An introduction to the Smarandache Square Complementary function

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Abstract

In this paper the main properties of Smarandache Square Complementary function has been analyzed. Several problems still unsolved are reported too.

The Smarandache square complementary function is defined as [4],[5]:

$$Ssc(n)=m$$

where m is the smallest value such that $m \cdot n$ is a perfect square.

Example: for n=8, m is equal 2 because this is the least value such that $m \cdot n$ is a perfect square.

The first 100 values of Ssc(n) function follows:

n	Ssc(n)	n	Ssc(n)	n	Ssc(n)	n	Ssc(n)
1	1	26	26	51	51	76	19
2	2	27	3	52	13	77	77
3	3	28	7	53	53	78	78
4	1	29	29	54	6	79	79
5	5	30	30	55	55	80	5
6	6	31	31	56	14	81	1
7	7	32	2	57	57	82	82
8	2	33	33	58	58	83	83
9	1	34	34	59	59	84	21
10	10	35	35	60	15	85	85
11	11	36	1	61	61	86	86
12	3	37	37	62	62	87	87
13	13	38	38	63	7	88	22
14	14	39	39	64	1	89	89
15	15	40	10	65	65	90	10
16	1	41	41	66	66	91	91
17	17	42	42	67	67	92	23
18	2	43	43	68	17	93	93
19	19	44	11	69	69	94	94
20	5	45	5	70	70	95	95
21	21	46	46	71	71	96	6
22	22	47	47	72	2	97	97
23	23	48	3	73	73	98	2
24	6	49	1	74	74	99	11
25	1	50	2	75	3	100	1

Let's start to explore some properties of this function.

Theorem 1: $Ssc(n^2)=1$ *where n=1,2,3,4...*

In fact if $k=n^2$ is a perfect square by definition the smallest integer m such that $m \cdot k$ is a perfect square is m=1.

Theorem 2: *Ssc(p)=p where p is any prime number*

In fact in this case the smallest m such that $m \cdot p$ is a perfect square can be only m=p.

Theorem 3: $Ssc(p^n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ p & \text{if } n \text{ is odd} \end{cases}$ *where p is any prime number.*

First of all let's analyze the even case. We can write:

$p^n = p^2 \cdot p^2 \cdot .... \cdot p^2 = \left| p^{\frac{n}{2}} \right|^2$ and then the smallest m such that $p^n \cdot m$ is a perfect square is 1.

Let's suppose now that n is odd. We can write:

$$p^n = p^2 \cdot p^2 \cdot .... \cdot p^2 \cdot p = \left| p^{\left\lfloor \frac{n}{2} \right\rfloor} \right|^2 \cdot p = p^{2 \cdot \left\lfloor \frac{n}{2} \right\rfloor} \cdot p$$

and then the smallest integer m such that $p^n \cdot m$ is a perfect square is given by m=p.

Theorem 4: $Ssc(p^a \cdot q^b \cdot s^c \cdot .... \cdot t^x) = p^{odd(a)} \cdot q^{odd(b)} \cdot s^{odd(c)} \cdot .... \cdot t^{odd(x)}$ *where p ,q, s, …,, t are distinct primes and the odd function is defined as:*

$$odd(n) = \begin{cases} 1 & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

Direct consequence of theorem 3.

Theorem 5: *The Ssc(n) function is multiplicative, i.e. if (n,m)=1 then* $Ssc(n \cdot m) = Ssc(n) \cdot Ssc(m)$

Without loss of generality let's suppose that $n = p^a \cdot q^b$ and $m = s^c \cdot t^d$ where p, q, s, t are distinct primes. Then:

$$Ssc(n \cdot m) = Ssc(p^a \cdot q^b \cdot s^c \cdot t^d) = p^{odd(a)} \cdot q^{odd(b)} \cdot s^{odd(c)} \cdot t^{odd(d)}$$

according to the theorem 4.

On the contrary:

$$Ssc(n) = Ssc(p^a \cdot q^b) = p^{odd(a)} \cdot q^{odd(b)}$$

$$Ssc(m) = Ssc(s^c \cdot t^d) = s^{odd(c)} \cdot t^{odd(d)}$$

This implies that: $Ssc(n \cdot m) = Ssc(n) \cdot Ssc(m)$ qed

Theorem 6: *If* $n = p^a \cdot q^b \cdot .... \cdot p^s$ *then* $Ssc(n) = Ssc(p^a) \cdot Ssc(p^b) \cdot ...... \cdot Ssc(p^s)$ *where p is any prime number.*

According to the theorem 4:

$$Ssc(n) = p^{odd(a)} \cdot p^{odd(b)} \cdot ...... \cdot p^{odd(s)}$$

and:

$$Ssc(p^a) = p^{odd(a)}$$
$$Ssc(p^b) = p^{odd(b)}$$

and so on. Then:

$$Ssc(n) = Ssc(p^a) \cdot Ssc(p^b) \cdot ...... \cdot Ssc(p^s) \qquad \text{qed}$$

Theorem 7: *Ssc(n)=n if n is squarefree, that is if the prime factors of n are all distinct. All prime numbers, of course are trivially squarefree [3].*

Without loss of generality let's suppose that $n = p \cdot q$ where p and q are two distinct primes. According to the theorems 5 and 3:

$$Ssc(n) = Ssc(p \cdot q) = Ssc(p) \cdot Ssc(q) = p \cdot q = n \quad \text{qed}$$

Theorem 8: *The Ssc(n) function is not additive.*:

In fact for example: Ssc(3+4)=Ssc(7)=7<>Ssc(3)+Ssc(4)=3+1=4

Anyway we can find numbers m and n such that the function Ssc(n) is additive. In fact if:

m and n are squarefree
k=m+n is squarefree.

then Ssc(n) is additive.
In fact in this case Ssc(m+n)=Ssc(k)=k=m+n and Ssc(m)=m Ssc(n)=n according to theorem 7.

Theorem 9: $\sum_{n=1}^{\infty} \frac{1}{Ssc(n)}$ *diverges*

In fact:

$$\sum_{n=1}^{\infty} \frac{1}{Ssc(n)} > \sum_{p=2}^{\infty} \frac{1}{Ssc(p)} = \sum_{p=2}^{\infty} \frac{1}{p} \quad \text{where p is any prime number.}$$

So the sum of inverse of Ssc(n) function diverges due to the well known divergence of series [3]:

$$\sum_{p=2}^{\infty} \frac{1}{p}$$

Theorem 10: *Ssc(n)>0 where n=1,2,3,4 ...*

This theorem is a direct consequence of Ssc(n) function definition. In fact for any n the smallest m such that $m \cdot n$ is a perfect square cannot be equal to zero otherwise $m \cdot n$=0 and zero is not a perfect square.

Theorem 11: $\sum_{n=1}^{\infty} \frac{Ssc(n)}{n}$ *diverges*

In fact being $Ssc(n) \geq 1$ this implies that:

$$\sum_{n=1}^{\infty} \frac{Ssc(n)}{n} > \sum_{n=1}^{\infty} \frac{1}{n}$$

and as known the sum of reciprocal of integers diverges. [3]

Theorem 12: $Ssc(n) \leq n$

Direct consequence of theorem 4.

Theorem 13: *The range of Ssc(n) function is the set of squarefree numbers.*

According to the theorem 4 for any integer n the function Ssc(n) generates a squarefree number.

Theorem 14: $0 < \frac{Ssc(n)}{n} \leq 1$ *for n>=1*

Direct consequence of theorems 12 and 10.

Theorem 15: $\frac{Ssc(n)}{n}$ *is not distributed uniformly in the interval]0,1]*

If n is squarefree then Ssc(n)=n that implies $\frac{Ssc(n)}{n} = 1$

If n is not squarefree let's suppose without loss of generality that $n = p^a \cdot q^b$ where p and q are primes.

Then:

$$\frac{Ssc(n)}{n} = \frac{Ssc(p^a) \cdot Ssc(p^b)}{p^a \cdot q^b}$$

We can have 4 different cases.

1) a even and b even

$$\frac{Ssc(n)}{n} = \frac{Ssc(p^a)\cdot Ssc(p^b)}{p^a \cdot q^b} = \frac{1}{p^a \cdot q^b} \leq \frac{1}{4}$$

2) a odd and b odd

$$\frac{Ssc(n)}{n} = \frac{Ssc(p^a)\cdot Ssc(p^b)}{p^a \cdot q^b} = \frac{p \cdot q}{p^a \cdot q^b} = \frac{1}{p^{a-1} \cdot q^{b-1}} \leq \frac{1}{4}$$

3) a odd and b even

$$\frac{Ssc(n)}{n} = \frac{Ssc(p^a)\cdot Ssc(p^b)}{p^a \cdot q^b} = \frac{p \cdot 1}{p^a \cdot q^b} = \frac{1}{p^{a-1} \cdot q^b} \leq \frac{1}{4}$$

4) a even and b odd

Analogously to the case 3 .

This prove the theorem because we don't have any point of Ssc(n) function in the interval]1/4,1[

Theorem 16: *For any arbitrary real number* $e > 0$, *there is some number n>=1 such that:*

$$\frac{Ssc(n)}{n} < e$$

Without loss of generality let's suppose that $q = p_1 \cdot p_2$ where p_1 and p_2 are primes such that $\frac{1}{q} < e$ and e is any real number grater than zero. Now take a number n such that:

$$n = p_1^{a_1} \cdot p_2^{a_2}$$

For a_1 and a_2 odd:

$$\frac{Ssc(n)}{n} = \frac{p_1 \cdot p_2}{p_1^{a_1} \cdot p_2^{a_2}} = \frac{1}{p_1^{a_1-1} \cdot p_2^{a_2-1}} < \frac{1}{p_1 \cdot p_2} < \boldsymbol{e}$$

For a_1 and a_2 even:

$$\frac{Ssc(n)}{n} = \frac{1}{p_1^{a_1} \cdot p_2^{a_2}} < \frac{1}{p_1 \cdot p_2} < \boldsymbol{e}$$

For a_1 odd and a_2 even (or viceversa):

$$\frac{Ssc(n)}{n} = \frac{p_1}{p_1^{a_1} \cdot p_2^{a_2}} = \frac{1}{p_1^{a_1-1} \cdot p_2^{a_2}} < \frac{1}{p_1 \cdot p_2} < \boldsymbol{e}$$

Theorem 17: $Ssc(p_k\#) = p_k\#$ where $p_k\#$ is the product of first k primes (primordial) [3].

The theorem is a direct consequence of theorem 7 being $p_k\#$ a squarefree number.

Theorem 18: *The equation* $\dfrac{Ssc(n)}{n} = 1$ *has an infinite number of solutions.*

The theorem is a direct consequence of theorem 2 and the well-known fact that there is an infinite number of prime numbers [6]

Theorem 19: *The repeated iteration of the Ssc(n) function will terminate always in a fixed point (see [3] for definition of a fixed point).*

According to the theorem 13 the application of Scc function to any n will produce always a squarefree number and according to the theorem 7 the repeated application of Ssc to this squarefree number will produce always the same number.

Theorem 20: *The diophantine equation Ssc(n)=Ssc(n+1) has no solutions.*

We must distinguish three cases:

1) n and n+1 squarefree
2) n and n+1 not squareefree
3) n squarefree and n+1 no squarefree and viceversa

Case 1. According to the theorem 7 Ssc(n)=n and Ssc(n+1)=n+1 that implies that Ssc(n)<>Ssc(n+1)

Case 2. Without loss of generality let's suppose that:

$$n = p^a \cdot q^b$$
$$n+1 = p^a \cdot q^b + 1 = s^c \cdot t^d$$

where p,q,s and t are distinct primes.

According to the theorem 4:

$$Ssc(n) = Ssc(p^a \cdot q^b) = p^{odd(a)} \cdot q^{odd(b)}$$
$$Ssc(n+1) = Ssc(s^c \cdot t^d) = s^{odd(c)} \cdot t^{odd(d)}$$

and then Ssc(n)<>Ssc(n+1)

Case 3. Without loss of generality let's suppose that $n = p \cdot q$. Then:

$$Ssc(n) = Ssc(p \cdot q) = p \cdot q$$
$$Ssc(n+1) = Ssc(p \cdot q + 1) = Ssc(s^a \cdot t^b) = s^{odd(a)} \cdot t^{odd(b)}$$

supposing that $n+1 = p \cdot q + 1 = s^a \cdot t^b$

This prove completely the theorem.

Theorem 21: $\sum_{k=1}^{N} Ssc(k) > \frac{6 \cdot N}{\pi^2}$ *for any positive integer N.*

The theorem is very easy to prove. In fact the sum of first N values of Ssc function can be separated into two parts:

$$\sum_{k_1=1}^{N} Ssc(k_1) + \sum_{k_2=1}^{N} Ssc(k_2)$$

where the first sum extend over all k_1 squarefree numbers and the second one over all k_2 not squarefree numbers.
According to the Hardy and Wright result [3], the asymptotic number Q(n) of squarefree numbers $\leq N$ is given by:

$$Q(N) \approx \frac{6 \cdot N}{\pi^2}$$

and then:

$$\sum_{k=1}^{N} Ssc(k) = \sum_{k_1=1}^{N} Ssc(k_1) + \sum_{k_2=1}^{N} Ssc(k_2) > \frac{6 \cdot N}{\pi^2}$$

because according to the theorem 7, $Ssc(k_1) = k_1$ and the sum of first N squarefree numbers is always greater or equal to the number Q(N) of squarefree numbers $\leq N$, namely:

$$\sum_{k_1=1}^{N} k_1 \geq Q(N)$$

Theorem 22: $\sum_{k=1}^{N} Ssc(k) > \frac{N^2}{2 \cdot \ln(N)}$ *for any positive integer* N.

In fact:

$$\sum_{k=1}^{N} Ssc(k) = \sum_{k'=1}^{N} Ssc(k') + \sum_{p=2}^{N} Ssc(p) > \sum_{p=2}^{N} Ssc(p)$$

because by theorem 2, Ssc(p)=p. But according to the result of Bach and Shallit [3], the sum of first N primes is asymptotically equal to:

$$\frac{N^2}{2 \cdot \ln(N)}$$

and this completes the proof.

Theorem 23: *The diophantine equations* $\frac{Ssc(n+1)}{Ssc(n)} = k$ *and* $\frac{Ssc(n)}{Ssc(n+1)} = k$ *where k is any integer number have an infinite number of solutions.*

Let's suppose that n is a perfect square. In this case according to the theorem 1 we have:

$$\frac{Ssc(n+1)}{Ssc(n)} = Ssc(n+1) = k$$

On the contrary if n+1 is a perfect square then:

$$\frac{Ssc(n)}{Ssc(n+1)} = Ssc(n) = k$$

Problems.

1) Is the difference |Ssc(n+1)-Ssc(n)| bounded or unbounded?

2) Is the Ssc(n) function a Lipschitz function ?
A function is said a Lipschitz function [3] if:

$$\frac{|Ssc(m) - Ssc(k)|}{|m-k|} \geq M$$ where M is any integer

3) Study the function FSsc(n)=m. Here m is the number of different integers k such that Ssc(k)=n.

4) Solve the equations Ssc(n)=Ssc(n+1)+Ssc(n+2) and Ssc(n)+Ssc(n+1)=Ssc(n+2). Is the number of solutions finite or infinite?

5) Find all the values of n such that $Ssc(n) = Ssc(n+1) \cdot Ssc(n+2)$

6) Solve the equation $Ssc(n) \cdot Ssc(n+1) = Ssc(n+2)$

7) Solve the equation $Ssc(n) \cdot Ssc(n+1) = Ssc(n+2) \cdot Ssc(n+3)$

8) Find all the values of n such that $S(n)^k + Z(n)^k = Ssc(n)^k$ where S(n) is the Smarandache function [1], Z(n) the Pseudo-Smarandache function [2] and k any integer.

9) Find the smallest k such that between Ssc(n) and Ssc(k+n), for n>1, there is at least a prime.

10) Find all the values of n such that Ssc(Z(n))-Z(Ssc(n))=0 where Z is the Pseudo Smarandache function [2].

11) Study the functions Ssc(Z(n)), Z(Ssc(n)) and Ssc(Z(n))-Z(Ssc(n)).

12) Evaluate $\lim_{k \to \infty} \frac{Ssc(k)}{q(k)}$ where $q(k) = \sum_{n \le k} \ln(Ssc(n))$

13) Are there m, n, k non-null positive integers for which $Ssc(m \cdot n) = m^k \cdot Ssc(n)$?

14) Study the convergence of the Smarandache Square complementary harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{Ssc^a(n)}$$

where a>0 and belongs to R

15) Study the convergence of the series:

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{Ssc(x_n)}$$

where x_n is any increasing sequence such that $\lim_{n\to\infty} x_n = \infty$

16) Evaluate:

$$\lim_{n\to\infty} \frac{\sum_{k=2}^{n} \frac{\ln(Ssc(k))}{\ln(k)}}{n}$$

Is this limit convergent to some known mathematical constant?

17) Solve the functional equation:

$$Ssc(n)^r + Ssc(n)^{r-1} + \dots\dots + Ssc(n) = n$$

where r is an integer ≥ 2.

18) What about the functional equation:

$$Ssc(n)^r + Ssc(n)^{r-1} + \dots\dots + Ssc(n) = k \cdot n$$

where r and k are two integers ≥ 2.

19) Evaluate $\sum_{k=1}^{\infty} (-1)^k \cdot \frac{1}{Ssc(k)}$

20) Evaluate $\frac{\sum_n Ssc(n)^2}{\left|\sum_n Ssc(n)\right|^2}$

21) Evaluate:

$$\lim_{n\to\infty}\left|\sum_n \frac{1}{Ssc(f(n))} - \sum_n \frac{1}{f(Ssc(n))}\right|$$

for f(n) equal to the Smarandache function S(n) [1] and to the Pseudo-Smarandache function Z(n) [2].

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