Fractal Operators in Non-Equilibrium Field Theory

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Abstract

Relativistic quantum field theory (QFT) describes fundamental interactions between elementary particles occurring in an energy range up to several hundreds GeV. Extending QFT beyond this range needs to account for the imbalance produced by unsuppressed quantum fluctuations and for the emergence of non-equilibrium phase transitions. Our underlying premise is that fractal operators become mandatory tools when exploring evolution from low-energy physics to the non-equilibrium regime of QFT. Canonical quantization using fractal operators leads to the concept of "complexon", a fractional extension of quantum excitations and a likely candidate for non-baryonic Dark Matter. A discussion on the duality between this new field-theoretic framework and General Relativity is included.

Key words: complexity, nonlinear dynamics, canonical quantization, fractal operators.

PACS numbers: 12.60.-i, 05.40. +j, 05.45. +b

1. Introduction and motivation

Quantum field theory (QFT) is an approximate description of particle phenomena occurring in an energy range below few hundreds GeV. For this reason, QFT is considered an *effective field theory* which deliberately ignores the substructure and the degrees of freedom observable above this upper bound [1]. A number of recent studies have suggested, from a variety of standpoints, that physics in the TeV regime of QFT may be a manifestation of *complex dynamics* [2-10 and related references 22-30]. For example, it has been argued that the onset of large and impulsive vacuum fluctuations, along with strong-gravity effects emerging from the short-distance behavior of QFT, warrant the passage from the standard tools of classical calculus to fractional calculus [8-10]. In general, use of conventional differential operators rests on the tacit assumption that a clear separation exists between the macroscopic and the microscopic levels of physical description. Implicit in this assumption is the condition that dynamical processes on the microscopic scale are *stable*. If this condition fails to be true, dynamical instabilities can develop on arbitrarily long time-scales and the macroscopic description of phenomena in terms of ordinary differential operators is no longer valid [13, 17]. Such a scenario may be typical for physics in the TeV regime where far-from equilibrium statistical processes are expected to dominate. Let us briefly elaborate on this point with the help of an idealized quantum-mechanical experiment. Consider an isolated two-state quantum system whose state vector $|\psi\rangle$ at time $t=0$ is given by [21]

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \tag{1}$$

where $|0\rangle$ and $|1\rangle$ denote the two orthogonal states and c_0, c_1 are complex numbers. Assume that the quantum vacuum, acting as source of large and steady fluctuations, may be modeled as a two-state reservoir with vectors $|\mathrm{v}_0\rangle$ and $|\mathrm{v}_1\rangle$. Bring the quantum vacuum in contact with the system at some instant $t_0 > 0$ and maintain the contact for an interval $t_{INT} > t_0$. The coupling of the two objects through unitary evolution leads to a time-dependent state

$$|\psi(t)\rangle = c_0|0\rangle \otimes |\mathrm{v}_0\rangle + c_1|1\rangle \otimes |\mathrm{v}_1\rangle \tag{2}$$

where $\otimes$ stands for the tensor product and $t_{INT} \geq t > t_0$. It is seen that the system and vacuum become *entangled* on a time-scale commensurate with t_{INT} and one can no longer treat (1) as describing an object with a well-defined quantum identity. In contrast to the low-energy regime of quantum theory, the high-energy dynamics of the vacuum is characterized by a large number of time-scales that are not reducible to a single average through coarse-graining. Under the most general circumstances, the vacuum dynamics may be regarded as an exotic stochastic process that can be accordingly modeled as Levy noise [11-12]. A characteristic attribute of the Levy distribution is that it has infinite second moments and gives rise to long-range correlated dynamics. It follows from these considerations that the ensemble system + vacuum evolves on *multiple scales.* This line of reasoning reproduces, in essence, the statistical physics argument for replacing ordinary derivatives and integrals with fractal operators [13]. As noted, the primary motivation for assuming that quantum fluctuations follow Levy statistics lies on the continuous regime of impulsive excitations generated by the short-distance limit of QFT.

To the best of our knowledge, this work represents the first attempt to build a field theory on the basis of fractional differential and integral operators. We caution that our contribution is meant to serve as an informal introduction and not as a rigorous and comprehensive treatment of the topic.

The paper is organized in the following way: sections 3 and 4 develop canonical quantization for free scalar and Dirac fields using fractal operators. Section 5 is devoted to a brief discussion on the dual aspect of the new framework and General Relativity. Last section contains a short summary of results and future prospects.

2. Notation, assumptions and conventions

We introduce here the main notations and assumptions that underlie the remainder of the paper:

i) the summation convention is applied on repeated upper and lower indices and Planck's constant is set to $\hbar = 1$.

ii) motivated by the growing evidence for complexity in field theory, our focus is the behavior of fractional *dynamical systems* [14]. These systems are characterized by non-integer powers of generalized coordinates and momenta

$$q^{\alpha} = |q|^{\alpha}, \quad p^{\alpha} = |p|^{\alpha} \tag{3}$$

in which $\alpha > 0$. As stated, we study the dynamics of free fractional scalar and Dirac fields. To avoid cluttering the notation, the corresponding field variables are respectively designated as

$$\varphi = q^{\alpha}, \quad \psi = q^{\alpha} \tag{4}$$

iii) the hat symbol "^" is used to indicate operators.

iv) analysis is limited to real or complex functions of the dimensionless field variable $q \geq 0$ for which fractional derivatives and integrals exist. We adopt hereafter the regularized expression for fractional derivative [18-19, 25]

$$D^{\alpha}[f(q)] = \frac{\partial^{\alpha}[f(q)]}{\partial q^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{q} \frac{\partial f(\xi)}{\partial \xi} \frac{d\xi}{(q-\xi)^{\alpha}} \tag{5}$$

where $0 < \alpha < 1$. The fractional momentum operator is introduced in Appendix A by analogy with conventional formulation of quantum mechanics. It may be shown that fractional momentum is linear and hermitean. The latter property follows from an extended definition of the conjugate operator, as detailed in (A7) through (A10).

v) the generalized Lagrangian for a classical fractional system depending on n fields, their fractional derivatives of orders ω_l and n locally defined exponents $\alpha_l(x)$ $(l = 1, 2, \ldots, n)$ is defined by

$$L_G{}^{\alpha_1(x),\alpha_2(x)...,\alpha_n(x)} = L_G(q_1^{\alpha_1(x)}, q_2^{\alpha_2(x)}, ..., q_n^{\alpha_n(x)}; D^{\omega_1} q_1^{\alpha_1(x)}, D^{\omega_2} q_2^{\alpha_2(x)}, ..., D^{\omega_n} q_n^{\alpha_n(x)}; t) \qquad (6)$$

vi) quantum field theory based on (6) is abbreviated throughout as c-QFT.

vii) the commutator and anti-commutator for any pair of arbitrary operators ($\hat{f}, \hat{g}$) are, respectively

$$[\hat{f}, \hat{g}] = \hat{f}\hat{g} - \hat{g}\hat{f} \qquad (7)$$

$$\{\hat{f}, \hat{g}\} = \hat{f}\hat{g} + \hat{g}\hat{f} \qquad (8)$$

viii) state vectors and inner products are formulated using Dirac notation.

ix) the vacuum state is considered empty and is labeled with the zero-particle ket $|0\rangle$.

x) dynamical processes described by c-QFT are Markovian and, as such, have no time memory.

xi) greek letters $\mu, \nu, \sigma = 0,1,2,3$ denote space-time indices whereas Latin letters $i, j, k = 1,2,3$ label the set of three spatial coordinates.

3. Scalar bosons in c-QFT

The classical Lagrangian for the free scalar field theory in 3+1 dimensions reads [16, 20, 26-28]

$$L = \partial^\mu \varphi \, \partial_\mu \varphi - m^2 \varphi^2 \qquad (9)$$

and leads to the following expression for the field momentum

$$\pi = \frac{\partial L}{\partial(\frac{\partial \varphi}{\partial t})} = \frac{\partial \varphi}{\partial t} \qquad (10)$$

It is known that the standard technique of canonical quantization promotes a classical field theory to a quantum field theory by converting the field and momentum variables into operators. To gain full physical insight with minimal complications in formalism, we work below in 0+1 dimensions. Define the field and momentum operators as

$$\begin{aligned} \varphi &\to \hat{\varphi} = \varphi \\ \pi &\to \hat{\pi}^{\alpha} = -i\frac{\partial^{\alpha}}{\partial|\varphi|^{\alpha}} \equiv -i D^{\alpha} \end{aligned} \tag{11}$$

Without loss of generality, we set $m = 1$ in (9). The Hamiltonian becomes

$$H \to \hat{H}^{\alpha} = -\frac{1}{2}D^{2\alpha} + \frac{1}{2}\varphi^{2} = \frac{1}{2}(\hat{\pi}^{2\alpha} + \varphi^{2}) \tag{12}$$

The state of the field in the Schrödinger representation is described by a complex-valued wavefunction $\Psi(\varphi) = \langle \varphi | \Psi \rangle$ whose conjugate-square is the probability density for φ. This wavefunction evolves according to the time-dependent Schrödinger equation

$$-i\partial_{t}\Psi(\varphi) = \hat{H}^{\alpha}\Psi(\varphi) \tag{13}$$

The commutation relations corresponding to (11) may be written as (per Appendix B)

$$\begin{aligned} &[\varphi, \varphi] = 0 \\ &\left[\hat{\pi}^{\alpha}, \hat{\pi}^{\alpha}\right] = [D^{\alpha}, D^{\alpha}] = 0 \\ &[\hat{\varphi}, \hat{\pi}^{\alpha}] = i\alpha\hat{\pi}^{(\alpha-1)} \end{aligned} \tag{14}$$

By analogy with the standard treatment of harmonic oscillator in quantum mechanics, it is convenient to work with the destruction and creation operators defined through [20, 27-28]

$$\begin{aligned} \hat{a}^{\alpha} &= \frac{1}{\sqrt{2}}[\hat{\varphi} + i\hat{\pi}^{\alpha}] \\ \hat{a}^{+\alpha} &= \frac{1}{\sqrt{2}}[\hat{\varphi} - i\hat{\pi}^{\alpha}] \end{aligned} \tag{15}$$

Straightforward algebra shows that these operators satisfy the following commutation rules

$$[\hat{a},\hat{a}]=[\hat{a}^{+\alpha},\hat{a}^{+\alpha}]=0$$
$$[\hat{a}^{+\alpha},\hat{a}^{\alpha}]=i[\hat{\varphi},\hat{\pi}^{\alpha}]=-\alpha\hat{\pi}^{(\alpha-1)} \tag{16}$$

The second relation in (16) leads to

$$\hat{H}^{\alpha}=\hat{a}^{+\alpha}\hat{a}^{\alpha}+\frac{1}{2}\alpha\hat{\pi}^{(\alpha-1)} \tag{17}$$

In the limit $\alpha=1$ we recover the quantum mechanics of the harmonic oscillator, namely

$$\hat{H}=\hat{a}^{+}\hat{a}+\frac{1}{2} \tag{18}$$

Next, consider the commutator $[\hat{N}^{\alpha},\hat{a}^{+}]$, where $\hat{N}\equiv\hat{a}^{+}\hat{a}$ designates the number operator. Making use of (16) and (17), we obtain:

$$[\hat{N}^{\alpha},\hat{a}^{+\alpha}]=\alpha\hat{\pi}^{(\alpha-1)}\hat{a}^{+\alpha} \tag{19}$$

Thus

$$\hat{N}^{\alpha}\hat{a}^{+\alpha}|0\rangle=(\hat{a}^{+\alpha}\hat{N}^{\alpha}+[\hat{N}^{\alpha},\hat{a}^{+\alpha}])|0\rangle=\alpha\hat{\pi}^{(\alpha-1)}\hat{a}^{+\alpha}|0\rangle \tag{20}$$

The eigenvalue equation corresponding to the above relation has the form

$$D^{(\alpha-1)}\hat{a}^{+\alpha}|0\rangle=\lambda^{(\alpha-1)}|0\rangle \tag{21}$$

and it is solved in Appendix C. Under the most general circumstances, $\lambda_n^{(\alpha-1)}>0$ $(n=1,2,..)$ form a set of real numbers. As a result, we are led to conclude that $\hat{a}^{+\alpha}|0\rangle$ represents an eigenvector of the number operator having fractional eigenvalues $\lambda_n^{(\alpha-1)}$. Stated differently, the action of the creation operator $\hat{a}^{+\alpha}$ on the empty vacuum is to produce a particle that carries a fractional quantum of energy. Following the general arguments of the Introduction, we may call these fractional excitations of the scalar field "*complexons*". According to Appendix C, since $\lambda_n^{(\alpha-1)}$ form a discrete set of eigenvalues, it is appropriate to regard the complexon as a *fractional particle with a discrete energy spectrum.* We close this section with the observation that, on account of (17) and (18), the term $\frac{1}{2}\alpha\pi^{(\alpha-1)}$ plays the role of a *zero-point operator*. In contrast with conventional quantum theory, it is apparent that the dynamical contribution of the background vacuum in c-QFT amounts to more than a constant additive term to the Hamiltonian.

4. Fermions in c-QFT

The classical Dirac equation describing free fermion fields in 3+1 dimension is [16, 20, 26-28]

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \tag{22}$$

where γ^{μ} are 4 x 4 matrices given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{23}$$

and ψ is a 4-component spinor which transforms under the spin ½ representation of the Lorentz group. The Dirac equation may be derived from the Lagrangian

$$L_D = \frac{i}{2}[\overline{\psi}\gamma^{\mu}(\partial_{\mu}\psi) - (\partial_{\mu}\overline{\psi})\gamma^{\mu}\psi] - m\overline{\psi}\psi \tag{24}$$

in which the adjoint spinor is defined as

$$\overline{\psi} = \psi^{+}\gamma^{0} \tag{25}$$

To simplify the formalism and capture the essentials of the argument, we choose to work again in 0+1 dimensions and set $m=1$. Let the spinor field be expanded in a basis containing the eigenstates of γ^0 that is

$$\psi = \psi_+ \left|+\right\rangle + \psi_- \left|-\right\rangle \tag{26}$$

where

$$\begin{gathered} \left|+\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \left|-\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \left\langle+\right| = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad \left\langle-\right| = \begin{pmatrix} 0 & 1 \end{pmatrix} \end{gathered} \tag{27}$$

The conjugate momentum of the spinor field is

$$\Pi = \frac{\partial L_D}{\partial(\frac{\partial \psi}{\partial t})} = i\psi^+ \tag{28}$$

Consider now the coordinate Schrödinger representation for Dirac fields, whereby an arbitrary state $\left|\Phi\right\rangle$ is represented by the wavefunction $\Phi(\psi) = \left\langle \psi \mid \Phi \right\rangle$. By analogy with the previous section, we cast Dirac field theory in the operator language. Let us take the state $\left|\psi\right\rangle$ to be an eigenstate of the field operator $\hat{\psi}$ with eigenvalue ψ

$$\hat{\psi}\left|\psi\right\rangle = \psi\left|\psi\right\rangle \tag{29}$$

The field conjugate momentum is then

$$\hat{\Pi} = i\hat{\psi}^+ = i\frac{\partial}{\partial \psi} \tag{30}$$

Creation and destruction operators are introduced as follows [27-28]

$$\begin{gathered} \hat{\psi} = \hat{b}\left|+\right\rangle + \hat{c}^+\left|-\right\rangle \\ \hat{\psi}^+ = \hat{b}^+\left\langle+\right| + \hat{c}\left\langle-\right| \end{gathered} \tag{31}$$

$$\bar{\vec{\psi}} = \hat{b}^{+}\langle + | - \hat{c}\langle - |$$

Here, $\hat{b}$ and $\hat{c}$ are the fermion and antifermion destruction operators, whereas $\hat{b}^{+}$ and $\hat{c}^{+}$ denote the fermion and antifermion creation operators. It is known that, to ensure that the total fermion energy is positive-definite, Dirac field theory is formulated using anticommutators rather than commutators [16, 26-28]. On account of (31) and of the fact that fermions and antifermions are always produced or annihilated in pairs, the field-momentum anti-commutator is given by

$$\{\vec{\psi}, \vec{\Pi}\} \equiv \{\vec{\psi}, i\vec{\psi}^{+}\} = \{\hat{b}, \hat{b}^{+}\} + \{\hat{c}, \hat{c}^{+}\} \tag{32}$$

Here, according to (8)

$$\{\vec{\psi}, \vec{\Pi}\} \equiv \vec{\psi}\,\vec{\Pi} + \vec{\Pi}\,\vec{\psi} \tag{33}$$

The Hamiltonian is proportional to the total number of particles $(\bar{\vec{\psi}}\vec{\psi})$ and assumes the form [27-28]

$$\vec{H}_{D} = \bar{\vec{\psi}}\vec{\psi} = (\hat{b}^{+}\hat{b} + \hat{c}^{+}\hat{c} - 1) \tag{34}$$

Moreover, a new operator may be introduced in the theory as being proportional to the difference of the number of fermions and number of antifermions. This is known as the charge operator and is represented by

$$\vec{Q} \equiv e\bar{\vec{\psi}}\gamma^{0}\vec{\psi} = e(\hat{b}^{+}\hat{b} - \hat{c}^{+}\hat{c} + 1) \tag{35}$$

where e is the electron charge. Proceeding in a way similar to the previous section, the conjugate momentum for Dirac c-QFT may be defined as

$$\vec{\Pi}^{\alpha} = i\vec{\psi}^{+\alpha} = i\frac{\partial^{\alpha}}{\partial|\psi|^{\alpha}} = iD^{\alpha} \tag{36}$$

From (33) and (36), the corresponding field momentum anti-commutator reads

$$\{\hat{\psi}, \hat{\Pi}^{\alpha}\} = 2\hat{\psi}\hat{\Pi}^{\alpha} + \alpha\hat{\Pi}^{(\alpha-1)} \tag{37}$$

Let us assume, for the sake of simplicity, that the contribution of the second term in (37) outweighs the contribution of the first term, i.e. $\left|\psi \frac{\partial^{\alpha}\Phi}{\partial\psi^{\alpha}}\right| \ll \left|\alpha \frac{\partial^{(\alpha-1)}\Phi}{\partial\psi^{(\alpha-1)}}\right|$. In addition, according to (32), there is a symmetric contribution of the fermion and anti-fermion number operators in the structure of $\{\hat{\psi}, \hat{\Pi}\}$. Thus we set

$$\{\hat{b}^{\alpha}, \hat{b}^{+\alpha}\} = \{\hat{c}, \hat{c}^{+}\} = \frac{1}{2}\alpha\hat{\Pi}^{(\alpha-1)} \tag{38}$$

A logical way to proceed from here is by writing down the anti-commutation relations involving the number and creation operators for fermions and anti-fermions. Retracing the sequence of steps (19) to (21), we arrive at the equation

$$D^{(\alpha-1)}\hat{b}^{+\alpha}\left|0\right\rangle = \eta^{(\alpha-1)}\left|0\right\rangle \tag{39}$$

whose eigenvalues $\eta^{(\alpha-1)}$ form a set of positive and fractional numbers. Considering the same arguments that lead to (21), fermion field excitations generated by $\eta^{(\alpha-1)}$ may be also interpreted as "*complexons*". Moreover, it follows from (35) and (39) that $\hat{Q}$ generates fractional fermion charges. The emergence of complexons and fractional Dirac charges may be seen as a dynamic manifestation of the high-energy regime that do not have a counterpart in conventional QFT.

5. Classical limit of c-QFT and General Relativity

Given the topological roots of exponent α [8, 29] and its dynamical role in the development of our model, it is of interest to explore how fractal attributes encoded by α may be mapped onto the underlying metric of the space-time manifold. To this end, consider the Lagrangian of the classical scalar field theory (9) in four-dimensional space-time

$$L = (\partial\varphi/\partial t)^2 - \sum_k (\partial\varphi/\partial x^k)^2 - m^2\varphi^2 \tag{40}$$

The generalized Lagrangian built from (40) assumes the form

$$L_G^\alpha = (\partial\varphi/\partial t)^2 - \sum_k D_{x_k}^{2\alpha}\varphi - m^2\varphi^2 \tag{41}$$

An alternate expression for the fractional derivative $D_{x_k}^\alpha \varphi$ is [18]

$$D_{x_k}^\alpha \varphi(x_k) = \frac{\varphi(0)x_k^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_0^{x_k} (x_k-\xi)^{-\alpha}\,\frac{\partial\varphi}{\partial x_k}(\xi)\,d\xi \tag{42}$$

Let the coordinate of point ξ_k ($0 < \xi_k \le x_k$) be defined as an arbitrary fraction of the endpoint coordinate $x_k \ll 1$, that is, $\xi_k = s x_k$, with $s \le 1$. Assuming, for the sake of simplicity, that the first term in (42) is negligible in comparison with the second term, we derive the following approximation

$$D_{x_k}^\alpha \varphi(\frac{\xi_k}{s}) \approx \frac{(1-s)^{-\alpha}}{s^{1-\alpha}\Gamma(1-\alpha)}\,\xi_k^{\,(1-\alpha)}\,\frac{\partial\varphi}{\partial x_k}(\xi_k) \tag{43}$$

and

$$D_{x_k}^{2\alpha}\varphi(\xi_k) \approx g_{ik}^\alpha(\xi_k)[\frac{\partial\varphi}{\partial x_i}(\xi_i)][\frac{\partial\varphi}{\partial x^k}(\xi_k)] \tag{44}$$

Up to a product of multiplicative factors independent of ξ_k, the metric $g_{ik}^\alpha(\xi_k)$ is given by

$$g_{ik}^\alpha(\xi_k) = \eta_{ik} g_{kk}^\alpha(\xi_k) \sim \eta_{ik}\xi_k^{2(1-\alpha)} \tag{45}$$

where η_{ik} is the Minkowski metric of Special Relativity.

On account of (44) and (45), we are led to conclude that the classical limit of c-QFT for free scalar bosons may be formally interpreted as a classical field theory *in curved space-*

time. It can be seen that (45) reduces to the metric of Special Relativity when the fractal topology of space-time makes the transition to a smooth topology, i.e. in the classical limit $\alpha \to 1$. The equivalent metric (45) is subject to the constraint briefly discussed in Appendix D.

6. Concluding remarks

We have laid out the groundwork for complex-Quantum Field Theory using the methodology of fractal differential and integral operators. Our framework has been developed with emphasis on canonical quantization and has led to the following conclusions: i) the Fock space of c-QFT includes *fractional numbers* of particles and antiparticles per state, and ii) classical limit of c-QFT is equivalent to field theory in *curved space-time*. The last finding suggests that c-QFT may be regarded as a natural bridge between conventional Quantum Field Theory and General Relativity. Future research efforts may be directed towards developing the complexon algebra, understanding the connection between fractional statistics [15, 20], non-commutative field theory and c-QFT, as well as formulating predictions that can be tracked and tested at the Large Hadron Collider and future accelerators.

Appendix A

Fractional derivative of order $0 < \alpha < 1$ described by (5) may be alternatively expressed as a convolution, i.e.

$$D_{>}^{\alpha} f(q) = f(q) * \Lambda_{\alpha}^{+}(q) \tag{A1}$$

where

$$\Lambda_{\alpha}^{+}(q) = \frac{q^{-\alpha}}{\Gamma(1-\alpha)} \quad (q > 0) \tag{A2}$$

Similarly we can introduce

$$D_{<}^{\alpha} f(q) = f(q) * \Lambda_{\alpha}^{-}(q) \tag{A3}$$

with

$$\Lambda_{\alpha}^{-}(q) = \Lambda_{\alpha}^{+}(-q) = \frac{q^{-\alpha}}{\Gamma(1-\alpha)} \quad (q < 0) \tag{A4}$$

Let

$$\hat{p}^{\alpha}\psi(q) = -iD_{>}^{\alpha}\psi(q) \tag{A5}$$

stand for the fractional momentum operator working on the wavefunction $\psi(q)$. In accordance with the standard formalism of quantum mechanics, its average is given by

$$\left\langle \hat{p}^{\alpha} \right\rangle = \int_{-\infty}^{\infty} \psi^{*}(q)(-iD_{>}^{\alpha})\psi(q)dq \tag{A6}$$

To keep the notation simple, we omit throughout the text the subscript "$>$". Hence we set

$$D_{>}^{\alpha} = D^{\alpha} \tag{A7}$$

Fractional momentum is a linear operator since it satisfies

$$\hat{p}^{\alpha}\psi_1 = \psi_2$$

$$\hat{p}^{\alpha}(\psi_1 + \psi_2) = \hat{p}^{\alpha}\psi_1 + \hat{p}^{\alpha}\psi_2 \tag{A8}$$

$$C\hat{p}^{\alpha}\psi = \hat{p}^{\alpha}(C\psi)$$

where C is an arbitrary constant. The integration by parts formula [19, 25]

$$\int_{-\infty}^{\infty} \psi^{*}(q)(-iD_{>}^{\alpha})\psi(q)dq = \int_{-\infty}^{\infty} \psi(q)(-iD_{<}^{\alpha})\psi^{*}(q)\, dq \tag{A9}$$

implies that the fractional momentum operator is hermitean if (and only if) we adopt the following definition

$$-iD_{<}^{\alpha} = (-iD_{>}^{\alpha})^{*} = i(D_{>}^{\alpha})^{*} \qquad \text{(A10)}$$

<u>Appendix B</u>: Derivation of the commutator $[\hat{\varphi}, \pi^{\alpha}] = i\alpha\pi^{(\alpha-1)}$

Start from (11) and the formal commutator definition

$$[\hat{\varphi}, \pi^{\alpha}]|\varphi\rangle = (-i)[\hat{\varphi} D^{\alpha}|\varphi\rangle - D^{\alpha}(\hat{\varphi}\cdot|\varphi\rangle)] \qquad \text{(B1)}$$

and apply the generalized Leibniz rule [13, 18]

$$D^{\alpha}(\hat{\varphi}\cdot|\varphi\rangle) = \sum_{m=0}^{\infty}\binom{\alpha}{m} D^{m}\varphi D^{\alpha-m}|\varphi\rangle = \varphi D^{\alpha}|\varphi\rangle + \binom{\alpha}{1} D^{(\alpha-1)}|\varphi\rangle \qquad \text{(B2)}$$

in which

$$\binom{\alpha}{m} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-m)\Gamma(1+m)} \qquad \text{(B3)}$$

Therefore

$$\binom{\alpha}{1} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha \qquad \text{(B4)}$$

From (B1) to (B4) we derive

$$[\hat{\varphi}, \pi^{\alpha}] = i\alpha\pi^{(\alpha-1)} \qquad \text{(B5)}$$

<u>Appendix C</u>: Fractional eigenvalue equation $\pi^{(\alpha-1)}\hat{a}^{+\alpha}|0\rangle = \frac{\lambda}{\alpha}\hat{a}^{+\alpha}|0\rangle$

Consider

$$D^{(\alpha-1)}\hat{a}^{+\alpha}|0\rangle = \lambda^{(\alpha-1)}\hat{a}^{+\alpha}|0\rangle \qquad \text{(C1)}$$

where

$$\lambda^{(\alpha-1)} = \frac{i\lambda}{\alpha} \qquad \text{(C2)}$$

Here, we have employed the notation

$$\hat{a}^{+\alpha} = \hat{a}^{+\alpha}(\varphi) \tag{C3}$$

The general solution of the above fractional eigenvalue equation subject to the boundary condition [18]

$$D^{(\alpha-2)}\hat{a}^{+\alpha} = (\hat{a}^{+\alpha})_1 \text{ as } \varphi \to 0 \tag{C4}$$

is represented by

$$\hat{a}^{+\alpha} = (\hat{a})_1^{+\alpha}\varphi^{\alpha-2}E_{\alpha-1,\alpha-1}[\lambda^{(\alpha-1)}\varphi^{(\alpha-1)}] \tag{C5}$$

in which $E_{\alpha,\beta}(x)$ denotes the Mittag-Lefler function of order α,β. To determine the eigenvalue spectrum $\lambda_n^{(\alpha-1)}$, we use a boundary condition that fixes the behavior of $\hat{a}^{+\alpha}$ and $(\hat{a}^{+\alpha})_1$ as the scalar field φ approaches its upper limit $\varphi \to \varphi_0$, namely

$$\hat{a}^{+\alpha}(\varphi_0)|0\rangle = A^{+\alpha}(\varphi_0)|0\rangle \tag{C6}$$

On the other hand we have, starting from the boundary condition definition (C4),

$$(\hat{a}^{+\alpha})_1(\varphi)|0\rangle = A_1^{+\alpha}(\varphi)|0\rangle \text{ as } \varphi \to 0 \tag{C7}$$

where it is assumed that

$$A_1^{+\alpha}(0) = A_1^{+\alpha}(\varphi_0) \tag{C8}$$

This ansatz leads to the following implicit equation for $\lambda_n^{(\alpha-1)}$

$$A^{+\alpha}(\varphi_0) = A_1^{+\alpha}(0)\,\varphi_0^{\alpha-2}E_{\alpha-1,\alpha-1}[\lambda^{(\alpha-1)}\varphi_0^{\alpha-1}] \tag{C9}$$

Appendix D

The equivalent metric must transform in a way that maintains invariance of the space-time interval under arbitrary coordinate changes $x \to \bar{x}(x)$. Hence, in general

$$\bar{g}^{\bar{\alpha}}_{\lambda\sigma}(\bar{x})(d\bar{x}^{\lambda})^{\bar{\alpha}}(d\bar{x}^{\sigma})^{\bar{\alpha}} = g^{\alpha}_{\mu\nu}(x)(dx^{\mu})^{\alpha}(dx^{\nu})^{\alpha} \tag{D1}$$

where $\bar{\alpha}$ labels the exponent corresponding to the reference frame $\bar{x}$. On account of (5), the partial derivative $\partial^{\bar{\alpha}}(x^{\mu})^{\alpha} \big/ \partial(\bar{x}^{\lambda})^{\bar{\alpha}}$ may be defined as

$$\frac{\partial^{\bar{\alpha}}(x^{\mu})^{\alpha}}{\partial(\bar{x}^{\lambda})^{\bar{\alpha}}} = \frac{1}{\Gamma(1-\bar{\alpha})} \int_{-\infty}^{\bar{x}^{\lambda}} \frac{\partial(x^{\mu})^{\alpha}}{\partial \bar{x}^{\lambda}} \frac{d\xi}{(\bar{x}^{\lambda}-\xi)^{\bar{\alpha}}} \tag{D2}$$

The formal connection between $\bar{g}^{\bar{\alpha}}_{\lambda\sigma}(\bar{x})$ and $g^{\alpha}_{\mu\nu}(x)$ may be consequently obtained upon replacing (D2) in (D1).

7. References

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