Probabilistic Interpretation of Quantum Mechanics with Schrödinger Quantization Rule

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Abstract

Quantum theory is a probabilistic theory, where certain variables are hidden or non-accessible. It results in lack of representation of systems under study. However, I deduce system's representation in probabilistic manner, introducing probability of existence w, and quantize it exploiting Schrödinger's quantization rule. The formalism enriches probabilistic quantum theory, and enables systems's representation in probabilistic manner.

keywords Schrödinger Operators • Probability • Hidden Variables

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1 Introduction

Classical physics is based on *mechanistic* perspective, where no contingencies appear [1, 2]. It results in a deterministic theory, where no *chances* appear, and systems are governed by mechanistic laws. On the contrary, quantum theory is a probabilistic theory [3, p. 260]. So is its interpretation [4]. Quantum theory is not based on mechanistic order [2]. Indeterminism, an ingredient part of the theory, appears due to some hidden variables [5, 6, 7]. In a non-deterministic (*acausal*) theory (like QM) certain variables are (*hidden*) non-accessible. It persists in lack of representation of the system under study.

However, we define system's existence in probabilistic manner. We assign a probability (w) in order to define a system in isolation. For $w = 1$ system is in *pure state* and all its variables are accessible, for $w \epsilon (0, 1)$ it is in *mixed state* as certain of variables are *hidden* or non-accessible (e.g. in presence of many type of interactions [8]). For $w = 0$ the system is in *forbidden state* and all its variables are hidden and system can be represented by none. We quantize this observable using Schrödinger's quantization rule and obtain $\widehat{w} = -i\hbar\partial/\partial s$. Exploiting usual formalism of QM [9, 10, 11] we further deduce quantum dynamical equations, based on non-commutativity between *probability* w and *dynamicals* $\mathcal{A}$.

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2 Probability Eigenvalue Formalism

We have a general form of Schrödinger's wavefunction[1] belonging to system's Hilbert space $\mathbb{H}$, in generalized perspective [10]

$$\psi(R(q_i,t), s(q_i,t)) := R(q_i,t) \exp\left(\frac{i}{\hbar} s(q_i,t)\right) , \qquad i = 1, 2, 3, \ldots, f \,, \tag{2.1}$$

which is orthonormalizable

$$\langle \psi_\alpha | \psi_\beta \rangle = \int_{-\infty}^{+\infty} \psi_\alpha^*(R(q_i,t), s(q_i,t)) \psi_\beta(R(q_i,t), s(q_i,t))\, d\tau = \delta_{\alpha\beta} \,, \tag{2.2}$$

where $d\tau$ $(= \prod_{i=1}^{f} h_i dq_i$, h being scale factor and f is degrees of freedom) is generalized volume element of the *configuration space*. [The system has all these variables, except ψ (and tacitly its space $\mathbb{H}$) in Praxic perspective]. Differentiate (2.1) partially *w.r.t.* $\mathcal{A}$ction $s(q_i,t)$ to obtain

$$\frac{\partial \psi(R(q_i,t), s(q_i,t))}{\partial s(q_i,t)} = \frac{i}{\hbar} \psi(R(q_i,t), s(q_i,t)) \,. \tag{2.3}$$

I entail a unit (zero-order differential) operator that satisfies for an ordinary function f as well as for wavefunction (See Appendix A)

$$\mathcal{I} f = f \,; \qquad \mathcal{I} \psi(R(q_i,t), s(q_i,t)) = \psi(R(q_i,t), s(q_i,t)) \,. \tag{2.4}$$

Following deduction (2.4) for (2.3), we obtain

$$\mathcal{I} \psi(R(q_i,t), s(q_i,t)) + i\hbar \frac{\partial \psi(R(q_i,t), s(q_i,t))}{\partial s(q_i,t)} = 0 \,, \tag{2.5}$$

which is in the form of eigenvalue equation. We deduce *Schrödinger unit operator* $\widehat{\mathcal{I}}$ [in the sense of Schrödinger's quantization rule] satisfying *unit eigenoperator equation* [13, Dwivedi 2005]

$$\widehat{\mathcal{I}} |\psi\rangle = \mathcal{I} |\psi\rangle \,; \qquad \widehat{\mathcal{I}} = -i\hbar \frac{\partial}{\partial s} \,. \tag{2.6}$$

Its expectation value is given by inner-product

$$\begin{aligned} \langle \widehat{\mathcal{I}} \rangle = \langle \psi | \widehat{\mathcal{I}} | \psi \rangle &= \int_{-\infty}^{+\infty} \psi^*(R(q_i,t), s(q_i,t)) \left(-i\hbar \frac{\partial \psi(R(q_i,t), s(q_i,t))}{\partial s(q_i,t)} \right) d\tau \\ &= \int_{-\infty}^{+\infty} |\psi(R(q_i,t), s(q_i,t))|^2 \, d\tau = Prob.\,(-\infty, +\infty) \,. \end{aligned} \tag{2.7}$$

[it could also be obtained alternatively using (2.4) and (2.6) in inner-product (2.7).] The operator $\widehat{\mathcal{I}}$, having *trace* $Prob.\,(-\infty,+\infty)$, entails properties of our probability operator $\widehat{w}$. For a system in isolation:

$$\begin{cases} Prob.\,(-\infty,+\infty) = w_{pure} = 1 & \text{for pure state;} \\ Prob.\,(-\infty,+\infty) = w_{mixed} \;\epsilon\; (0,1) & \text{for mixed state;} \\ Prob.\,(-\infty,+\infty) = w_{forbidden} = 0 & \text{for forbidden state.} \end{cases} \tag{2.8}$$

Thus $\widehat{\mathcal{I}}$ is essentially $\widehat{w}$ that satisfies *probability eigenvalue equation*

$$\widehat{w} |\psi_w\rangle = w |\psi_w\rangle \,; \qquad \widehat{w} = -i\hbar \frac{\partial}{\partial s} \,. \tag{2.9}$$

[1] It is notable that ψ is function of q_i and t implicitly as well as function of R and s explicitly. Although $\mathcal{A}$ction $s[\ldots]$ is not function of q_i and t necessarily, instead it is often functional of the path. It has been taken function of q_i and t here for mere convention, that does not hurt assertion. Nevertheless, the result holds intact if one prefers $\psi(R,s) := R \exp\left(\frac{i}{\hbar} s\right)$ over (2.1).

Or

$$w\psi_w(R(q_i,t),s(q_i,t)) + i\hbar\frac{\partial\psi_w(R(q_i,t),s(q_i,t))}{\partial s(q_i,t)} = 0\,, \tag{2.10}$$

having solution

$$\psi_w(R(q_i,t),s(q_i,t)) = A\exp\left(\frac{i}{\hbar}ws(q_i,t)\right)\,. \tag{2.11}$$

For now we will treat ψ as function of s solely, for mere convention. For *orthonormalization* we have the inner-product,

$$\begin{aligned}\langle\psi_{w'}|\psi_w\rangle &= \int_{-\infty}^{+\infty}\psi^*_{w'}(s)\psi_w(s)\,ds \\ &= |A|^2\int_{-\infty}^{+\infty}\exp\left(\frac{i}{\hbar}(w-w')s\right)\,ds = |A|^2 2\pi\hbar\delta(w-w')\,.\end{aligned} \tag{2.12}$$

For $A = 1/\sqrt{2\pi\hbar}$, we have

$$\psi_w(s) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{i}{\hbar}ws\right) \tag{2.13}$$

that follows *Dirac orthonormality*

$$\langle\psi_{w'}|\psi_w\rangle = \delta(w-w')\,. \tag{2.14}$$

However, these eigenfunctions form **complete** set $\left(\psi = \sum_w c_w\psi_w\right)$. For (square-integrable) function $\psi(s)$,[2]

$$\psi(s) = \int_0^1 c(w)\psi_w(s)\,dw = \frac{1}{\sqrt{2\pi\hbar}}\int_0^1 c(w)exp\left(\frac{i}{\hbar}ws\right)\,dw\,. \tag{2.15}$$

The expansion coefficient is obtained by *Fourier's trick*

$$\langle\psi_{w'}|\psi\rangle = \int_0^1 c(w)\langle\psi_{w'}|\psi_w\rangle\,dw = \int_0^1 c(w)\delta(w-w')\,dw = c(w')\,, \tag{2.16}$$

or

$$c(w) = \langle\psi_w|\psi\rangle\,. \tag{2.17}$$

Exploiting completeness (2.15), the amplitude R in (2.1) is obtained

$$R = \frac{1}{\sqrt{2\pi\hbar}}\int_0^1 c(w)\exp\left(\frac{i}{\hbar}s(w-1)\right)\,dw\,. \tag{2.18}$$

3 Quantum Dynamical Equations

Dynamics is a law relating physical quantities in course of *time* (or some *internal observables* [15]). In Praxic theory $\mathcal{A}$*ction* is a fundamental physical entity [14]. However, it could often be customary to deduce dynamics in course of $\mathcal{A}$*ction*. Let differentiate the inner-product,

$$\langle\widehat{\mathcal{A}}\rangle = \langle\psi|\widehat{\mathcal{A}}|\psi\rangle = \int_{-\infty}^{+\infty}\psi^*\widehat{\mathcal{A}}\psi\,d\tau\,, \tag{3.1}$$

exactly *w.r.t.* $\mathcal{A}$ction with differential-integral rule

$$\widehat{f}g(\kappa) = \widehat{f}\int_{-\infty}^{+\infty}\phi(\tau)\mathcal{K}(\kappa,\tau)\,d\tau = \int_{-\infty}^{+\infty}\widehat{f}\left\{\phi(\tau)\mathcal{K}(\kappa,\tau)\right\}\,d\tau\,, \tag{3.2}$$

we obtain (using chain rule for $\widehat{f} := \frac{\partial}{\partial s}$)

$$\frac{\partial}{\partial s}\langle\widehat{\mathcal{A}}\rangle = \langle\frac{\partial\psi}{\partial s}|\widehat{\mathcal{A}}|\psi\rangle + \langle\psi|\frac{\partial\widehat{\mathcal{A}}}{\partial s}|\psi\rangle + \langle\psi|\widehat{\mathcal{A}}|\frac{\partial\psi}{\partial s}\rangle\,. \tag{3.3}$$

[2] As Probability does not exist in the limit $(-\infty,0)\,\mathbb{U}\,(1,+\infty)$, we have omitted integration over this limit. It does not create trouble in formalism.

Considering probability eigenvalue equations

$$|\frac{\partial\psi}{\partial s}\rangle = \frac{i}{\hbar}|\widehat{w}\psi\rangle\,, \qquad \langle\frac{\partial\psi}{\partial s}| = -\frac{i}{\hbar}\langle\widehat{w}^{\dagger}\psi|\,, \tag{3.4}$$

we obtain

$$\frac{\partial}{\partial s}\langle\widehat{\mathcal{A}}\rangle = \langle\frac{\partial\widehat{\mathcal{A}}}{\partial s}\rangle - \frac{i}{\hbar}[\langle\widehat{w}^{\dagger}\psi|\widehat{\mathcal{A}}|\psi\rangle - \langle\psi|\widehat{\mathcal{A}}\widehat{w}|\psi\rangle]\,. \tag{3.5}$$

Here $\mathcal{A}$, defined by $\mathcal{A} = \langle\psi|\widehat{\mathcal{A}}|\psi\rangle$, is a *dynamical* [15] — an observable-valued-function of system's variables — $\mathcal{A}(q_i, t)$ as distinct from observables. Since probability is a real aspect of nature, i.e., in operator representation, it must be hermitian[3],

$$\langle\widehat{w}^{\dagger}\psi|\widehat{\mathcal{A}}|\psi\rangle = \langle\psi|\widehat{w}\widehat{\mathcal{A}}|\psi\rangle\,, \tag{3.6}$$

which yields

$$\frac{\partial}{\partial s}\langle\widehat{\mathcal{A}}\rangle = \langle\frac{\partial\widehat{\mathcal{A}}}{\partial s}\rangle - \frac{i}{\hbar}\langle[\widehat{w},\widehat{\mathcal{A}}]_-\rangle\,. \tag{3.7}$$

This is first order *quantum dynamical equation.* Following the analogy, we further obtain second and third order quantum dynamical equations

$$\frac{\partial^2}{\partial s^2}\langle\widehat{\mathcal{A}}\rangle = \langle\frac{\partial^2\widehat{\mathcal{A}}}{\partial s^2}\rangle - \frac{i}{\hbar}\Big\langle\Big\{[\widehat{w},\frac{\partial\widehat{\mathcal{A}}}{\partial s}]_- + \frac{\partial}{\partial s}[\widehat{w},\widehat{\mathcal{A}}]_- - \frac{i}{\hbar}[\widehat{w},[\widehat{w},\widehat{\mathcal{A}}]_-]_-\Big\}\Big\rangle\,, \tag{3.8}$$

and

$$\begin{aligned}\frac{\partial^3}{\partial s^3}\langle\widehat{\mathcal{A}}\rangle = \langle\frac{\partial^3\widehat{\mathcal{A}}}{\partial s^3}\rangle - \Big\langle\Big\{&[\widehat{w},\frac{\partial^2\widehat{\mathcal{A}}}{\partial s^2}]_- + \frac{\partial}{\partial s}[\widehat{w},\frac{\partial\widehat{\mathcal{A}}}{\partial s}]_- + \frac{\partial^2}{\partial s^2}[\widehat{w},\widehat{\mathcal{A}}]_- \\ &- \frac{i}{\hbar}\Big([\widehat{w},[\widehat{w},\frac{\partial\widehat{\mathcal{A}}}{\partial s}]_-]_- + [\widehat{w},\frac{\partial}{\partial s}[\widehat{w},\widehat{\mathcal{A}}]_-]_- + \frac{\partial}{\partial s}[\widehat{w},[\widehat{w},\widehat{\mathcal{A}}]_-]_- \\ &- \frac{i}{\hbar}[\widehat{w},[\widehat{w},[\widehat{w},\widehat{\mathcal{A}}]_-]_-]_-\Big)\Big\}\Big\rangle\,.\end{aligned} \tag{3.9}$$

For operators ($\frac{\partial^n\widehat{\mathcal{A}}}{\partial s^n}$, $n = 0, 1, 2, ...$) compatible with $\widehat{w}$, these equations follow Ehrenfest's theorem

$$\frac{\partial^n\langle\widehat{\mathcal{A}}\rangle}{\partial s^n} = \langle\frac{\partial^n\widehat{\mathcal{A}}}{\partial s^n}\rangle\,. \tag{3.10}$$

It holds good for observables having simultaneous eigenstates with probability w.

Appendix

A Unit Operator

Unit operator (eigenoperator), analogous to *identity matrix*, is deduced as a zero-order (ordinary or partial) differential operator (irrespective of with respect to what) defined as

$$\mathcal{I} := \partial^0_x = \frac{\partial^0}{\partial x^0}\,; \qquad (x = q, p, t, ...)\,. \tag{A.1}$$

We have observed in *mathematical analysis* that a zero-order differential operator does not change the function to which it is applied which leads to deduce it unit operator satisfying $\mathcal{I}f = f$. For example, in Ostrogradsky transformation, zero-order prime of generalized co-ordinate $\overset{(n)}{q}$, $(n = 0, 1, 2, 3, ...)$ for $n = 0$ is given by q. It may be extended to $\overset{(n)}{q} = \mathcal{I}q = q$ for $n = 0$ with $\mathcal{I} := \partial^0_t$. The deduction is less applicable in mathematical analysis but is very important to deal

[3] It also follows from counter-intuitive behavior of probability operator $\widehat{w}$.

with quantum problems. Unit operator is quantized to $\widehat{\mathcal{I}} := -i\hbar\frac{\partial}{\partial s}$ satisfying unit eigenoperator equation $\widehat{\mathcal{I}}|\psi\rangle = \mathcal{I}|\psi\rangle$ while treating quantum problems. For example, a quantum transformation with $\overset{(n)}{\psi}$, $(n = 0, 1, 2, 3, ...)$ (being n^{th}-order partial derivative of ψ w.r.t. any variable x) is extended for $n = 0$, $\overset{(n)}{\psi} = \mathcal{I}\psi = \psi$ with $\mathcal{I} := \partial_x^0$. This is a quantum problem and we quantize $\mathcal{I}$ to $\widehat{\mathcal{I}}$ which yields $\overset{(n)}{\psi} + i\hbar\frac{\partial\psi}{\partial s} = 0$, for $n = 0$.

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