SMARANDACHE PSEUDO- IDEALS

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Abstract

In this paper we study the Smarandache pseudo-ideals of a Smarandache ring. We prove every ideal is a Smarandache pseudo-ideal in a Smarandache ring but every Smarandache pseudo-ideal in general is not an ideal. Further we show that every polynomial ring over a field and group rings FG of the group G over any field are Smarandache rings. We pose some interesting problems about them.

Keywords:

Smarandache pseudo-right ideal, Smarandache pseudo-left ideal, Smarandache pseudo-ideal.

Definition [1]: A Smarandache ring is defined to be a ring A such that a proper subset of A is a field (with respect to the same induced operations). Any proper subset we understand a set included in A, different from the empty set, from the unit element if any, and from A.

For more about Smarandache Ring and other algebraic concepts used in this paper please refer [1], [2], [3] and [4].

Definition 1 Let $(A, +, \bullet)$ be a Smarandache ring. B be a proper subset of A $(B \subset A)$ which is a field. A nonempty subset S of A is said to be a Smarandache pseudo-right ideal of A related to B if

1. $(S, +)$ is an additive abelian group.
2. For $b \in B$ and $s \in S$, $s \bullet b \in S$.

On similar lines we define Smarandache pseudo-left ideal related to B. S is said to be a Smarandache pseudo-ideal (S-pseudo-ideal) related to B if S is both a Smarandache pseudo-right ideal and Smarandache pseudo-left ideal related to B.

Note: It is important and interesting to note that the phrase "related to B" is important for if the field B is changed to B' the same S may not in general be S-pseudo-ideal related to B' also. Thus the S-pseudo-ideals are different from usual ideal defined on a ring. Further we define S-pseudo-ideal only when the ring itself is a Smarandache ring. Otherwise we don't define them to be S-pseudo-ideal. Throughout this paper unless notified F[x] or R[x] will be polynomial of all degrees, $n \to \infty$.

Theorem 2 Let F be a field. F[x] be a polynomial ring in the variable x. F[x] is a Smarandache ring.

Proof: Clearly $F \subset F[x]$ is a field which is a proper subset of F[x], so F[x] is a Smarandache ring.

If F is a commutative ring then we have the following:

Theorem 3 Let R[x] be a polynomial ring. R be a commutative ring. R[x] is a Smarandache ring if and only if R is a Smarandache Ring.

Proof: If R is a Smarandache ring clearly there exists a proper subset S of R which is a field. So R[x] is a Smarandache ring.

Conversely if R[x] is a Smarandache ring we have $S \subset R$ such that S is a field. So R must be a Smarandache ring. Since $R[x] = \left\{ \sum_{i=0}^{\infty} r_i x^i \middle/ r_i \in R \right\}$. R[x] cannot contain any polynomial which has inverse. Hence the claim.

<u>*Example 1:*</u> Let Q[x] be the polynomial ring over the rationals. Clearly Q[x] is a Smarandache ring. Consider $S = \langle n(x^2+1) / n \in Q \rangle$ is generated under ' + '. Clearly QS $\subseteq$ S and SQ $\subseteq$ S. So S is a S-pseudo-ideal of Q[x] related to Q.

Theorem 4 Let R be any Smarandache ring. Any ideal of R is a S-pseudo-ideal of R related to some subfield of R but in general every S-pseudo-ideal of R need not be an ideal of R.

Proof: Given R is a Smarandache ring. So $\phi \neq B$, $B \subset R$ is a field. Now I is an ideal of R. So $IR \subseteq I$ and $RI \subseteq I$. Since $B \subset R$ we have $BI \subseteq I$ and $IB \subseteq I$. Hence I is a S-pseudo-ideal related to B.

To prove the converse, consider the Smarandache ring given in Example 1. S is a S-pseudo-ideal but S is not an ideal of Q[x] as xS is not contained in S. Hence the claim.

<u>*Example 2:*</u> Let $\mathfrak{R}$ be the field of reals. $\mathfrak{R}[x]$ be the polynomial ring. Clearly $\mathfrak{R}[x]$ is a Smarandache ring. Now $Q \subset \mathfrak{R}[x]$ and $\mathfrak{R} \subset \mathfrak{R}[x]$ are fields contained in $\mathfrak{R}[x]$. Consider $S = \langle n(x^2+1) / n \in Q \rangle$ generated additively as a group. Now S is a S-pseudo-ideal relative to Q but S is not a S-pseudo-ideal related to $\mathfrak{R}$. Thus this leads us to the following result.

Theorem 5 Let R be a Smarandache ring. Suppose A and B are two subfields of R. S be a S-pseudo-ideal related to A. S need not in general be a S-pseudo-ideal related to B.

Proof: The example 2 is an illustration of the above theorem.

Based on these properties we propose the following problems:

Problem 1 Find conditions on the Smarandache ring so that a S-pseudo-ideal which are not ideals of the ring related with every field is a S-pseudo-ideal irrelevant of the field under consideration.

Problem 2 Find conditions on the Smarandache ring so that every S-pseudo-ideal is an ideal.

Example 3 $Z_{12} = \{0, 1, 2, 3, ..., 11\}$ be the ring. Clearly Z_{12} is a Smarandache ring for $A = \{0, 4, 8\}$ is a field in Z_{12} with $4^2 = 4 \pmod{12}$ acting as the multiplicative identity. Now $S = \{0, 6\}$ is a S-pseudo-ideal related to A. But S is also an ideal of Z_{12}. Every ideal of Z_{12} is also a S-pseudo-ideal of Z_{12} related to A.

Problem 3 Find conditions on n for Z_n (n not a prime) to have all S-pseudo-ideals to be ideals.

Example 4 Let $M_{2\times2}$ be the set of all 2×2 matrices with entries from the prime field $Z_2 = \{0,1\}$.

$$M_{2\times2} = \left\{ \begin{matrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \text{and} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{matrix} \right\}$$ be the ring of matrices under usual matrix addition and multiplication modulo 2.

Now $M_{2\times2}$ is a Smarandache ring for $A = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ is a field of $M_{2\times2}$. Let $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$, S is a Smarandache pseudo-left ideal related to A but S is not a Smarandache pseudo-right ideal related to A for $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ as $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin$ S. Now $B = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is also a field. $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ is a left ideal related to B but not a right ideal related to B. $C = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ is a field. Clearly S is not a Smarandache pseudo-left ideal with respect to C. But S is a Smarandache pseudo-right ideal with respect to C.

Thus from the above example we derive the following observation.

Observation: A set S can be a Smarandache pseudo-left ideal relative to more than one field. For S = $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ is a Smarandache pseudo-left ideal related to both A and B. The same set S is not a Smarandache pseudo-left ideal with respect to the related field C = $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right\}$ but S is a Smarandache pseudo-right ideal related to C.

Thus the same set S can be Smarandache pseudo left or right ideal depending on the related field. Clearly S is a S-pseudo-ideal related to the field D = $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$.

Definition 6 Let R be a Smarandache ring. I be a S-pseudo-ideal related to A, $A \subset R$ (A a field). I is said to be a Smarandache minimal pseudo-ideal of R if I_1 is another S-pseudo-ideal related to A and $(0) \subseteq I_1 \subseteq I$ implies $I_1 = I$ or $I_1 = (0)$.

<u>**Note:**</u> The minimality of the ideal may vary in general for different related fields.

Definition 7 Let R be a Smarandache ring. M is said to be a Smarandache maximal pseudo-ideal related to a subfield A, $A \subset R$ if M_1 is another S-pseudo-ideal related to A and if $M \subseteq M_1$ then $M = M_1$.

Definition 8 Let R be a Smarandache ring. A S-pseudo-ideal I related to a field A, $A \subset R$ is said to be a Smarandache cyclic pseudo-ideal related to a field A, if I can be generated by a single element.

Definition 9 Let R be a Smarandache ring. A S-pseudo-ideal I related to a field A, $A \subset R$ is said to be a Smarandache prime pseudo-ideal related to A if $x \bullet y \in I$ implies $x \in I$ or $y \in I$.

<u>Example 5:</u> Let $Z_2 = (0, 1)$ be the prime field of characteristic 2. $Z_2[x]$ be the polynomial ring of all polynomials of degree less than or equal to 3, that is $Z_2[x] = \{0, 1, x, x^2, x^3, 1+x, 1 + x^2, 1 + x^3, x + x^2, x + x^3, x^2 + x^3, 1 + x + x^3, 1 + x + x^2, 1 + x^2 + x^3, x + x^2 + x^3, 1+x + x^2 + x^3\}$. Clearly $Z_2[x]$ is a Smarandache ring as it contains the field Z_2.

$S = \{0, (1 + x), (1 + x^3), (x + x^3)\}$ is a S-pseudo-ideal related to Z_2 and not related to $Z_2[x]$.

<u>Example 6:</u> Let $Z_2 = (0,1)$ be the prime field of characteristic 2. $S_3 = \{1, p_1, p_2, p_3, p_4, p_5\}$ be the symmetric group of degree 3. Here $1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. Z_2S_3 be the group ring of the group S_3 over Z_2. Z_2S_3 is a Smarandache ring. $Z_2S_3 = \{1, p_1, p_2, p_3, p_4, p_5, 1 + p_1, 1 + p_2, ..., p_1 + p_2 + p_3 + p_4 + p_5, 1 + p_1 + p_2 + p_3 + p_4 + p_5\}$. Now $A = \{0, p_4 + p_5\}$ is a

field $A \subset Z_2S_3$. Let $S = \{0, 1 + p_1 + p_2 + p_3 + p_4 + p_5\}$ be the subset of Z_2S_3. S is a S-pseudo-ideal related to A. S is also a S-pseudo-ideal related to Z_2.

Theorem 10 Let F be a field and G be any group. The group ring FG is a Smarandache ring.

Proof: F is a field and G any group FG the group ring is a Smarandache ring for $F \subset FG$ is a field of the ring FG. Hence the claim.

Theorem 11 Let $Z_2 = \{0,1\}$be the prime field of characteristic 2. G be a group of finite order say n. Then Z_2G has S-pseudo-ideals, which are ideals of Z_2G.

Proof: Take $Z_2 = \{0, 1\}$ as a field of Z_2G. Let $G = \{g_1, g_2, ..., g_{n-1}, 1\}$ be the set of all elements of G. Now $S = \{0, (1+g_1 + g_2 + ...+ g_{n-1})\}$ is a S-pseudo-ideal related to Z_2 and S is also an ideal of Z_2G. Hence the claim.

Problem 4 Find conditions on the group G and the ring R so that the group ring RG is a Smarandache ring?

References

[1] Padilla, Raul. *Smarandache Algebraic Structures,* Bulletin of Pure and Applied Sciences, Delhi, Vol. 17 E., No. 1, 119-121, (1998) http://www.gallup.unm.edu/~smarandache/ALG-S-TXT.TXT

[2] W.B.Vasantha Kandasamy, *Obedient Ideals in a Finite Ring*, J. Inst. Math. & Comp. Sci., Vol. 8, 217 - 219, (1995).

[3] W.B.Vasantha Kandasamy, *On idealy strong group rings,* The Mathematics Education, Vol. XXX, 71 - 72, (1996)

[4] W.B.Vasantha Kandasamy, *On Generalized Semi-ideals of a Group Ring,* The Journal of Qufu Normal University., Vol. 18, No. 4, (1992).