The basis of quantum mechanics' compatibility with relativity—whose impairment gives rise to the Klein-Gordon and Dirac equations

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Abstract

Solitary-particle quantum mechanics' inherent compatibility with special relativity is implicit in Schrödinger's postulated wave-function rule for the operator quantization of the particle's canonical three-momentum, taken together with his famed time-dependent wave-function equation that analogously treats the operator quantization of its Hamiltonian. The resulting formally four-vector equation system assures proper relativistic covariance for any solitary-particle Hamiltonian operator which, together with its canonical three-momentum operator, is a Lorentz-covariant four-vector operator. This, of course, is *always* the case for the quantization of the Hamiltonian of a properly relativistic *classical* theory, so the strong correspondence principle *definitely remains valid* in the relativistic domain. Klein-Gordon theory *impairs* this four-vector equation by *iterating* and contracting it, thereby injecting *extraneous* negative-energy solutions that are *not orthogonal* to their positive-energy counterparts of the same momentum, thus destroying the basis of the quantum probability interpretation. Klein-Gordon theory, which thus depends on the *square* of the Hamiltonian operator, is as well thereby cut adrift from Heisenberg's equations of motion. Dirac theory *confuses* the space-time symmetry of the *four-vector equation system* with such symmetry for its *time component alone*, which it fatuously *imposes*, thereby *breaching* the strong correspondence principle for the free particle and imposing the starkly unphysical *momentum-independence of velocity*. Physically sensible alternatives, *with* external electromagnetic fields, to the Klein-Gordon and Dirac equations are derived, and the simple, elegant symmetry-based approach to antiparticles is pointed out.

Introduction: quantum mechanics' inherent compatibility with relativity

The inherent compatibility of solitary-particle quantum mechanics with special relativity is a straightforward consequence Schrödinger's two basic postulates for the wave function [1, 2], i.e., for the quantum state vector in the Schrödinger picture in configuration representation, namely $\langle \mathbf{r}|\psi(t)\rangle$. The first Schrödinger wave-function postulate is his rule for the operator quantization of the particle's *canonical three-momentum*,

$$-i\hbar\nabla_{\mathbf{r}}(\langle\mathbf{r}|\psi(t)\rangle) = \langle\mathbf{r}|\widehat{\mathbf{p}}|\psi(t)\rangle, \tag{1a}$$

which is as well, of course, a result of Dirac's postulated canonical commutation relation [3]. The second Schrödinger wave-function postulate is his famed *time-dependent wave equation* [1, 3, 2],

$$i\hbar\partial(\langle\mathbf{r}|\psi(t)\rangle)/\partial t = \langle\mathbf{r}|\widehat{H}|\psi(t)\rangle, \tag{1b}$$

which formally treats the operator quantization of the particle's *Hamiltonian* in a manner analogous to the way Eq. (1a) treats the operator quantization of the particle's canonical three-momentum. The straightforward theoretical physics *implication* of Eqs. (1a) and (1b) is simply that the operators $\widehat{\mathbf{p}}$ and $\widehat{H}$ are the *generators* of the wave function's *infinitesimal space and time translations*, respectively. Therefore, in anticipation of the *restriction on such generators which special relativity imposes*, these two equations are usefully *combined* into the *single* formally *four-vector* Schrödinger equation for the wave function,

$$i\hbar\partial(\langle\mathbf{r}|\psi(t)\rangle)/\partial x_\mu = \langle\mathbf{r}|\widehat{p^\mu}|\psi(t)\rangle, \tag{1c}$$

where the *contravariant four-vector space-time partial derivative operator* $\partial/\partial x_\mu$ is defined as $\partial/\partial x_\mu \stackrel{\text{def}}{=} (c^{-1}\partial/\partial t, -\nabla_{\mathbf{r}})$, and the *formal* "contravariant four-vector" energy-momentum operator $\widehat{p^\mu}$ is defined as $\widehat{p^\mu} \stackrel{\text{def}}{=} (\widehat{H}/c, \widehat{\mathbf{p}})$. Since special relativity *requires* the contravariant space-time partial derivative four-vector operator $\partial/\partial x_\mu$ to transform between inertial frames in *Lorentz-covariant* fashion, it is apparent from Eq. (1c) that the *Hamiltonian operator* $\widehat{H}$ will be *compatible with special relativity* if it is related to the canonical three-momentum operator $\widehat{\mathbf{p}}$ in such a way that *also* makes the energy-momentum operator $\widehat{p^\mu}$ a contravariant four-vector which transforms between inertial frames in *Lorentz-covariant* fashion. This property of the Hamiltonian operator will, of course, be satisfied *automatically* if it is *the quantization of the Hamiltonian of a properly relativistic classical theory. Therefore the strong correspondence principle definitely remains valid in the relativistic domain.*

Now for a completely *free* solitary particle of nonzero mass m, the logic of the Lorentz transformation from its *rest frame*, where it has four-momentum $(mc, \mathbf{0})$, to a frame where it has velocity $\mathbf{v}$ (where $|\mathbf{v}| < c$) leaves *no freedom at all in the choice of its classical Hamiltonian.* That Lorentz boost takes this particle's four-momentum to,

$$(mc(1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}},\, m\mathbf{v}(1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}}) = (E(\mathbf{v})/c,\, \mathbf{p}(\mathbf{v})), \tag{2a}$$

which, together with the *identity*,

$$mc^2(1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}} = \sqrt{m^2c^4 + |cm\mathbf{v}|^2(1 - |\mathbf{v}|^2/c^2)^{-1}}, \tag{2b}$$

implies that,

$$E(\mathbf{v}) = \sqrt{m^2c^4 + |c\mathbf{p}(\mathbf{v})|^2} = H_{\text{free}}(\mathbf{p}(\mathbf{v})). \tag{2c}$$

Therefore, for the completely *free* relativistic solitary particle of nonzero mass m, theoretically systematic, conservative adherence to the *strong correspondence principle* flatly determines the relativistic Hamiltonian operator to be the square-root operator,

$$\widehat{H}_{\text{free}} = \sqrt{m^2c^4 + |c\widehat{\mathbf{p}}|^2}. \tag{3}$$

This conclusion *even* extends to the *free* spin-$\frac{1}{2}$ particle of nonzero mass: notwithstanding that spin-$\frac{1}{2}$ itself is a nonclassical attribute, the nonrelativistic Pauli Hamiltonian operator for such a particle *automatically reduces* to the *usual* nonrelativistic *purely kinetic-energy Hamiltonian* operator in the *free-particle* limit, and one can *always* find an inertial frame of reference in which a *free particle of nonzero mass* is *completely nonrelativistic*, i.e., for the completely *free* particle one can *always* find an inertial frame of reference in which the nonzero-mass relativistic free-particle Hamiltonian operator $\sqrt{m^2c^4 + |c\widehat{\mathbf{p}}|^2}$ *arbitrarily* well approximates $mc^2 + |\widehat{\mathbf{p}}|^2/(2m)$, which is the *Pauli* free-particle Hamiltonian operator, offset by the merely *constant* rest-mass energy term mc^2.

Since, as we shall shortly see, the Dirac free-particle Hamiltonian operator *is very much at odds with the relativistic free-particle square-root Hamiltonian operator of* Eq. (3) [2], *even notwithstanding the complete compatibility of* Eq. (3) with the free-particle Pauli theory, it will be imperative to understand exactly *how and why* the Dirac Hamiltonian operator comes to be *in conflict with the fundamental requirement of relativistic quantum mechanics*, namely that $\widehat{p^\mu} = (\widehat{H}/c, \widehat{\mathbf{p}})$ *must transform between inertial frames as a Lorentz-covariant four-vector.* First, however, we turn to analysis of the Klein-Gordon theory, which *rejects the fundamental quantum mechanical* Eq. (1c) *in that precise form*, instead *substituting in its place* Eq. (1a) together with a once-iterated and Lorentz-contracted version of Eq. (1c).

Klein-Gordon theory's impairment of quantum mechanics

Very strongly motivated by considerations of perceived calculational ease, which we briefly discuss in the next section, *rather than by those of quantum mechanics*, Klein, Gordon and Schrödinger *rejected* the *natural* time-dependent Schrödinger equation of Eq. (1b) in favor of a *once-iterated* and then Lorentz-contracted Lorentz-scalar version of Eq. (1c) [2, 4, 5], which yields,

$$\partial^2(\langle\mathbf{r}|\psi(t)\rangle)/(\partial x^\mu \partial x_\mu) + (\langle\mathbf{r}|\widehat{p_\mu}\,\widehat{p^\mu}|\psi(t)\rangle)/\hbar^2 = 0, \tag{4a}$$

where, of course,

$$\widehat{p_\mu}\,\widehat{p^\mu} = (\widehat{H}^2 - |c\widehat{\mathbf{p}}|^2)/c^2, \tag{4b}$$

so that *in the special case of the free particle*, where $\widehat{H} = \widehat{H}_{\text{free}}$, which is given by Eq. (3), it is readily seen that the *general* Klein-Gordon equation of Eq. (4a) above *reduces* to,

$$(\partial^2/(\partial x^\mu \partial x_\mu) + (mc/\hbar)^2)\langle\mathbf{r}|\psi(t)\rangle = 0. \tag{5}$$

To each stationary eigensolution $e^{-i\sqrt{m^2c^4+|c\mathbf{p}|^2}\;t/\hbar}\langle\mathbf{r}|\mathbf{p}\rangle$ of eigenmomentum $\mathbf{p}$ of the natural time-dependent relativistic *free* solitary-particle Schrödinger equation, which is Eq. (1b) for the case that the Hamiltonian operator $\widehat{H}$ is equal to $\widehat{H}_{\text{free}}$, Eq. (5) *adds* an *extraneous negative-energy partner solution* $e^{+i\sqrt{m^2c^4+|c\mathbf{p}|^2}\;t/\hbar}\langle\mathbf{r}|\mathbf{p}\rangle$ of the *same* momentum, whose *sole reason for existing* is the *entirely gratuitous iteration* of Eq. (1c)! These *completely* extraneous *negative* "free solitary-particle" energies, $-\sqrt{m^2c^4 + |c\mathbf{p}|^2}$, do *not* correspond to *anything* that exists in the *classical* dynamics of a free relativistic solitary particle, and by their negatively *unbounded* character threaten to spawn unstable runaway phenomena should the *free* Klein-Gordon equation be sufficiently perturbed (the Klein paradox) [2]. Due to the fact that the Klein-Gordon equation *lacks* a corresponding Hamiltonian—it depends on only the *square* of a Hamiltonian, as is seen from Eq. (4b)—it turns out, as is easily verified, that the *two* solutions of the *same momentum* $\mathbf{p}$ which have *opposite-sign* energies, i.e., $\pm\sqrt{m^2c^4 + |c\mathbf{p}|^2}$, *fail to be orthogonal to each other*, which *outright violates a key property* of orthodox quantum mechanics! *Without this property* the probablity interpretation of quantum mechanics *cannot be sustained*, and the Klein-Gordon equation is unsurprisingly *diseased* in that regard, yielding, inter alia, *negative probabilities* [2].

Furthermore, Klein-Gordon theory, which depends on the *square* of a Hamiltonian operator (see Eq. (4b)) rather than on the Hamiltonian operator *itself*, is thereby *cut adrift* from the normal quantum mechanical relationship to the Heisenberg picture, Heisenberg's equations of motion and the Ehrenfest theorem. In a nutshell, *substitution* of the *gratuitous iteration* of Eq. (1c) for the Eq. (1b) *part* of Eq. (1c), which is *exactly* what Klein-Gordon theory *does*, grievously *impairs and degrades* the *unexceptionable quantum-mechanical nature* of Eq. (1c) *itself*! The need to carefully respect *all* the components of Eq. (1c) rather than to opportunistically try to "bend" aspects of its time component (i.e., Eq. (1b)) "at the edges" unfortunately *did not register at all* with Dirac ahead of his attempt to "repair" the problems of the Klein-Gordon theory.

Dirac theory's fatuous *imposition* of "space-time coordinate symmetry"

The probability disease of the Klein-Gordon theory prompted Dirac to sensibly *reinstate* Eq. (1b), the time-dependent Schrödinger equation. Very unfortunately, being motivated *above all else* by the *same non-physics considerations of perceived calculational ease* as Klein, Gordon and Schrödinger [2, 5], Dirac hit upon a completely misguided but nonetheless *plausible-sounding* "reason", ostensibly emanating from the "relativistic need" for Eq. (1b) to *by itself* exhibit "space-time coordinate symmetry" [6, 7, 2], to *linearize* the square-root

Hamiltonian operator $\widehat{H}_{\text{free}}$ of Eq. (3) for the nonzero-mass free particle [8, 6, 5], *notwithstanding* the fact that Eqs. (2) *effectively produce the specifically square-root form of* H_{free} from *nothing more* than the nonzero-mass free particle's Lorentz transformation properties!

Unfortunately *not being aware of the specifically four-vector form of* Eq. (1c), Dirac focused *myopically on only* Eq. (1b), the time-dependent Schrödinger equation, whose *theoretical physics essence* is, of course, that the Hamiltonian operator is the *generator* of infinitesimal *time translations* of the wave function. This theoretical-physics *core essence* of Eq. (1b) is, in isolation, *supremely indifferent* to the typical space-time coordinate symmetry that is such an ubiquitous characteristic of special relativity. It is *only* on contemplating the *whole* of the four-vector equation *system* of Eq. (1c), *which Dirac was not aware enough to do*, that its *global* symmetry between space and time coordinates snaps *immediately* into focus. Very unfortunately *deprived* of this revelation, and *intently* focused on *just* Eq. (1b), Dirac "concluded" that it must *of itelf* be *compelled* to manifest the "missing" space-time coordinate symmetry! If Dirac had even been *fully cognizant* of the *core theoretical physics role* of Eq. (1b), i.e., that of putting the Hamiltonian operator in the driver's seat of infinitesimal *time translations* and *only time translations*, of the wave function, he definitely would have been *very much less certain* that *forcing* Eq. (1b) to manifest space-time coordinate symmetry was *at all* a sensible idea! Such salutary doubts and second thoughts were, however, *unfortunately never to occur to Dirac.*

So because Eq. (1b) is *linear* in the time partial derivative operator, Dirac shallowly concluded that "space-time coordinate symmetry" *compels* it's relativistic version *to also be linear in the space gradient*, which implies that the Hamiltonian operator on its right-hand side is *likewise* linear in the particle three-momentum. Had he been thinking *at all* about the *physical implications* of this, Dirac would have *quickly* noticed a *plethora of red flags* inherent in *any such linearity.* In the nonrelativistic limit of *small* particle three-momentum, it is obvious that the Hamiltonian *must* become a *quadratic* function of the particle three-momentum, which is a *mathematical impossibility* for a Hamiltonian *linear* in the three-momentum! Note, however, that the relativistic *square-root* Hamiltonian $\widehat{H}_{\text{free}}$ of Eq. (3), from whose clutches Dirac, like Klein, Gordon, Schrödinger before him, was very strongly resolved, for non-physics reasons of calculational ease, to *escape*, in fact *passes* this *small* three-momentum *quadratic-dependence* test with flying colors! Furthermore, since the square-root Hamiltonian operator's *classical precursor* H_{free} is established beyond *any shadow of a doubt for a nonzero-mass free particle* (e.g., see Eqs. (2)), Dirac was effectively boxed into the *very* fraught position of *implicitly disowning the strong correspondence principle* in order to *cling* to his *linearized* Hamiltonian! Furthermore, it is an elementary observation from the Heisenberg equation of motion that *any* Hamiltonian which is *linear* in the particle three-momentum operator produces a particle *velocity* operator (and thus also a particle speed operator) which is *completely independent* of the particle three-momentum operator! This obviously *cannot make sense* in the free-particle *nonrelativistic limit*, where it is clear that the particle velocity operator *must* be *proportional* to the particle momentum operator, i.e., $d\widehat{\mathbf{r}}/dt = \widehat{\mathbf{p}}/m$. That free-particle velocity should be *independent* of free-particle momentum is in fact totally devoid of *physical* sense in *any* regime pertaining to the *relativistic* free particle: it is *entirely clear* that relativistic free-particle velocity and momentum are *always directionally aligned* (this *even* is true for the ultra-relativistic massless photon).

In utter contrast to this miasma of impending *physically senseless* "predictions" *irrespective* of *what* the *coefficients* of the three-momentum components in the *linearized* Dirac Hamiltonian are ultimately chosen to be, the Heisenberg equation of motion in *conjunction* with the free-particle relativistic *square-root* $\widehat{H}_{\text{free}}$ Hamiltonian operator of Eq. (3) yields *only physically impeccable* relativistic free-particle results, *including*, in particular, the result for the *relativistic velocity operator*! We shall, of course, be adhering to Dirac's choice of coefficients for the components of the three-momentum in his linearized Hamiltonian, but that particular choice *doesn't* alleviate *any* of the unphysical results just pointed out; in fact, it turns out to *sharpen* them, with the momentum-independent particle *speed* coming out to be a *universal c-number* whose value is more than 70% *greater* than that of light! As if this were not enough, the Dirac choice of coefficients (which turns out to involve mutually anticommuting matrices whose squares are unity) results in a *staggering violation of Newton's first law of motion*, with the free electron predicted to have a spontaneous "Compton acceleration" whose *minimum* magnitude, of order $m_e c^3/\hbar$, is around $10^{28}g$, where g is the acceleration of gravity at the earth surface! Almost needless to say, the relativistic free-particle *square-root* Hamiltonian operator $\widehat{H}_{\text{free}}$ of Eq. (3), in conjunction with Heisenberg's equation of motion yields, in *flawless* contrast, *utterly strict adherence* to Newton's first law of motion!

In fact, the only obvious "success" that the Dirac theory's *completely misdirected* imposition of space-time coordinate symmetry on Eq. (1b) can claim is the *non-physics* one of *calculational ease*, which, as we have pointed out, was its *overriding motivation* from the *very start.* In configuration representation, the *physically sensible* relativistic free-particle *square root* Hamiltonian operator $\widehat{H}_{\text{free}}$ of Eq. (3) is a *non-local integral* *operator*, which certainly quashes any hoped-for "separation of variables for partial differential equations" technology for solving the *relativistic* version of the hydrogen atom. Contemplation of this fact of physics seems to have thoroughly rattled Klein, Gordon, Schrödinger and, in due course, Dirac. Quite *desperately* wanting that standard partial differential equation technology to *still be applicable* in the relativistic domain, they first invented the quantum-mechanically deficient Klein-Gordon theory and then Dirac's astonishingly unphysical linearized Hamiltonian. Schrödinger, the *inventor* of bound-state perturbation theory, *ought to have realized* that his perturbation method is *very well suited* to the patently *small* relativistic corrections to the *nonrelativistic model* of the hydrogen atom, and called off the jointly shared slighting of theoretical-physics best practice which was tacit in the above-noted quite desperate efforts to *preserve the applicability* of standard partial differential equation solution technologies in the relativistic domain. To be sure, the very *calculational ease* that is the key aspect of Dirac's unphysical linearized Hamiltonian *also* makes it a relatively easy target to pick apart in detail!

We turn now to Dirac's choice of coefficients for his linearized relativistic mass-m free-particle Hamiltonian operator $\widehat{H}_D$ and the consequences of that choice. Having *rejected* the strong correspondence principle in favor of imposing "space-time coordinate symmetry" on Eq. (1b) and thus *linearity* in the components of the three-momentum operator on $\widehat{H}_D$, Dirac adopted a *severely weakened correspondence principle* for $\widehat{H}_D$ in relation to the relativistic free-particle square-root Hamiltonian operator $\widehat{H}_{\text{free}}$ of Eq. (3), namely,

$$(\widehat{H}_D)^2 = (\widehat{H}_{\text{free}})^2 = m^2c^4 + c^2|\widehat{\mathbf{p}}|^2, \tag{6a}$$

which ensures that any solutions of the time-dependent Schrödinger equation, i.e., Eq. (1b), with Hamiltonian operator $\widehat{H}$ equal to Dirac's relativistic free-particle $\widehat{H}_D$ will *also* be solutions of the free-particle Klein-Gordon equation, namely of Eq. (5). Now expressing the linearized $\widehat{H}_D$ in terms of four dimensionless Hermitian matrix coefficients $(\beta, \vec{\alpha})$, where $\vec{\alpha} \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, \alpha_3)$, in the form,

$$\widehat{H}_D = c\vec{\alpha}\cdot\widehat{\mathbf{p}} + \beta mc^2, \tag{6b}$$

it is seen that the weak correspondence principle of Eq. (6a) implies that β and all the components of $\vec{\alpha}$ mutually *anticommute* and have squares equal to unity. Their anticommutation relations imply that they are traceless [2], and therefore $\widehat{H}_D$ is traceless as well, and thus has a *negative* energy eigenvalue to match every positive one. So the eigenenergies of Dirac's linearized $\widehat{H}_D$ turn out to *include* all the extraneous *negative* energies of the Klein-Gordon equation. While the negative-energy eigenstates of $\widehat{H}_D$ *are* properly *orthogonal* to their positive-energy counterparts, the *other* inherent issues which the presence of these negative-energy solutions raise in the context of a free solitary particle, such as total lack of classical correspondence and the Klein paradox remain unresolved [4]. Upon applying the Heisenberg equations of motion to the free-particle Dirac Hamiltonian operator $\widehat{H}_D$, as given by Eq. (6b), we obtain the velocity operator,

$$\widehat{\mathbf{v}} = d\widehat{\mathbf{r}}/dt = c\vec{\alpha}, \tag{7a}$$

which we see, as pointed out above, is *completely independent* of the three-momentum operator $\widehat{\mathbf{p}}$. Since the three Hermitian matrix components of $\vec{\alpha}$ all have squares equal to unity, we obtain for the free Dirac particle speed operator the *universal* c-number value,

$$|\widehat{\mathbf{v}}| = c\sqrt{3}, \tag{7b}$$

which *exceeds* c, the speed of light, by 73%! Also, notwithstanding its nonzero mass m, the free Dirac particle *has no inertial rest frame*! These results *show conclusively* that the *imposition* on Eq. (1b) of "space-time coordinate symmetry" with its resulting *linearized* Hamiltonian operator, that is *crystallized* as the free-particle Dirac Hamiltonian operator $\widehat{H}_D$ of Eq. (6b) upon imposition of the severely weakened correspondence principle of Eq. (6a), *violates* special relativity!

To better grasp this point, let us note that if we use the relativistic square-root Hamiltonian operator $\widehat{H}_{\text{free}}$ *instead* of $\widehat{H}_D$ in the Heisenberg equations of motion, we obtain,

$$\widehat{\mathbf{v}} = c^2\widehat{\mathbf{p}}/(m^2c^4 + |c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}},$$

which is a momentum-dependent velocity operator whose magnitude is always strictly smaller than the speed of light c, and which will have the value zero for an eigenstate of $\widehat{\mathbf{p}}$ which has three-momentum eigenvalue equal to $\mathbf{0}$. So the particle rest frame certainly exists. We are actually in a position to now *repeat* the Eqs. (2) Lorentz boost of a nonzero-mass free particle's four-momentum from its rest frame to an inertial frame in which it has nonvanishing velocity, but this time *using for that velocity the operator value* $\widehat{\mathbf{v}}$ *that arises from* $\widehat{H}_{\rm free}$, which *operator* $\widehat{\mathbf{v}}$ is displayed just above, *instead of using for that velocity the c-number value* $\mathbf{v}$ of Eqs. (2). If we *simply follow* the steps of Eqs. (2), albeit using this particular *operator* $\widehat{\mathbf{v}}$, we end up with the *operator four-momentum* $((m^2c^2 + |\widehat{\mathbf{p}}|^2)^{\frac{1}{2}}, \widehat{\mathbf{p}})$, which *accords perfectly* with the *three-momentum operator* $\widehat{\mathbf{p}}$ and the *Hamiltonian operator* $(m^2c^4 + |c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}}$.

Now let us try to *emulate* the above successful "Lorentz boost of particle four-momentum out of the particle rest frame using the *operator* $\widehat{\mathbf{v}}$ that arises from $\widehat{H}_{\rm free}$" exercise by *instead* using the simple operator $\widehat{\mathbf{v}}$ that arises from the *Dirac* $\widehat{H}_D$, which of course is $\widehat{\mathbf{v}} = c\vec{\alpha}$. For $\widehat{\mathbf{p}} = \mathbf{0}$, we have that $\widehat{H}_D = \beta mc^2$. If we now try to follow the steps of Eqs. (2) from this starting point by using the Dirac $\widehat{\mathbf{v}} = c\vec{\alpha}$, we end up with the quite senseless operator four-momentum $(\beta mc(-i/\sqrt{2}\,), \beta\vec{\alpha}mc(-i/\sqrt{2}\,))$ that bears little resemblance to the *desired* operator four-momentum result $(\vec{\alpha}\cdot\widehat{\mathbf{p}} + \beta mc, \widehat{\mathbf{p}})$. In other words, for the Dirac theory Hamiltonian operator $\widehat{H}_D$, the attempt to show Lorentz covariance has ended in disaster. The Dirac theory contravenes special relativity just as the Klein-Gordon theory contravenes quantum mechanics.

We have pointed out that for the free-particle nonrelativistic Pauli theory the Hamiltonian operator has only the kinetic energy term $|\widehat{\mathbf{p}}|^2/(2m)$, which implies that *orbital* angular momentum $\widehat{\mathbf{r}}\times\widehat{\mathbf{p}}$ is *exactly* conserved. In contrast, the free-particle Dirac theory has a very strong spin-orbit coupling which almost certainly is not physically sensible. From Eq. (7a) it is clear that in the free-particle Dirac theory,

$$d(\widehat{\mathbf{r}}\times\widehat{\mathbf{p}})/dt = c\vec{\alpha}\times\widehat{\mathbf{p}}, \tag{8a}$$

and therefore the magnitude of the spin-orbit *torque* is,

$$|d(\widehat{\mathbf{r}}\times\widehat{\mathbf{p}})/dt| = c|\widehat{\mathbf{p}}|\sqrt{2}\,, \tag{8b}$$

Now the particle's kinetic energy is $((m^2c^4 + |c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}} - mc^2)$. If we take the *dimensionless ratio* of the particle's spin-orbit torque magnitude to its kinetic energy, we obtain $((1 + (mc/|\widehat{\mathbf{p}}|)^2)^{\frac{1}{2}} + (mc/|\widehat{\mathbf{p}}|))\sqrt{2}$, which increases monotonically *without bound* as $|\widehat{\mathbf{p}}|$ *decreases.* This is, of course, *not consistent* with the free-particle Pauli theory, where this ratio always *vanishes identically.* So the Dirac theory *does not* reduce to the Pauli theory merely by going to small values of momentum. This was already clear, of course, from the fact that the Dirac particle's *speed* always has the value $c\sqrt{3}$ *irrespective* of its momentum, which *doesn't accord* with the free-particle Pauli theory at all.

We now present the details of the free Dirac particle's staggering violation of Newton's first law of motion. Using Heisenberg's equations of motion, the Dirac free-particle Hamiltonian $\widehat{H}_D$ operator of Eq. (6b), and the simple Dirac free particle velocity operator $\widehat{\mathbf{v}} = c\vec{\alpha}$, we obtain the Dirac free-particle spontaneous acceleration operator,

$$\widehat{\mathbf{a}} = d\widehat{\mathbf{v}}/dt = (2mc^3/\hbar)(i\beta\vec{\alpha} + (\widehat{\mathbf{p}}\times\vec{\sigma})/(mc)), \tag{9a}$$

where,

$$\vec{\sigma} \stackrel{\rm def}{=} (-i/2)(\vec{\alpha}\times\vec{\alpha}), \tag{9b}$$

which means,

$$\sigma_i \stackrel{\rm def}{=} (-i/2)\epsilon_{ijk}\alpha_j\alpha_k. \tag{9c}$$

Since $|i\beta\vec{\alpha}|^2 = 3$, $|\widehat{\mathbf{p}}\times\vec{\sigma}|^2 = 2|\widehat{\mathbf{p}}|^2$, and $(i\beta\vec{\alpha})\cdot(\widehat{\mathbf{p}}\times\vec{\sigma}) = -(\widehat{\mathbf{p}}\times\vec{\sigma})\cdot(i\beta\vec{\alpha})$, we obtain for the magnitude of the spontaneous acceleration,

$$|\widehat{\mathbf{a}}| = (2\sqrt{3}\,mc^3/\hbar)(1 + (2/3)(|\widehat{\mathbf{p}}|/(mc))^2)^{\frac{1}{2}}, \tag{9d}$$

whose minimum value, $(2\sqrt{3}\,mc^3/\hbar)$, is, for the case of the electron, well in excess of $10^{28}g$, where g is the acceleration of gravity at the earth's surface. This dumbfounding spontaneous acceleration certainly *drives home the point* that the "free" Dirac particle *has no inertial rest frame.* Note that matters *don't even improve* as the Dirac particle is made more massive; the minimum spontaneous acceleration is *proportional* to the particle mass! The systematics of this unphysical absurdity show *just how profoundly* the Dirac theory *bungles special relativity.* As previously mentioned, the relativistic free-particle *square-root* Hamiltonian operator $\widehat{H}_{\rm free}$ *strictly adheres* to Newton's first law of motion, yielding $\widehat{\mathbf{a}} = \mathbf{0}$.

Finally, it is a *built-in property* of special relativity that the character of the physics it describes becomes Galilean/Newtonian if the speed of light c is taken to be *asymptotically large.* For example, if we subtract from the relativistic free-particle square-root Hamiltonian operator $\widehat{H}_{\rm free}$ of Eq. (3) its value at $\widehat{\mathbf{p}} = \mathbf{0}$, which is mc^2, the limit of that difference as $c \to \infty$ is, of course, the *nonrelativistic free-particle kinetic energy operator* $|\widehat{\mathbf{p}}|^2/(2m)$. But if we carry out *exactly same steps* with the Dirac Hamiltonian operator $\widehat{H}_D$, we face *formal divergence* as $c \to \infty$! Likewise, if we take the particular velocity operator $\widehat{\mathbf{v}}$ discussed in the paragraph *which follows* the one in which Eq. (7b) occurs, which is obtained from the Heisenberg equation of motion *in conjunction with* $\widehat{H}_{\rm free}$, and is explicitly given by,

$$\widehat{\mathbf{v}} = c^2\widehat{\mathbf{p}}/(m^2c^4 + |c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}},$$

it is clear that as $c \to \infty$, $\widehat{\mathbf{v}} \to \widehat{\mathbf{p}}/m$, i.e., *in this limit* $\widehat{\mathbf{v}}$ *becomes the nonrelativistic free-particle velocity operator.* On the other hand, for the Dirac free-particle velocity operator $\widehat{\mathbf{v}}$ of Eq. (7a) we *again* face *formal divergence* as $c \to \infty$! The clear lesson from these instances is that a *linearized* Hamiltonian operator such as the Dirac $\widehat{H}_D$ is *inherently incapable of incorporating the physics of special relativity.* It is obvious the *mathematical presence of a square root* is *essential* to correctly capturing the *the properties of relativistic physics. Even* in the case of the Klein-Gordon equation, which "squares out" the mathematical presence of the square root, it is *impossible* to show that the "physics" it describes becomes properly nonrelativistic as $c \to \infty$. Looking in detail at the solution space of the Klein-Gordon equation, we readily see that this is due the *gratuitous presence* of of the *extraneous* negative-energy solutions, which are, of course, *unphysical detritus.*

Correct relativistic quantum mechanics with an external electromagnetic field

It is clear that, for the nonzero-mass solitary relativistic free particle, the Klein-Gordon theory, which disrupts its quantum mechanics, and the Dirac theory, which dumbfoundingly abolishes its rest frame, *must both be abandoned in favor of the straightforward square-root Hamiltonian operator* $\widehat{H}_{\rm free}$ of Eq. (3). For the free photon, which is massless, the time-dependent Schrödinger equation (i.e, Eq. (1b)) *in conjunction with the zero-mass case of* $\widehat{H}_{\rm free}$ turns out to be *already implicit in the source-free case of Maxwell's equations* [5].

We shall now develop the *extension* of $\widehat{H}_{\rm free}$ to the case of a particle of charge e and nonzero mass m when an external electromagnetic potential $A^\mu(\mathbf{r}, t)$ is present, first for a spin-0 particle and then for a spin-$\frac{1}{2}$ particle. For additional background and detail concerning the derivation of those two Hamiltonians see reference [4]. The guiding concept is that accurate understanding of the physics experienced by a nonzero-mass solitary particle in an inertial frame where it is instantaneously traveling arbitrarily slowly translates, via a continuous sequence of successive Lorentz transformations, into the accurate understanding of that physics in an *arbitrary inertial frame.* Therefore a tested and trusted *nonrelativistic* theory of a nonzero-mass solitary particle's behavior ought to always be reasonably straightforwardly upgradable to a correct *relativistic* one that explicitly reduces to the underlying nonrelativistic one in any inertial frame in which that solitary particle happens to be instantaneously moving at nonrelativistic speed. The approach to attempting to carry out such a program which is followed here is to try to associate *individual terms* of the solitary particle's *nonrelativistic Hamiltonian* with fully Lorentz-covariant energy-momentum four-vectors whose *time components reduce to those individual nonrelativistic Hamiltonian terms* in inertial frames where that particle is traveling at *arbitrarily slow speed.* The *individual* Lorentz-covariant energy-momentum four-vectors so determined are then, of course, *summed* to produce the solitary particle's Lorentz-covariant *relativistic total energy-momentum four-vector.* The resulting solitary-particle *relativistic total three-momentum* is obviously identified as the *generator of the solitary particle's space translations* and thus as the solitary particle's *relativistic canonical three-momentum.* Of course the solitary particle's *relativistic total energy*, when expressed as *function of its relativistic canonical three-momentum, the time, and that particle's three space coordinates* comprises that particle's *relativistic Hamiltonian.* Initially, of course, the *individual terms* contributing to the solitary particle's relativistic total energy-momentum four-vector will be couched in the language of its three space coordinates, the time, and that particle's relativistic *kinetic* three-momentum. Upon *identification* of the particle's relativistic *canonical* (i.e., *total*) three-momentum, it is then *necessary to solve* for its relativistic *kinetic* three-momentum as a *function* of that relativistic canonical three-momentum *in order to be able to reexpress its total relativistic energy as its Hamiltonian.* Regrettably, a very conceivable "fly in the ointment" is *that there is no guarantee* that the particle's relativistic *kinetic* three-momentum can be obtained as a *function of its relativistic canonical three-momentum in closed form.* Thus the solitary particle's relativistic Hamiltonian *itself* may conceivably *only be available as a sequence of approximations.* Such a state of affairs is obviously *not* what one would desire, but *unlike* Klein, Gordon, Schrödinger and Dirac, the *physically-motivated*, conservative theorist is, like Einstein, obliged to coexist with whatever undesirable *calculational* consequences that physically-based, conservative theory happens to carry with it. The art of the *appropriate approximation*, one carefully attuned to a particular problem at hand, is surely yet another vital skill the theoretical physicist is called upon to hone.

Let us now apply the the above program to a spin-0 solitary particle of mass m and charge e in the presence of an external electromagnetic potential $A^\mu(\mathbf{r}, t)$. It will be recalled that all *magnetic* effects of such a potential on the particle's motion *vanish entirely* in the particle's rest frame, and are, more generally, of order $O(1/c)$, whereas in nonrelativistic physics the speed of light c is regarded as an asymptotically large parameter. Thus the *strictly nonrelativistic* Hamiltonian operator for this particle involves *only* the electromagnetic potential's time component $A^0(\mathbf{r}, t)$,

$$\widehat{H}^{(\mathrm{NR})}_{\mathrm{EM};0} = |\widehat{\mathbf{p}}|^2/(2m) + eA^0(\widehat{\mathbf{r}}, t). \tag{10a}$$

Because of the technical issue regarding the choice of ordering of noncommuting operators (whose resolution we allude to below), it will be convenient to develop the relativistic energy-momentum four-vector as a function of *classical* $(\mathbf{r}, \mathbf{p})$ phase space *rather* than as a function of the *already quantized* $(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})$ phase space of Eq. (10a). The solitary particle's nonrelativistic kinetic energy $|\mathbf{p}|^2/(2m)$, plus its rest mass energy mc^2, is well-known to correspond to c times its Lorentz-covariant *free-particle kinetic energy-momentum four-vector* p^μ,

$$p^\mu \stackrel{\text{def}}{=} ((m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}, \mathbf{p}),$$

where, of course, $\mathbf{p}$ is the particle's relativistic *kinetic* three-momentum, which was *distinguished* in the above discussion from its relativistic *total* (i.e., *canonical*) three-momentum. It is apparent that in the nonrelativistic limit $|\mathbf{p}| \ll mc$, the time component times c of p^μ does indeed, as just mentioned, behave as,

$$cp^0 \approx mc^2 + |\mathbf{p}|^2/(2m).$$

The *potential energy* term $eA^0(\mathbf{r}, t)$ of $H^{(\mathrm{NR})}_{\mathrm{EM};0}$, divided by c, is obviously the time component of the Lorentz-covariant energy-momentum four-vector $eA^\mu(\mathbf{r}, t)/c$. Therefore adding eA^μ/c to p^μ produces a fully Lorentz-covariant *total* energy-momentum four-vector whose time component times c reduces, in any inertial frame in which the nonzero-mass charged spin-0 solitary particle instantaneously has arbitrarily small speed, to that particle's nonrelativistic classical Hamiltonian $H^{(\mathrm{NR})}_{\mathrm{EM};0}$ (which corresponds to the quantized Hamiltonian operator $\widehat{H}^{(\mathrm{NR})}_{\mathrm{EM};0}$ of Eq. (10a)) plus that particle's rest mass energy mc^2. We therefore regard,

$$P^\mu \stackrel{\text{def}}{=} p^\mu + eA^\mu/c, \tag{10b}$$

as that solitary particle's *fully relativistic* energy-momentum four-vector. Thus the particle's *relativistic total three-momentum* is,

$$\mathbf{P} = \mathbf{p} + e\mathbf{A}(\mathbf{r}, t)/c, \tag{10c}$$

and its *relativistic total energy* is,

$$E(\mathbf{r}, \mathbf{p}, t) = cP^0 = (m^2c^4 + |c\mathbf{p}|^2)^{\frac{1}{2}} + eA^0(\mathbf{r}, t). \tag{10d}$$

Here we are in the fortunate position of being able to *solve* Eq. (10c) for the particle's relativistic *kinetic* three-momentum $\mathbf{p}$ as a *function* of its relativistic *total* (i.e., *canonical*) three-momentum $\mathbf{P}$ in *closed form*,

$$\mathbf{p}(\mathbf{P}) = \mathbf{P} - e\mathbf{A}(\mathbf{r}, t)/c, \tag{10e}$$

which result for $\mathbf{p}(\mathbf{P})$, i.e., $\mathbf{p}$ *as a function of* $\mathbf{P}$, we must now *substitute* into Eq. (10d) for the relativistic *total energy* in order to *reexpress* that total energy as the *relativistic Hamiltonian* $H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r}, \mathbf{P}, t)$, i.e.,

$$H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r}, \mathbf{P}, t) \stackrel{\text{def}}{=} E(\mathbf{r}, \mathbf{p}(\mathbf{P}), t).$$

Therefore Eqs. (10d) and (10e) yield the following *fully relativistic classical Hamiltonian* $H^{(\mathrm{REL})}_{\mathrm{EM;0}}(\mathbf{r},\mathbf{P},t)$, *which corresponds to our original nonrelativistic Hamiltonian operator* $\widehat{H}^{(\mathrm{NR})}_{\mathrm{EM;0}}$ *of* Eq. (10a),

$$H^{(\mathrm{REL})}_{\mathrm{EM;0}}(\mathbf{r},\mathbf{P},t) = (m^2c^4 + |c\mathbf{P} - e\mathbf{A}(\mathbf{r},t)|^2)^{\frac{1}{2}} + eA^0(\mathbf{r},t). \tag{10f}$$

Because of the *presence of the square root* in Eq. (10f) for $H^{(\mathrm{REL})}_{\mathrm{EM;0}}(\mathbf{r},\mathbf{P},t)$, there could conceivably be an issue regarding the *ordering* of the mutually noncommuting operators $\widehat{\mathbf{r}}$ and $\widehat{\mathbf{P}}$ when one attempts *quantize* this classical Hamiltonian $H^{(\mathrm{REL})}_{\mathrm{EM;0}}(\mathbf{r},\mathbf{P},t)$ to become the *Hamiltonian operator* $\widehat{H}^{(\mathrm{REL})}_{\mathrm{EM;0}}$. Use of the *Hamiltonian phase-space path integral* [9] with $H^{(\mathrm{REL})}_{\mathrm{EM;0}}(\mathbf{r},\mathbf{P},t)$ in its *classical form as given by* Eq. (10f) provides one *definitive solution* to any such operator-ordering issue. Another *completely equivalent* solution to this issue lies with a *natural slight strengthening* of Dirac's canonical commutation rule such that *it remains self-consistent* [10]. From *both* of these approaches the resulting *unambiguous operator-ordering rule* turns out to be the one of Born and Jordan [11].

It is well worth noting that the relativistic classical Hamiltonian $H^{(\mathrm{REL})}_{\mathrm{EM;0}}(\mathbf{r},\mathbf{P},t)$ of Eq. (10f) for the solitary spin-0 charged particle, when inserted into Hamilton's classical equations of motion, yields, upon taking proper account of Eq. (10c), *the fully relativistic version of the Lorentz-force law.* In other words, the Hamiltonian $H^{(\mathrm{REL})}_{\mathrm{EM;0}}(\mathbf{r},\mathbf{P},t)$ embodies *nothing more or less* than the well-known classical relativistic physics of the charged particle developed by H. A. Lorentz [12]. It is *truly a lamentable pity* that Klein, Gordon and Schrödinger went off on their *tangent which plays such havoc with quantum mechanics* instead of simply *going forward in workmanlike fashion with the quantization of Lorentz' utterly transparent relativistic legacy*, namely the $H^{(\mathrm{REL})}_{\mathrm{EM;0}}(\mathbf{r},\mathbf{P},t)$ of Eq. (10f) above. Of course, upon taking the particle charge e to zero, $H^{(\mathrm{REL})}_{\mathrm{EM;0}}(\mathbf{r},\mathbf{P},t)$ simply becomes the H_{free} of Eq. (2c), as it indeed must.

We turn now to the spin-$\frac{1}{2}$ charged particle, whose nonrelativistic Hamiltonian $H^{(\mathrm{NR})}_{\mathrm{EM;}\frac{1}{2}}$ is the *same* as $H^{(\mathrm{NR})}_{\mathrm{EM;0}}$ *except for an additional interaction energy between the particle's intrinsic magnetic moment and the external magnetic field*, i.e., its Pauli energy. Notwithstanding that this Pauli magnetic dipole energy is customarily *formally written* as including a factor of $(1/c)$, *it must nonetheless be kept in the nonrelativistic limit* because it *fails to vanish* in the spin-$\frac{1}{2}$ particle's *rest frame*,

$$H^{(\mathrm{NR})}_{\mathrm{EM;}\frac{1}{2}} = |\mathbf{p}|^2/(2m) + (ge/(mc))(\hbar/2)\vec{\sigma}\cdot(\nabla_{\mathbf{r}}\times\mathbf{A}(\mathbf{r},t)) + eA^0(\mathbf{r},t). \tag{11a}$$

The Pauli magnetic dipole energy contribution to $H^{(\mathrm{NR})}_{\mathrm{EM;}\frac{1}{2}}$ is the *only part* of this Hamiltonian which is *not a multiple of the two-by-two identity matrix* in the spin-$\frac{1}{2}$ two-by-two-matrix degrees of freedom. Now the relativistic energy-momentum four-vectors we shall be developing will of course *themselves* be two-by-two matrices, but this should not present difficulties so long as their four components *all mutually commute.* To *ensure* that this is the case, we shall "quarantine" the non-identity Pauli energy matrix into a *Lorentz scalar*, which we can furthermore render *dimensionless* by dividing by mc^2. If we now *multiply* this *dimensionless Lorentz scalar* by the particle's *kinetic* energy-momentum four-vector $p^\mu = ((m^2c^2 + |\mathbf{p}|^2)^{\frac{1}{2}}, \mathbf{p})$, we will indeed have the desired energy-momentum contribution whose *time component times* c reduces to the Pauli energy matrix in the particle rest frame.

There remains, of course, the challenging problem of turning the complicated Pauli energy term into a Lorentz scalar. In relativistic tensor language, the magnetic field axial vector $(\nabla_{\mathbf{r}}\times\mathbf{A}(\mathbf{r},t))$ that appears in the Pauli term comprises a certain *three-dimensional part* of the four-dimensional relativistic second-rank antisymmetric electromagnetic *field tensor* $F^{\mu\nu}(\mathbf{r},t) = \partial^\mu A^\nu(\mathbf{r},t) - \partial^\nu A^\mu(\mathbf{r},t)$. If we can manage to reexpress the three-dimensional axial spin-$\frac{1}{2}$ vector $(\hbar/2)\vec{\sigma}$ as a "matching" three-dimensional part of a four-dimensional relativistic second-rank antisymmetric tensor $s^{\mu\nu}$ as well, hopefully the Pauli energy will end up being proportional to to their *Lorentz-scalar contraction* $s^{\mu\nu}F_{\mu\nu}(\mathbf{r},t)$. As a first step, we define the natural *three-dimensional* second-rank antisymmetric spin-$\frac{1}{2}$ tensor S^{ij} in terms of the spin-$\frac{1}{2}$ axial vector,

$$S^{ij} \stackrel{\mathrm{def}}{=} (\hbar/2)\epsilon^{ijk}\sigma^k,$$

and take note that the complicated factor in the Pauli energy term neatly reduces to a contraction of S^{ij} with the well-known *magnetic-field* three-dimensional part $F^{ij}(\mathbf{r},t)$ of $F^{\mu\nu}(\mathbf{r},t)$,

$$(\hbar/2)\vec{\sigma}\cdot(\nabla_{\mathbf{r}}\times\mathbf{A}(\mathbf{r},t)) = (1/2)S^{ij}F^{ij}(\mathbf{r},t),$$

which allows us to reexpress the nonrelativistic Eq. (11a) in the *relativistically more suggestive form*,

$$mc^2 + H^{(\mathrm{NR})}_{\mathrm{EM};\frac{1}{2}} = mc^2[1 + |\mathbf{p}|^2/(2m^2c^2) + (g/2)(e/(m^2c^3))S^{ij}F^{ij}(\mathbf{r},t)] + eA^0(\mathbf{r},t). \tag{11b}$$

It is now time to work out the nonzero-mass spin-$\frac{1}{2}$ particle's fully covariant *four-dimensional* antisymmetric spin tensor $s^{\mu\nu}$. In the *particle rest frame*, namely in the special inertial frame where the *particle kinetic three-momentum* $\mathbf{p}$ *vanishes*, the nine space-space components of $s^{\mu\nu}$ must clearly be the nine components of S^{ij}, and its remaining seven components must be *filled out with zeros*, i.e.,

$$s^{\mu\nu}(\mathbf{p}=\mathbf{0}) \stackrel{\text{def}}{=} \delta^\mu_i \delta^\nu_j S^{ij},$$

which *ensures* that, in the particle rest frame,

$$s^{\mu\nu}(\mathbf{p}=\mathbf{0})F_{\mu\nu}(\mathbf{r},t) = S^{ij}F^{ij}(\mathbf{r},t).$$

Once a tensor is *fully determined* in *one* inertial frame, it is fully determined in *all* inertial frames by application of the appropriate Lorentz transformation to its indices. To get from the particle rest frame to the inertial frame where the particle has kinetic three-momentum $\mathbf{p}$ simply requires the appropriate Lorentz-boost four-dimensional matrix $\Lambda^\mu_\alpha(\mathbf{v}(\mathbf{p})/c)$ that is characterised by the corresponding dimensionless scaled relativistic particle velocity,

$$\mathbf{v}(\mathbf{p})/c = (\mathbf{p}/(mc))/(1 + |\mathbf{p}/(mc)|^2)^{\frac{1}{2}},$$

and its accompanying dimensionless time-dilation factor,

$$\gamma(\mathbf{p}) = (1 + |\mathbf{p}/(mc)|^2)^{\frac{1}{2}},$$

so that, in general,

$$s^{\mu\nu}(\mathbf{p}) = \Lambda^\mu_i(\mathbf{v}(\mathbf{p})/c)\Lambda^\nu_j(\mathbf{v}(\mathbf{p})/c)S^{ij},$$

which, of course, *ensures* that $s^{\mu\nu}(\mathbf{p})F_{\mu\nu}(\mathbf{r},t)$ is a Lorentz scalar that *Lorentz-invariantly carries the particle's rest-frame value of* $S^{ij}F^{ij}(\mathbf{r},t)$.

With that, we are in the position to be able to write down the Lorentz-covariant *total* energy-momentum four-vector P^μ for the spin-$\frac{1}{2}$ particle that corresponds to its nonrelativistic Eq. (11b) in the *same way* that the total energy-momentum four-vector of Eq. (10b) for the spin-0 particle corresponds to *its* nonrelativistic Eq. (10a),

$$P^\mu \stackrel{\text{def}}{=} p^\mu[1 + (g/2)(e/(m^2c^3))s^{\alpha\beta}(\mathbf{p})F_{\alpha\beta}(\mathbf{r},t)] + eA^\mu(\mathbf{r},t)/c. \tag{11c}$$

From P^μ we obtain the particle's relativistic total energy,

$$E(\mathbf{r},\mathbf{p},t) = cP^0 = (m^2c^4 + |c\mathbf{p}|^2)^{\frac{1}{2}}[1 + (g/2)(e/(m^2c^3))s^{\mu\nu}(\mathbf{p})F_{\mu\nu}(\mathbf{r},t)] + eA^0(\mathbf{r},t), \tag{11d}$$

and also its relativistic total three-momentum,

$$\mathbf{P} = \mathbf{p}[1 + (g/2)(e/(m^2c^3))s^{\mu\nu}(\mathbf{p})F_{\mu\nu}(\mathbf{r},t)] + e\mathbf{A}(\mathbf{r},t)/c. \tag{11e}$$

It is obvious from Eq. (11e) that we *cannot solve* for $\mathbf{p}(\mathbf{P})$ in *closed form*, but we *can* write $\mathbf{p}(\mathbf{P})$ in "iteration-ready" form as,

$$\mathbf{p}(\mathbf{P}) = (\mathbf{P} - e\mathbf{A}(\mathbf{r},t)/c)[1 + (g/2)(e/(m^2c^3))s^{\mu\nu}(\mathbf{p}(\mathbf{P}))F_{\mu\nu}(\mathbf{r},t)]^{-1}, \tag{11f}$$

and, of course, from $E(\mathbf{r},\mathbf{p}(\mathbf{P}),t)$, we also obtain the schematic form of the particle's relativistic Hamiltonian,

$$H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t) = (m^2c^4 + |c\mathbf{p}(\mathbf{P})|^2)^{\frac{1}{2}}[1 + (g/2)(e/(m^2c^3))s^{\mu\nu}(\mathbf{p}(\mathbf{P}))F_{\mu\nu}(\mathbf{r},t)] + eA^0(\mathbf{r},t). \tag{11g}$$

If we take the limit $g \to 0$ in Eqs. (11f) and (11g), then $H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t) \to H^{(\mathrm{REL})}_{\mathrm{EM};0}(\mathbf{r},\mathbf{P},t)$, as is easily checked from Eq. (10f). Of course it is nothing more than the *most basic common sense* that fully relativistic spin-$\frac{1}{2}$ theory simply reduces to fully relativistic spin-0 theory when the spin coupling of the single particle to the external field is switched off, but analogous cross-checking between the Dirac and Klein-Gordon theories is never so much as discussed! It is certainly possible to add a term to the Dirac Hamiltonian that *cancels out* it's *supposed* $g = 2$ spin coupling to the magnetic field, but the result of doing this bears *very little resemblance* to the Klein-Gordon equation in the presence of the external electromagnetic potential! Elementary consistency checks are obviously *not* the strong suit of those two "theories"!

It is unfortunate that Eq. (11f) for $\mathbf{p}(\mathbf{P})$ is not amenable to closed-form solution, but if we assume that the spin coupling term, $(g/2)(e/(m^2c^3))s^{\mu\nu}(\mathbf{p}(\mathbf{P}))F_{\mu\nu}(\mathbf{r},t)$, which is a dimensionless Hermitian two-by-two matrix, effectively has the magnitudes of both of its eigenvalues much smaller than unity (which should be a very safe assumption for atomic physics), then we can approximate $\mathbf{p}(\mathbf{P})$ via successive iterations of Eq. (11f), which produces the approximation $(\mathbf{P} - e\mathbf{A}(\mathbf{r},t)/c)$ for $\mathbf{p}(\mathbf{P})$ through zeroth order in the spin coupling and,

$$\mathbf{p}(\mathbf{P}) \approx (\mathbf{P} - e\mathbf{A}(\mathbf{r},t)/c)[1 + (g/2)(e/(m^2c^3))s^{\mu\nu}(\mathbf{P} - e\mathbf{A}(\mathbf{r},t)/c)F_{\mu\nu}(\mathbf{r},t)]^{-1},$$

through first order in the spin coupling. We wish to interject at this point that since $s^{\mu\nu}(\mathbf{p}(\mathbf{P}))$ is an anti-symmetric tensor, the tensor contraction $s^{\mu\nu}(\mathbf{p}(\mathbf{P}))F_{\mu\nu}(\mathbf{r},t)$ is equal to $2s^{\mu\nu}(\mathbf{p}(\mathbf{P}))\partial_\mu A_\nu(\mathbf{r},t)$, which is often a more transparent form. Now if we simply use the approximation $(\mathbf{P} - e\mathbf{A}(\mathbf{r},t)/c)$ through zeroth order in the spin coupling for $\mathbf{p}(\mathbf{P})$, we obtain the following approximation to $H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}$,

$$H^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}(\mathbf{r},\mathbf{P},t) \approx (m^2c^4 + |c\mathbf{P} - e\mathbf{A}(\mathbf{r},t)|^2)^{\frac{1}{2}}[1 + (ge/(m^2c^3))s^{\mu\nu}(\mathbf{P} - e\mathbf{A}(\mathbf{r},t)/c)\partial_\mu A_\nu(\mathbf{r},t)] + eA^0(\mathbf{r},t). \quad (11\mathrm{h})$$

Antiparticle partners from field theoretic symmetry

If one were to write down a purely electromagnetic (and thus parity-conserving) quantum field theory that treats *only* positive-energy electrons and photons, which would be possible via second quantization of $\widehat{H}^{(\mathrm{REL})}_{\mathrm{EM};\frac{1}{2}}$ and quantization of the electromagnetic field, it is clear that one would then have a quantum field theory which exhibits no invariance whatsoever under charge conjugation. Simply imposing charge conjugation invariance on such a theory *forces positrons to exist.* This is a *very familiar picture indeed*: the imposition of symmetries on quantum field theories *not at all infrequently forces families of particles to exist.* Note that the positrons that would be forced into existence by the imposition of charge conjugation invariance would *also be purely positive-energy particles*, at least when free. "Reinterpretation" of unbounded-below, free-particle negative-energy spectra plays *no role whatsoever* in the existence of these positrons: their existence is driven theoretically *entirely by the enforcement of the symmetry*!

The imposition of charge conjugation invariance—or, more generally, CP invariance—on solitary-particle quantum mechanics that has been taken to the quantum field theory level does not *of itself*, however, *ensure* that external fields can mediate the basic physical processes of particle-antiparticle pair production and annihilation [13], processes which *cannot*, of course, *be accommodated in the context of a strictly solitary-particle idealization.* To accommodate these processes, the imposition of CP invariance at the field theory level *needs to be accompanied* by the *additional* imposition of invariance under the interchange of particle absorption/emission with, respectively, antiparticle emission/absorption, which we term "CP-*equivalence*" symmetry. To give one crude, schematic sketch of an imposition of the dual symmetries of CP invariance and "CP-equivalence", suppose we have in hand the quantum mechanics Hamiltonian operator $\widehat{H}$ for a solitary relativistic charged particle interacting with external fields. The straightforward second quantization of $\widehat{H}$ produces the quantum-field-theoretic schematic quantized Hamiltonian density $\psi^\dagger\widehat{H}\psi$, which has the particle emission field $\psi^\dagger$ and the particle absorption field ψ, but *no antiparticle fields*, and therefore is completely CP *noninvariant.* This is readily remedied by introducing the antiparticle emission field $\psi^\dagger_{\mathrm{CP}}$ and the antiparticle absorption field ψ_{CP}, which correspond to the CP-transformed version of $\widehat{H}$ that we denote as $\widehat{H}_{\mathrm{CP}}$ (a prototypical difference between $\widehat{H}$ and $\widehat{H}_{\mathrm{CP}}$ is, of course, in the sign of the solitary particle's charge). Using these entities, one simple way to impose CP invariance symmetry is to replace the Hamiltonian density $\psi^\dagger\widehat{H}\psi$ by $(\psi^\dagger\widehat{H}\psi + \psi^\dagger_{\mathrm{CP}}\widehat{H}_{\mathrm{CP}}\psi_{\mathrm{CP}})$, which can permit external fields to mediate both particle and antiparticle scattering, but not yet particle-antiparticle pair production or annihilation. To include the latter physical phenomena we shall *additionally* impose the "CP-equivalence" symmetry which was pointed out above by *also* requiring the Hamiltonian density to be invariant under the particle field interchanges $\psi \leftrightarrow \psi^\dagger_{\mathrm{CP}}$ and $\psi^\dagger \leftrightarrow \psi_{\mathrm{CP}}$. It is apparent that one straightforward upgrade of the solitary-particle Hamiltonian density $\psi^\dagger\widehat{H}\psi$ which incorporates the *dual symmetries* of CP invariance and "CP-equivalence" that we are stipulating here is given by,

$$\psi^\dagger\widehat{H}\psi \to (\psi + \psi^\dagger_{\mathrm{CP}})^\dagger\widehat{H}(\psi + \psi^\dagger_{\mathrm{CP}}) + (\psi_{\mathrm{CP}} + \psi^\dagger)^\dagger\widehat{H}_{\mathrm{CP}}(\psi_{\mathrm{CP}} + \psi^\dagger),$$

which clearly can permit external fields to mediate pair production and annihilation, as well as particle and antiparticle scattering.

The role of symmetries in accounting for and compelling the existence of energy degeneracies has been a robust and *highly fruitful* theme of quantum physics since its *very earliest days*; the names of such pioneers as Wigner and Weyl come readily to mind. The existence of antiparticles is a *classic example of energy degeneracy* in the context of quantum field theory, *and there is absolutely no reason whatsoever that it should not take its rightful place in the theoretical physics pantheon of symmetry-driven phenomena.* Unbounded-below negative energies that *require* "reinterpretation" are a staggeringly bizarre manifestation of *undiluted metaphysics* that physical science would do extremely well indeed to shed forever, *especially* in light of the fact that their *origin* resides in gratuitously-generated *entirely extraneous* "solutions" (Klein-Gordon theory) or an equally *gratuitous weakening* of the correspondence principle that springs from extraordinary insistence on, for a physically *nonviable* reason, *dealing with the squares of dynamical variables instead of those dynamical variables themselves* (Dirac theory). The "reinterpretation" of the egregiously metaphysical unbounded-below negative-energy spectra of the Klein-Gordon and Dirac theories is a quintessential instance of the syndrome that afflicted Einstein's knight, "who dyed his whiskers green, and then used a large fan so that they should not be seen."

The *breaking* of symmetries has become a very strong theme in the last seven decades, and the breaking of the antiparticle-associated CP invariance has been empirically firmly established. Indeed a glance at the gross particle-antiparticle "nonsymmetry" that makes our galaxy's existence *possible* powerfully suggests that there is *much* that remains to be *learned* about the breaking of this symmetry. A theory that assigns *two independent fields* to the description of the particle and its distinguishable antiparticle makes it *far more straightforward* to model what are no doubt extremely important effects pertaining to that symmetry breaking. With two *independent* fields one can essay such basics as a *slight mass difference* between particle and antiparticle, which is certainly *not available* if particle and antiparticle are construed as merely somehow reflecting *different parts* of the *same operator's energy spectrum.*

Conclusion

It has been made very clear in the above that Schrödinger's simple postulates regarding the solitary particle's wave function specify that the generators of that wave function's infinitesimal space and time translations are, respectively, the canonical three-momentum operator and the Hamiltonian operator, and that this fact provides an extremely satisfactory basis, which is *completely compatible with the strong correspondence principle*, for the *fully relativistic version of solitary-particle quantum mechanics.* From the standpoint of *systematic, conservative* physical theory, the utterly clear implication of the above observations, in light of the strong correspondence principle, is that the correct Hamiltonian operator for the solitary relativistic free particle is given by the square-root operator of Eq. (3), and that the interaction of a spinless nonzero-mass solitary charged relativistic particle with an external electromagnetic potential in the context of quantum mechanics must be described by the quantization of the Hamiltonian that corresponds to the fully relativistic version of H. A. Lorentz' electromagnetic force law, specifically the Hamiltonian of Eq. (10f).

Unfortunately, the pellucid implications for *the relativistic quantum mechanics of the solitary particle* of Schrödinger's postulates for the wave function *never dawned* on Schrödinger himself, nor on Klein, Gordon, nor Dirac. That this is the case is made *painfully clear* by Dirac's simultaneously superfluous and extremely damaging imposition of "space-time coordinate symmetry" on Schrödinger's *time-dependent* wave function equation (Eq. (1b)), notwithstanding that *precisely this particular appearance of space-time coordinate symmetry* is the *most striking feature* of Schrödinger's *four-vector* wave function equation (Eq. (1c))! Although these pioneers failed to appreciate even the *very existence* of Eq. (1c), *notwithstanding* that it is a *mere summary* of Schrödinger's wave-function postulates, they *did* have an appreciation of the correspondence principle, albeit they regarded it as a rather more *plastic* concept than the definitive *strong form* which the *facts* of both the Hamiltonian phase-space path integral [9] and the *completed* version of Dirac's canonical commutation rule [10] *very clearly reveal.* But, extremely unfortunately, because of perceived issues of calculational convenience, these pioneers gave the correspondence principle short shrift indeed, and instead chose ostensible "relativistic quantum mechanics" routes that *terribly distort* the quantum mechanical implications and impact of the underlying *classical special relativistic mechanics* of the solitary particle which had been so ably developed and expounded by Lorentz; the *gratuitous injection* of *totally extraneous* unphysical, unbounded-below negative-energy "solutions" is, of course, a prime example of this. For any half-way serious disciple of the correspondence principle, these *patently absurd* "solutions", which are at *complete loggerheads* with the *classical* relativistic solitary-particle mechanics of Lorentz, would *of themselves* have been *reason enough to immediately call off* the attemped tweaking of the time-dependent Schrödinger equation and/or of the ostensibly "relativistic" Hamiltonians being fed to it, and to *return forthwith* to *strictly Lorentzian classical relativistic basics* as the *correct physical foundation* upon which to erect relativistic quantum mechanics.

Notwithstanding subsequent "reinterpretation" of these unbounded-below negative-energy "solutions" to "accommodate" the physical existence of mass-degenerate antiparticles, which is universally acclaimed as a "triumph" [2], it is *vastly less conceptually tortuous* and equally vastly *more in keeping with all of the rest of known quantum physics* to account for the existence of mass-degenerate antiparticles as the *utterly straightforward consequence* of *CP invariance of the overlying quantum field theory.* Since this invariance is, indeed, *a slightly broken one* in the manner of so many *other* symmetries of physics—a fact that would appear to be of critical importance to the *very existence* of the familiar physical world of our experience—it would seem all the more important to *have available* the theoretical tool of *two completely independent quantum fields* (with each having *bounded-below energy*, of course) for the descriptions, respectively, of a particle and its distinguishable antiparticle, which permits their masses, for example, to be very slightly different, *instead* of being *tied* to the "reinterpreted negative energy spectrum" *consequence* of having certain attributes of particle and antiparticle being *irrevocably identical.* It is *symmetries* and *not* "reinterpretations" of blatantly gratuitous and absurdly unphysical equation "solutions" that are reasonably and plausibly held to be responsible for the existence of mass-degenerate partner particles in *all instances other than that of distinguishable antiparticles.* It is surely now *well past time* for the theoretical physics treatment of mass-degenerate distinguishable antiparticles to be put on a track which is *completely parallel* to the sensible symmetry-based handling accorded *as a matter of standard routine* to *all other* ostensibly "understood" particle mass degeneracies.

Departure from the theoretical physics scene of the physically misbehaving Klein-Gordon and Dirac equations would *not only* restore physical good sense to relativistic solitary-particle quantum mechanics, it would unveil physics' harmonious hierarchical integrity: the *underlying physical essence* of the relativistic Lorentzian Hamiltonian of Eq. (10f) is *explicitly* its *simple nonrelativistic counterpart* of Eq. (10a), and the *same role* is played by the nonrelativistic Pauli theory of Eq. (11a) in relation to *its* fully relativistic upgrade that is given by Eqs. (11g) and (11f). Likewise, when the upgrade of these solitary-particle theories to fully interacting field theories is made via second quantization and the imposition of the dual symmetries of CP invariance and "CP-equivalence", the physical essence of the *multi-particle interactions* is obviously *firmly anchored* in those of the *merely solitary relativistic particle with external fields*, which interactions, in turn, as we have *just noted*, may be no more than the *relativistic upgrade* of such *nonrelativistic basics* as a stationary charged particle's interaction with an electric potential or a stationary magnetic dipole's interaction with a magnetic field! Likewise, the properties of any quantized Hamiltonian are highly sensitive to those of its *classical precursor*, as the key *stationary phase path for the path integral* amply attests. And of course both the presence *and absence* of symmetries is of crucial importance to the *very character* of the physics. It is *precisely because* of the *profound underlying linkages* inherent in what at first glance may appear to be *very diverse* aspects of physics that the fatuous mathematical "modifications" unwarily introduced into relativistic quantum mechanics by Klein, Gordon, Schrödinger, and Dirac are so very damaging.

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