ON THE MEAN VALUE OF THE ADDITIVE ANALOGUE OF SMARANDACHE FUNCTION*

Yi Yuan and Zhang Wenpeng

Research Center for Basic Science, Xi'an Jiaotong University
Xi'an, Shaanxi, P.R.China

Abstract. For any positive integer n, let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined the smallest $m \in N^+$, where $n|m!$. In this paper, we study the mean value properties of the additive analogue of $S(n)$, and give an interesting mean value formula for it.

1. Introduction and results

For any positive integer n, let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined the smallest $m \in N^+$, where $n|m!$. In paper [2], Jozsef Sandor defined the following analogue of Smarandache function:

$$S_1(x) = \min\{m \in N : x \le m!\}, \quad x \in (1, \infty), \tag{1}$$

which is defined on a subset of real numbers. Clearly $S_1(x) = m$ if $x \in ((m-1)!, m!]$ for $m \ge 2$ (for $m = 1$ it is not defined, as $0! = 1! = 1!$), therefore this function is defined for $x > 1$.

About the arithmetical properties of $S(n)$, many people had studied it before (see reference [3]). But for the mean value problem of $S_1(n)$, it seems that no one have studied it before. The main purpose of this paper is to study the mean value properties of $S_1(n)$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. *For any real number $x \ge 2$, we have the mean value formula*

$$\sum_{n \le x} S_1(n) = \frac{x \ln x}{\ln \ln x} + O\left(\frac{x(\ln x)(\ln \ln \ln x)}{(\ln \ln x)^2}\right).$$

Key words and phrases. Smarandache function; Additive Analogue; Mean Value formula.
* This work is supported by the Zaizhi Doctorate Foundation of Xi'an Jiaotong University.

2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. That is,

Lemma. *For any fixed positive integers* m *and* n *, if* $(m-1)! < n \leq m!$*, then we have*

$$m = \frac{\ln n}{\ln \ln n} + O\left(\frac{(\ln n)(\ln \ln \ln n)}{(\ln \ln n)^2}\right).$$

Proof. From $(m-1)! < n \leq m!$ and taking the logistic computation in the two sides of the inequality, we get

$$\sum_{i=1}^{m-1} \ln i < \ln n \leq \sum_{i=1}^{m} \ln i. \tag{2}$$

Then using the Euler's summation formula we have

$$\sum_{i=1}^{m} \ln i = \int_1^m \ln t dt + \int_1^m (t - [t])(\ln t)' dt = m \ln m - m + O(\ln m) \tag{3}$$

and

$$\sum_{i=1}^{m-1} \ln i = \int_1^{m-1} \ln t dt + \int_1^{m-1} (t - [t])(\ln t)' dt = m \ln m - m + O(\ln m). \tag{4}$$

Combining (2),(3) and (4), we can easily deduce that

$$\ln n = m \ln m - m + O(\ln m). \tag{5}$$

So

$$m = \frac{\ln n}{\ln m - 1} + O(1). \tag{6}$$

Similarly, we continue taking the logistic computation in two sides of (6), then we also have

$$\ln m = \ln \ln n + O(\ln \ln m), \tag{7}$$

and

$$\ln \ln m = O(\ln \ln \ln n). \tag{8}$$

Hence, by (6), (7) and (8) we have

$$m = \frac{\ln n}{\ln \ln n} + O\left(\frac{(\ln n)(\ln \ln \ln n)}{(\ln \ln n)^2}\right).$$

This completes the proof of Lemma.

Now we use Lemma to complete the proof of Theorem. For any real number $x \geq 2$, by the definition of $S_1(n)$ and Lemma we have

$$\begin{aligned}\sum_{n\leq x} S_1(n) &= \sum_{\substack{n\leq x\\ (m-1)!<n\leq m!}} m \\ &= \sum_{n\leq x}\left(\frac{\ln n}{\ln\ln n} + O\left(\frac{(\ln n)(\ln\ln\ln n)}{(\ln\ln n)^2}\right)\right) \\ &= \sum_{n\leq x}\frac{\ln n}{\ln\ln n} + O\left(\frac{x(\ln x)(\ln\ln\ln x)}{(\ln\ln x)^2}\right). \qquad (8)\end{aligned}$$

By the Euler's summation formula, we deduce that

$$\begin{aligned}\sum_{n\leq x}\frac{\ln n}{\ln\ln n} &= \int_2^x \frac{\ln t}{\ln\ln t}dt + \int_2^x (t-[t])\left(\frac{\ln t}{\ln\ln t}\right)' dt + \frac{\ln x}{\ln\ln x}(x-[x]) \\ &= \frac{x\ln x}{\ln\ln x} + O\left(\frac{x}{\ln\ln x}\right). \qquad (9)\end{aligned}$$

So, from (8) and (9) we have

$$\sum_{n\leq x} S_1(n) = \frac{x\ln x}{\ln\ln x} + O\left(\frac{x(\ln x)(\ln\ln\ln x)}{(\ln\ln x)^2}\right).$$

This completes the proof of Theorem.

References

1. F.Smarandache, *Only Problems, not Solutions*, Xiquan Publ.House, Chicago, 1993.
2. Jozsef Sandor, *On an additive analogue of the function S*, Smaramche Notes Journal **13** (2002), 266-270.
3. Jozsef Sandor, *On an generalization of the Smarandache function*, Notes Numb.Th.Discr.Math. **5** (1999), 41-51.
4. Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.