf{E}|\mathbf{e_x} = 0. \qquad (64)$$

On the other hand, if we put the electromagnetic wave having another advancing direction, then the cross terms are alive and cause interactions between those waves. Therefore, the $\mathfrak{p}$ wave pushes out the electromagnetic wave. In the same way, from (52), the $\mathfrak{g}$ wave also pushes out the electromagnetic wave. So, if the gravitational wave is the $\mathfrak{p}$ or $\mathfrak{g}$ wave, then the electromagnetic wave including the visible light is kicked out by the gravitational wave.

C. Interaction between materials outputting scalar waves

Now, we think about a material outputting $\mathfrak{p}$ waves at the origin of the coordinates. In one dimensional case, those $\mathfrak{p}$ waves in front and back of the material become

$$\xi_{\text{front}} = \cos(\omega(t-z)), \tag{65}$$
$$\xi_{\text{back}} = \cos(\omega(t+z)), \tag{66}$$

respectively, and an incoming wave with the same frequency becomes

$$\xi_{\text{in}} = \cos(\omega(t-z)). \tag{67}$$

Then, the resulting waves are those superpositions,

$$\xi_{\text{front}} + \xi_{\text{in}} = 2\cos(\omega t)\cos(\omega z), \tag{68}$$
$$\xi_{\text{back}} + \xi_{\text{in}} = 2\cos(\omega(t+z)). \tag{69}$$

From these equations, we know that there is a standing wave in front of the material. So, the energy flow in front of the material equals zero. On the other hand, there is the still alive backflow in backward of the material. Consequently, the material receives the force because of the energy conservation law and, then, move to the direction of the source of the incomming wave. Thus, the materials outputting $\mathfrak{p}$ waves of the same frequency attract each other. In the same way, the materials outputting $\mathfrak{g}$ waves also pull in each other. Interestingly, there are no interactions between the material outputting $\mathfrak{p}$ waves and that outputting $\mathfrak{g}$ waves because those waves can not be merged. So, the $\mathfrak{p}$ and $\mathfrak{g}$ waves have a possibility to be the gravitational wave. However, those attracting each other must output the waves with the same frequency, ω_{G}, which is so-called the gravitational frequency depending on the source materials like the plasma frequency.

D. Relation with weight

We discuss about weight with assuming that the gravitational wave is the $\mathfrak{g}$ wave of the frequency ω_{G}. So, we think about a spherical wave originating from a material,

$$\eta = \frac{\eta_0 \cos(\omega_{\text{G}}(t-r))}{4\pi r^2}. \tag{70}$$

Since the gravitational force works in proportion to the amplitude of η, we can explain the weight of the material as

$$\sqrt{\varepsilon_0}\eta_0 = \sqrt{C_{\text{M}}} M c, \tag{71}$$

where M is the weight (kg) and $\sqrt{C_{\text{M}}}$ is a proportionality constant ($\sqrt{\text{m/kg}}$).

IV. SUMMARY

We explained that the generalized Maxwell equations in vacuum are basically the same with Dirac's extended Maxwell equations, in which there are the intrinsic charges and currents defined by the time differential and gradation of the scalar fields, respectively. From the tensor form of these equations, we derived the conservation laws, which explain that either the magnetoscalar or electroscalar wave has a possibility to be our gravitational wave with the gravitational frequency. Moreover, it is also explained that the electromagnetic wave including the visible light is kicked out by the gravitational wave. Furthermore, these equations expect the possibility of the existence of the materials outputting another gravitational wave which is different from ours. From those materials, we do not feel any gravitational forces although the electromagnetic waves are pushed out by those gravitational waves. However, there is still a problem that materials must output the gravitational wave constantly, which has to be solved.

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