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JUSTIFICATION OF THE ZETA REGULARIZATION PROCEDURE FOR THE INTEGRALS $\int_0^\infty x^{m-s}dx$

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- ABSTRACT: In this paper we review and try to justify some results we gave before concerning the zeta regularization of integrals $\int_0^\infty x^{m-s}dx$ via the zeta regularization of the divergent series $\sum_{i=0}^\infty i^{m-s}$ and the zeta function $\zeta(m-s)$

REGULARIZATION OF DIVERGENT INTEGRALS:

In a previous paper [6] we gave a method to regularize divergent integrals of the form $\int_0^\infty x^m dx$ for m a positive integer , via the zeta regularization method , that attach a finite meaning $\zeta(-m)$, to a power-law divergent series $\sum_{i=0}^\infty i^m$, via the analytic continuation of the zeta function of Riemann to negative exponents $\zeta(1-s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$. First we took the Euler-Bernoulli summation formula with $f(x)=x^m$, and used the property for the k-th derivative of the function x^m, $\frac{d^k\left(x^m\right)}{dx^k} = \frac{\Gamma(m+1)}{\Gamma(m-k+1)}x^{m-k}$ together with the zeta regularization of the series $\sum_{i=0}^\infty i^m \to \zeta(-m)$ to find the following

$$\int_0^\infty x^m dx = \frac{m}{2}\int_0^\infty x^{m-1}dx + \zeta(-m) - \sum_{r=1}^{\infty}\frac{B_{2r}m!(m-2r+1)}{(2r)!(m-2r+1)!}\int_0^\infty x^{m-2r}dx \qquad (1)$$

For ‘m’ a positive integer , this is a recurrence formula to get the value of $\int_0^\infty x^m dx$ for m=01,2,3,4,..... from the starting value $\int_0^\infty dx = 1+1+1+...... = -\frac{1}{2}$, this is explained on section (2) in our previous paper [6] , now we would like to compare our method with another well-known method in theoretical physics to calculate divergent integrals , the ‘dimensional regularization method’.

- *Dimensional regularization of divergent integrals:*

The method of dimensional regularization , assumes we can define a d-dimensional space and d-dimensional polar coordinates , so we can write any integral as $\frac{(2\pi)^{d/2}}{\Gamma(d/2)}\int_0^\infty dx f(x)x^{d-1} = I(d)$, here ‘d’ is the dimension of the space R^d , with ‘d’ any arbitrary real or complex number, in many cases if the function f can be expanded into a power series, $f(x) = \sum_{n=0}^{\infty}(-1)^n \phi(n)$ for some function $\phi(n)$ that can also be defined for negative argument s+n=0 , then ‘Ramanujan Theorem’ asserts that for any value of ‘d’ this integral will be equal to $\frac{\pi\phi(-s)}{sin(s\pi)}\frac{(2\pi)^{d/2}}{\Gamma(d/2)} = I(d)$. (2)

Here it comes the first problem, for d=4 (space-time dimension) the integral I(d) would be divergent in many cases, if we recall the Gamma function with functional equation $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$ for s=d integer this function is divergent , hence ‘dimensional regularization’ introduces a pole or divergent quantity by expanding the Gamma function near $\varepsilon = 0$, $\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon)$, with γ being the Euler-Mascheroni constant.

The link of ‘dimensional regularization’ with our method of calculation of integrals come when we wish to calculate divergent integrals of the form $\int_a^\infty dx f(x)$ as $x \to \infty$ which is just the case s=0 inside $\int_a^\infty dx f(x)x^{-s} = \hat{F}(1-s)$ a > 0 (Mellin transform) , the idea is to choose a big enough ‘s’ so the integrand $f(x)x^{-s} \approx \frac{1}{x^2}$ for big x making the integral finite , then expand the function into a Convergent Laurent series for |x| >0 in the form $f(x)=\sum_{n=-\infty}^{\infty} c_n x^n \quad c\in R$ and perform the term-by-term integration ,after that we could use the identity based on Euler-Maclaurin summation formula (valid for real or complex 's')

$$\int_a^\infty x^{m-s}dx=\frac{m-s}{2}\int_a^\infty x^{m-1-s}dx+\zeta(s-m)+H(s,m,a)$$
$$-\sum_{r=1}^{\infty}\frac{B_{2r}\Gamma(m-s+1)}{(2r)!\Gamma(m-2r+2-s)}\left(a^{m-2r+1-s}+(m-2r+1-s)\int_a^\infty x^{m-2r-s}dx\right) \qquad a>0 \quad (3)$$

$$H(s,m,a)=a^{m-s}-\sum_{k=1}^{a}k^{m-s}+\sum_{r=1}^{\infty}\frac{B_{2r}\Gamma(m-s+1)}{(2r)!\Gamma(m-2r+2-s)}a^{m-s-2r+1} \qquad H(s,m,0)=0 \quad (4)$$

To express any divergent integral $\int_a^\infty x^{m-s}dx=\sum_i a_i(s)\zeta(s-i)$, that is, we can relate using formula (3) a divergent integral $\int_a^\infty x^{m-s}dx$ to a divergent series $\zeta(s-m)=\sum_{i=0}^{\infty} i^{m-s}$, $s\to 0^+$, in practice if 'm' is an integer as s tends to 0 due to the poles of the Gamma function $\Gamma(x)$ at negative integers (3) is no longer infinite and 'r' runs only from $r=1$ to $r=\left[\frac{m+2}{2}\right]$, so for integer (positive) 'm' we can relate the divergent integral, using formula (1) and (3) to a linear combination involving , Bernoulli Numbers B_{2r} and negative values of Zeta function $\zeta(-r)$ $r=0,1,2,3,....,m$, from the definition of the functional equation for the Riemann Zeta, $\zeta(-2n)=0 \quad n\in N$.

There is still a problem with the logarithmic divergence $\int\frac{dx}{x}$, since the Riemann zeta has a pole at s=1 (Harmonic series) , this is the main drawback of our theory, we can regularize every divergent integral by using the negative values of the Riemann zeta , but we can not regularize with our method (unless we introduce some correction) the logarithmic divergencies, this is the objective of our next section in the paper.

- *The logarithmic divergence of $\zeta(1-s)$ with $s\to 0^+$:*

The case $\zeta(1-s)$, $s\to 0^+$ can not be handeld with formulae (1) or (3) due to a pole of Riemann Zeta at s= 1 , this is the main serious drawback of our regularization method , we can not use Euler-Maclaurin summation formula to regularize $\int_0^\infty\frac{dx}{x+a}$ due to the fact that the sum $\sum_{n=0}^{\infty}\frac{1}{n+a}$ is divergent, there are several possibilities to get a finite result indeed

- The finite part $F.p\int_0^{\infty}\frac{dx}{x+a}=-\log(a)$ in the sense of 'Hadamard integral' [10] exists and depends on the value of 'a'
- We could differentiate respect to 'a' to get $-\int_0^{\infty}\frac{dx}{(x+a)^2}=-\frac{1}{a}$, then performing integration with respect to 'a' again we get $-\log(a)+c_a$, here c_a is a physical parameter (mass, charge,..) that must be determined by experiments to fit the calculations.

Both methods yield to the same result if we set $c_a=0$, which can be imposed as an scale-invariance (invariance under a change of variable y=ax for any $a\in R$) of the integral $\int_0^{\infty}\frac{dx}{x}=2c_a=0$, this results has the physical meaning that quantities (mass, charge) should not depend on the Energy scale we are performing the researchs.

The last (but not the less powerful) method is based on two results of mathematical analysis, the Abel-Plana formula, relating an integral to a series

$$\sum_{n=0}^{\infty} f(n)-\int_0^{\infty} f(x)dx=\frac{1}{2}f(0)+i\int_0^{\infty}\frac{dt}{e^{2\pi t}-1}.\left(f(it)-f(-it)\right) \qquad (5)$$

And the 'Ramanujan resummation' (Finite part) of the series $\sum_{n=0}^{\infty}\frac{1}{n+a}\to-\frac{\Gamma'}{\Gamma}(a)$ [ ] . Combining both formulae we can give a finite meaning to the logarithmic divergence

$$-\frac{\Gamma'}{\Gamma}(a)-\left(\int_0^{\infty}\frac{dx}{x+a}\right)_R=\frac{1}{2a}+2\int_0^{\frac{\pi}{2}}\frac{d\theta}{e^{2\pi a\tan(\theta)}-1}.\tan(\theta) \qquad (6)$$

Expression (6) is finite for $a\neq 0$ and can be regarded as the 'regularized' value of the logarithmic divergent integral $\int_0^{\infty}\frac{dx}{x+a}$

One of the advantages of dealing with logarithmic divergences is that in case we can regularize $\int_0^{\infty}\frac{dx}{x+a}=-\log(a)+c_a$ for any fixed 'a' different from 0 adding and substracting terms and using the fact that $\int_0^{\infty}\left(\frac{1}{x+a}-\frac{1}{x+b}\right)dx=\log(b/a)$, then for any 'b' different from a and bigger than 0 $\int_0^{\infty}\frac{dx}{x+b}=-\log(b)+c_a$, an special case is whenever a=0 and we have that $\int_0^\infty \frac{dx}{x} = 2c_a$, then if we impose the physical condition of scale-invariance again $c_a = 0$,as the only possible alternative.

A mathematical justification of this comes from 'Ramanujan resummation' of the Harmonic series $\sum_{n=1}^\infty \frac{1}{n} = \gamma = 0.57721...$ (Euler-Mascheroni constant) , however we know from the definition of the constant above that $\lim_{k\to\infty}\left(\sum_{n=1}^k \frac{1}{n} - \int_1^k \frac{dx}{x}\right) = \gamma$, so Ramanujan resummation that takes only the finite part of the series into account ignores the infinite quantity $\log(\infty)$, a justification of this is the following, if we want to regularized the integral $\int_0^\infty dx \frac{1}{x+a}$, we can integrate with respect to 'a' and then use the fact (regularization of functional determinants) $\sum_{n=0}^\infty \log(n+a) = -\partial_s \zeta_H(0,a)$, using the expression for teh Hurwitz zeta along s=0 $\frac{\partial \zeta_H(0,a)}{\partial s} = \log\Gamma(a) - \log\sqrt{2\pi}$ and finally differentiation with respect to 'a' gives $\sum_{n=0}^\infty \frac{1}{n+a} = -\frac{\Gamma'}{\Gamma}(a)$, now using Euler-Maclaurin summation formula again we can get a finite regularization for $\int_0^\infty dx \frac{1}{x+a}$

- *Zeta regularization and the cut-off* Λ *:*

Another method involved in the calculation of divergent integrals is the following, for any integral $\int_a^\infty dx f(x)$ we can introduce a 'cut-off' $\int_a^\Lambda dx f(x)$ to make it finite , and after calculations we take the limit $\Lambda \to \infty$, expanding the function f into a convergent Laurent series for |x| > a , we can use the Euler-Maclaurin formulae to stablish a recurrence between the powers of this cut-off

$$I(m,\Lambda) = (m/2)I(m-1,\Lambda) + \zeta(-m) - \sum_{r=1}^\infty \frac{B_{2r}}{(2r)!} a_{mr}(m-2r+1)I(m-2r,\Lambda) \quad (7)$$

$a_{mr} = \frac{\Gamma(m+1)}{\Gamma(m-2r+2)}$, and $I(m,\Lambda) = \int_0^\Lambda x^m dx = \frac{a^{m+1}}{m+1} + \int_a^\Lambda x^m dx$, from here we can obtain finite results, even in the limit $\Lambda \to \infty$, for example

$$\left(\int_a^\infty dx \frac{x^2}{1+x}\right)_R = \frac{-1-\zeta(0)}{2} + \zeta(-1) - \log(a) + a - \frac{a^2}{2} + \sum_{j=3}^\infty \frac{(-1)^j}{j-2} a^{2-j} \quad a > 1 \quad \Lambda \to \infty \quad (8)$$

We simply have expanded for a >1 the integrand into the convergent Laurent series $x-1+\frac{1}{x}+\sum_{j=3}^{\infty}(-1)^j x^{1-j}$ and performing term-by-term integration to obtain formulae (8)

This cut-off regularization can be achieved imposing the following condition $\zeta_H(-s,\Lambda)=0$ as $\Lambda\to\infty$ for every positive 's' , this is connected with the expression for the sum of the s-th powers of 'n' $\sum_{i=1}^{\Lambda-1} i^s=\zeta(-s)-\zeta_H(-s,\Lambda)$

Here we have introduced the Hurwitz Zeta function $\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}=\zeta_H(s,a)$, $\zeta_H(s,1)=\zeta(s)$ we can define also a similar 'regularization' for this Hurwitz Zeta via a functional equation.

Equation (8) tells us an important thing, although initially $\int_a^{\infty}\frac{dx}{x+1}x^2$ is DIVERGENT, using our model of Zeta regularization we have managed to give it a finite value using $\zeta(0)=-\frac{1}{2}$ and $\zeta(-1)=-\frac{1}{12}$, this is the main difference with 'dimensional regularization' that has remaining infinities due to the poles of the Gamma function $\Gamma(z)$, however using the Zeta regularization plus the Ramanujan resummation for the Harmonic series $\sum_{n=0}^{\infty}\frac{1}{(n+a)}$, we only have finite quantities, this is the main advantage (including the simplicity of applying our method) of our Zeta regularization algorithm for divergent integrals.

However, we think we made a mistake in paper [6] and the real value for the case m=0 should be $\int_0^{\infty}dx=1+\zeta(0)$, since for f(x)=1 the term $\frac{f(0)+f(\infty)}{2}=1$, the Euler-Maclaurin summation formula with a cutt-off gives

$$\int_0^{\Lambda}x^m dx=\frac{m}{2}\int_0^{\Lambda}x^{m-1}dx+\zeta(-m)-\zeta(-m,\Lambda)-\sum_{r=1}^{\infty}\frac{B_{2r}m!(m-2r+1)}{(2r)!(m-2r+1)!}\int_0^{\Lambda}x^{m-2r}dx \qquad (9)$$

The powers of Λ coming from $\int_0^{\Lambda}x^m dx$ should cancel the ones coming from the Bernoulli polynomials $\zeta_H(-m,x)=-\frac{B_{m+1}(x)}{m+1}$, form m=0 $\int_0^{\Lambda}dx=1+\zeta(0)+\Lambda-\frac{1}{2}$, for m=1 $\int_0^{\Lambda}xdx=\frac{1+\zeta(0)}{2}+\frac{\Lambda^2}{2}+\zeta(-1)-\frac{1}{4}+\frac{1}{12}$, and so on so we think that the regularization $\int_0^{\infty}dx=1+\zeta(0)$ is better defined for the divergent integral with m=0 , from the expression above we get the well-known results $\zeta(0)=-\frac{1}{2}$ and $\zeta(-1)=-\frac{1}{12}$

, so our method of regularization for integrals is compatible with the usual zeta regularization algorithm. This value '1' could be viewed as the Residue in the limit $s \to 0$ inside $\frac{s}{2}\int_0^\infty \frac{dx}{x^{1-s}} = 1$ obtained from $\zeta(1+s) = \frac{1}{s} + \gamma$, this 'residue' appears setting m=0 inside the recurrence (1) , so from (1) we can also expect $\int_0^\infty dx = 1 + \zeta(0)$

- *Scale invariance and Zeta regularization:*

One could ask if there is any physical justification for the Zeta regularization method of series and integrals $\sum_{n=1}^{\infty} n^s = \zeta(-s)$, the main idea is that when we are performing physical calculations of parameters such as mass 'm' these paremeters can not rely on the energy scale Λ , this means that if we have two energy scales related by a dilation $\overline{\Lambda} = a\Lambda$, $a \in R^+$ then the integral $\int_0^\Lambda dxf$ must be independent of the cut-off as $\Lambda \to \infty$, so if we use the Euler-Maclaurin summation formula in order to express divergent integrals as a linear combination of divergent series $\sum_{n=1}^{\infty} n^r$ r = 0,1,2,3,..... we should impose the condition $\sum_{n=1}^{\Lambda-1} n^s = \sum_{n=1}^{\overline{\Lambda}-1} n^s \quad \forall s$, if we consider the case of infinitesimal dilations $\overline{\Lambda} = (1+\varepsilon)\Lambda$ as $\varepsilon \to 0$ from the definition $\sum_{i=1}^{\Lambda-1} i^s = \zeta(-s) - \zeta_H(-s,\Lambda)$ and imposing scale invariance for the series

$$\sum_{n=1}^{\overline{\Lambda}-1} n^s - \sum_{n=1}^{\Lambda-1} n^s = 0 = \zeta(-s) - \zeta(-s) - \zeta_H(-s,\overline{\Lambda}) + \zeta_H(-s,\Lambda) = 0 =$$
$$\lim_{\varepsilon\to\infty}\left(-\zeta_H(-s,(1+\varepsilon)\Lambda) + \zeta_H(-s,\Lambda)\right) = -\Lambda \frac{\partial \zeta_H(-s,\Lambda)}{\partial \Lambda} = s\Lambda\zeta_H(-s+1,\Lambda) = 0 \qquad (9)$$

In the last step inside (9) we have used the shift property of the Hurwitz Zeta function, the last condition in (9) is equivalent to the statement $\zeta_H(-q,\Lambda) = 0$ as $\Lambda \to \infty$, with $q = s-1$, this is precisely why $\sum_{n=1}^{\infty} n^k = \zeta(-k)$, since the value $\zeta(-k)$ does not depend on the energy scale or cut-off Λ , equation (9) is a physical justification of our procedure of Zeta regularization of integrals , we impose the condition $\zeta_H(-q,\Lambda) = 0$ as $\Lambda \to \infty$, in order our theory to have an scale invariance , for the case q=0 or q=-1 (Harmonic series) the terms $\log(1+\varepsilon)$ and $\varepsilon\Lambda$ vanish as $\varepsilon \to 0$, providing the linear term $\varepsilon\Lambda \to 0$ as $\varepsilon \to 0$. For 'm' integer $-\Lambda \frac{\partial \zeta_H(-m,\Lambda)}{\partial \Lambda} = \Lambda \frac{\partial B_m(\Lambda)}{\partial \Lambda}$, and the dependence of hte series $\sum_{n=1}^{\Lambda-1} n^k$ on the cut-off is made explicit using the Hurwitz zeta function , for m=0 there is a extra factor of 1 so for our regularized integral we must take $\int_0^\infty dx = 1+\zeta(0)$ since inside Euler-Maclaurin the upper limit is $\Lambda - 1$

CONCLUSIONS AND FINAL REMARKS:

After having read the paper and learned about Zeta regularization of divergent integrals or concepts such as 'Ramanujan resummation' one can have the wrong idea that this methods are only mere curiosities without applications , the value $\zeta(-1) = -\frac{1}{12}$, is used in string theory, and years before the value $\zeta(-3) = -\frac{1}{120}$ was used by Casimir and others to calculate the force between two plates , the 'Zeta regularization' algorithm is beyond being a simple trick in order to sweep the infinities its importance come specially in Quantum Field Theory (QFT) when one does perturbation theory in order to calculate masses and other physical parameters , even though we have only considered integrals of the form $\int_0^\infty x^m dx$ for positive 'm' this technique can be applied (by means of a change of variable $x = \frac{1}{y}$) to integrals with a divergence as $x \to 0$, $\int_0^\infty \frac{dx}{x^\alpha}$ with $\alpha \ge 2$ and integer, this is justified by the fact that if we introduce a cut-off Λ , then $\int_0^\infty \frac{dx}{x^\alpha} = \lim_{\Lambda\to\infty}\left(\int_{\Lambda^{-1}}^{\Lambda} \frac{dx}{x^\alpha}\right)$ (again with the same condition for parameter alpha) , hence $\int_{\Lambda^{-1}}^{\Lambda} \frac{dx}{x^\alpha} = -\int_{\Lambda^{-1}}^{\Lambda} x^{\alpha-2}dx$ as the cut-off goes to infinite, so we can aplly (1) , (3) or (8).

For further reading on what is Zeta regularization or the sum of divergent series beyond this paper I would strongly recommend 'Divergent series' [7] by G.H Hardy (a bit old fashioned but easy to read) .One of the best introductions to Zeta regularization is found on E. Elizalde's " Zeta function techniques with applications" [5] , for a survey on "Ramanujan resummation" and other stuff discovered by Ramanujan on divergent series we have references [2] and [3] as best sources, other formulae introduced are kindly explained on the reference books by T. Apostol (Number theory) explaining what is exactly the analytic continuation for the Hurwitz and Riemann Zeta [1] , another interesting books about the Zeta function, regularization and Mathematical Analysis are [4] , [8] and [9] the 'Dimensional regularization' method including examples is explained in detail in the paper by Gerard t'Hooft and M. Veltmann or in the excellent book by Zeidler with several practical mathematical examples [10]. Also in Zeidler's book one can see the further applications of divergent series to QFT and the importance of Zeta regularization procedure in String theory or in calculations of the 'Casimir effect'. Note also the equivalence (except for a minus sign) of the Zeta regularization expansion near the pole s=1 and the expansion of the Gamma function near its poles

$$\zeta(1+\varepsilon) = \frac{1}{\varepsilon} + \gamma + O(\varepsilon) \qquad \Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon) \qquad \varepsilon \to 0+ \quad (10)$$

Zeta regularization also avoids the 'unphysical' change of dimension (dimension is an analytic parameter and it is only set to d=4 after final calculations) used in dimensional regularization.

Finally we recall that our method can be extended for multiple integrals in the form

$$\int d^4k_1.....\int d^4k_n J(k_1,k_2,.........,k_n)\prod_{i=1}^{n}\frac{1}{\left(m_i^2+k_i^2\right)} \qquad (11)$$

The idea is to consider (11) as an integral on R^{4n} , then we make a change of variable to 4n- dimensional polar coordinates to rewrite (11) as

$$\int d\Omega_{4n-1}\int_0^\infty drr^{4n-1}J(r,\Omega_{4n-1})\prod_{i=1}^{n}\frac{1}{\left(m_i^2+r^2 f_i(\Omega_{4n-1})\right)} \qquad r^2=\sum_{i=1}^{n}k_i^2 \qquad (12)$$

Again we would add and substract terms of the form $\int d\Omega_{4n-1}\int_0^\infty drr^k g_k(\Omega_{4n-1})$ in order to make (11) converge, the integrals $\int_0^\infty drr^k$ again can be regularized using (1) except for the case k=-1. The pole at s=1 of $\zeta(s)$ seems a big deal in our method to regularize infinities a final thought about this will be the following, let us suppose we could expand $f(r)=\frac{1}{r^{-1}+a}=\sum_{m=-\infty}^{\infty}\frac{D^{k+\alpha}f(0)}{\Gamma(\alpha+k+1)}r^{k+\alpha}$ $0<\alpha<1$ [11] valid for example for Re(r) <1 , with $\alpha\in R$ an arbitrary real number different form an integer, then making the change of variable $r\to 1/r$ into the generalized Taylor series (assuming we can generalize the derivatives to fractional arbitrary orders) and taking the integral term-by-term we will find expressions of the form $\int_1^\infty r^{k\pm\alpha}dr$ k >0, if α is different from an integer, then using formulae (1) and (3), (4) we can relate this divergent integrals to negative values of zeta function and get the series $\sum_{k=0}^{\infty}a_k\zeta(-k\mp\alpha)$, the problem here is that (1) and (3) are no longer finite recurrence equations , also since negative values of Zeta function are related to Bernoulli's number, then by stirling formula $|B_{2n}|\approx 4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}$ we can (unfortunately) expect that the sum $\sum_{k=0}^{\infty}a_k\zeta(-k\mp\alpha)$ will be divergent and will only be well-defined in the sense of Borel-resummation

$$\int_0^\infty dte^{-t}g(t) \text{ with } \sum_{k=0}^{\infty}\frac{a_k}{k!}\zeta(-k\mp\alpha)t^k=g(t) \qquad (13)$$

So by performing a Taylor/Laurent series with fractional powers $x^{k+\alpha}$ we can avoid the pole at s=1 replacing it with the evaluation (regularization) of a divergent power series, such (13) note that although recurrence (3) will have an infinite number of terms, for m >1 we can use the identity $\int_1^\infty \frac{dx}{x^m} = \frac{1}{m-1}$ hence only a finite number of divergent integrals will appear inside (3)

APPENDIX A: RAMANUJAN RESUMMATION AND FINITE PART INTEGRAL

In order to calculate the divergent integral as $x \to 0$ $\int_0^1 \frac{\varphi(x)}{x^k} dx$ with $\varphi(x) \in C^k[0,1]$,

we define the Truncated Taylor Polynomial of order ‘k’ $T_k[\varphi(0)](x) = \sum_{i=0}^{k} \frac{D^i \varphi(0)}{i!} x^i$

$$\int_0^1 \frac{\varphi(x)}{x^k} dx = \lim_{\varepsilon \to 0} \left(\int_\varepsilon^1 \frac{\varphi(x) - T_k[\varphi(0)](x)}{x^k} dx + \sum_{i=0}^{k} \frac{D^i \varphi(0)}{i!(i+1)} \left(1^{i+1} - \varepsilon^{i+1}\right) - \log(\varepsilon) \right) \qquad (14)$$

Hadamard’s definition of the finite part is just dropping down the terms $\log \varepsilon$ and ε^{-m} $m \in R$ as $\varepsilon \to 0$ to get the ‘finite’ value

$$F.p\left(\int_0^1 \frac{\varphi(x)}{x^k} dx \right) = \int_0^1 \frac{\varphi(x) - T_k[\varphi(0)](x)}{x^k} dx + \sum_{i=0}^{k} \frac{D^i \varphi(0)}{i!(i+1)} \qquad (15)$$

A better definition and a generalization for further functions can be found on Zeidler [10] , a few examples are $F.p\int_0^a \frac{1}{x} dx = \log a$, $F.p\int_0^a \frac{1}{x^k} dx = \frac{a^{1-k}}{1-k}$, $F.p\int_0^\infty x^m dx = 0$

If we combine this definition of Hadamard integral and the Euler-Maclaurin summation

$$\int_0^{\varepsilon^{-1}} x^m dp = \frac{\varepsilon^{-m}}{2} + \sum_{i=1}^{\left[\varepsilon^{-1}\right]} i^m - \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} a_{mr} \varepsilon^{-m+2r-1} \qquad a_{mr} = \frac{\Gamma(m+1)}{\Gamma(m-2r+2)} \qquad (16)$$

For ‘m’ being a positive integer , if we consider the divergent integral in Hadamard’s sense then $\int_0^\infty x^m dx = 0$ and drop the terms ε^{-k} ,except $\varepsilon^0 = 1$ formula (12) gives

$\sum_{i=1}^{\infty} i^m = -\frac{B_{m+1}}{m+1} = \zeta(-m)$, which is just the definition of Zeta regularization for a divergent series $m \neq -1$, for the case of the logarithmic integral $\int_1^\infty \frac{1}{x} dx$ taken as ‘finite part’ is 0 , so the Euler-Maclaurin formula for this case gives $\sum_{n=1}^{\infty} \frac{1}{n} = \gamma = 0.55721...$ (‘Euler-Mascheroni’ constant) this last equation can not be obtained by Zeta regularization , however it can be seen from the definition of the Harmonic series $\zeta(1)=\gamma-\log(\varepsilon)$ that in the sense of 'Finite part' $F.p\left(\zeta(1)\right)=\gamma$

The Finite part definition for the Mellin transform $\int_0^\infty x^{s-1}dx=0 \ \ \forall s\in R$ inserted in the Euler-Maclaurin summation gives an analytic method to calculate the sum of divergent series $\sum_{n=1}^\infty n^{s-1}$ via the 'regularized' expression $\sum_{n=1}^\infty n^{s-1}\begin{cases}\zeta(1-s) & s\in R-\{0\} \\ \gamma=0.55721... & s=0\end{cases}$ this is precisely the definition Ramanujan gave [3] for the sum of the series $\sum_{n=1}^\infty n^{s-1}$.

To end this appendix we shall give two more justifications of the identity $\int_0^\infty \frac{dx}{x+a}=-\log a$ to regularize the logarithmic divergence

- If we insert the finite part of the logarithmic divergence $\int_0^\infty \frac{dx}{x+a}=-\log a$, and use the Euler-Maclaurin summation formula, we recover the Ramanujan resummation value for the Harmonic series $\sum_{n=0}^\infty \frac{1}{n+a}=-\frac{\Gamma'}{\Gamma}(a)$
- If we replace $\int_0^\infty \frac{dx}{x+a}$ by $\int_{-\infty}^\infty \frac{H(x-a)dx}{x}$, $H(x-a)\begin{cases}0 & \text{x< a} \\ 1 & \text{x > a}\end{cases}$ incluidng the Heaviside step function , using the convolution theorem for Fourier integrals we would get $\int_{-\infty}^\infty \frac{H(x-a)dx}{x}=-\log(a)+f(0)$, with $f(x)=\frac{d\,|x|}{dx}$ $f(0)=0$ in this case so our 'regularization' is consistent , another equivalent formulation of this would be to differentiate with respect to 'a' inside $H(x-a)$ to get a Dirac delta distribution $-\delta(x-a)$, using the property $\int_{-\infty}^\infty dxf(x)\delta(x-a)=f(a)$ and integrating again with respect to a with zero constant of integration we get the required result $-\log(a)$

- If we integrate with respect to 'a' inside $\int_0^\infty \frac{dx}{x+a}$ we get the still divergent integral $\int_0^\infty \log(x+a)dx$, using Euler-Maclaurin summation formula plus zeta regularization of the series $\sum_{n=0}^\infty \log(n+a)=-\partial_s\zeta_H(0,a)$ and differentiation with respect to variable 'a' we can also get a finite result for logarithmic integrals based on the regularization for the Harmonic series $\sum_{n=0}^\infty \frac{1}{n+a}=-\frac{\Gamma'}{\Gamma}(a)$

- Adding a counterterm inside the Lagrangian to provide integrals of the form $\int \frac{d^4k}{(m^2+k^2)^2}$, which cancel logarithmic divergencies in d=4 can help to get finite measurable results by 'renormalization'
- Another method is to replace or 'regularize' our integral $\int_0^{\Lambda} \frac{dx}{(x+a)^{1+\varepsilon}}$ with $\varepsilon \to 0^+$, $\Lambda \to \infty$ and $\Lambda^{\varepsilon} \to 1$ so $\varepsilon \log \Lambda \to 0$, in this case using the Power series expansion $x^k = \sum_{n=0}^{\infty} \frac{k^n \log^n(x)}{n!}$, Integral Calculus gives

$$\int_0^{\Lambda} \frac{dx}{(x+a)^{1+\varepsilon}} = \lim_{\Lambda\to\infty,\ \varepsilon\to 0^+} \frac{\Lambda^{\varepsilon}-a^{\varepsilon}}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{1-a^{\varepsilon}}{\varepsilon} = -\log(a) \qquad (17)$$

To Resume, if we have the divergent integral $\int_a^{\infty} x^{m-s}dx$ for $m \in N \cup \{0\}$, using formula (3) to regularize this divergent integral using the values $\zeta(s-r)$ r = 0,1,2,... as $s \to 0^+$, for m = -1 , we can use either the Abel-Plana summation formula (5) or the regularized value $\int_0^{\infty} \frac{dx}{(x+a)^{1+\varepsilon}} \to$ -log(a) or $-\log a + c_a$, with 'c' and adjustable free parameter, which would be the only one free parameter in our theory. The final question is why does this work ? , the idea is that perhaps whenever taking physical consideration using the Zeta reguarlization algorithm we can 'substract' the infinite from the sum or integral to obtain finite results , in our example the integral in question

$$\left(\int_a^{\infty} dx \frac{x^2}{1+x}\right)_R \to \int_a^{\infty} dx \frac{x^2}{1+x} - F(\infty) = \frac{-1-\zeta(0)}{2} + \zeta(-1) - \log(a) + a - \frac{a^2}{2} + \sum_{j=3}^{\infty} \frac{(-1)^j}{j-2} a^{2-j}$$
(18)

With $\frac{dF}{dx} = \frac{x^2}{x+1}$, so we have the equation $\frac{d}{d\Lambda}\left(\int_a^{\Lambda} dx \frac{x^2}{x+1} - F(\Lambda)\right) = 0 \quad \Lambda \to \infty$ from the Euler-Maclaurin summation formula $F(\infty) = c_{-1}\log\infty + \sum_n c_n \infty^n$, this is perhaps why the zeta regularization algorithm works . In the case of a logarithmic divergence we could take the Abel-Plana formula to obtain the finite-part

$$f.p\left(\sum_{n=0}^{\infty} f(n)\right) - \int_0^{\infty} dx f(x) = \frac{f(0)}{2} + i\int_0^{\infty} dx \frac{f(ix)-f(-ix)}{e^{2\pi x}-1} \qquad \int_0^{\Lambda} dx f(x) \approx \log\Lambda + c \qquad (19)$$

Here f.p means that we ignore the divergent term $\log\Lambda$ inside the series, for example $f.p\left(\sum_{n=1}^{\infty} n^{-1}\right) = \gamma = 0.57721..$ (Harmonic series) , this would come apparently from imposing the condition for every 's' $\zeta(-s+1,\Lambda) = 0 \quad \Lambda \to \infty$

APPENDIX B: ZETA REGULARIZATION AND AN INTEGRAL REPRESENTATION FOR THE RIEMANN ZETA FUNCTION

Riemann found the following integral representation for his Zeta function

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i}\oint_C \frac{s^{z-1}}{e^{-s}-1}ds \qquad \Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \quad (20)$$

This expression is valid for every complex 's' except s=1 (pole) and C is a closed curve which encircles all the poles of $e^{-s}-1$, a direct application of Cauchy's residue theorem gives

$$\zeta(z) = \sum_{m=-\infty}^{\infty}(2\pi i m)^{z-1}\Gamma(1-z) = \left(\sum_{m=-\infty}^{-1}(2\pi i m)^{z-1} + \sum_{m=1}^{\infty}(2\pi i m)^{z-1}\right)\Gamma(1-z) \quad (21)$$

However these series are divergent for z >0 , if we apply zeta regularization to the divergent sums inside (A.2)

$$\left(\sum_{m=-\infty}^{-1}(2\pi i m)^{z-1} + \sum_{m=1}^{\infty}(2\pi i m)^{z-1}\right)\Gamma(1-z) = \left(2\pi\right)^{z-1}\Gamma(1-z)\zeta(1-z)(i^{z-1}+(-i)^{z-1}) \quad (22)$$

From the Euler's formula for cosine we obtain for (A.3) the functional equation

$$\zeta(z) = \Gamma(1-z)\zeta(1-z)\sin\left(\frac{\pi z}{2}\right) = \frac{\pi}{\cos\left(\frac{\pi z}{2}\right)}.\frac{\zeta(1-z)}{\Gamma(z)}\left(2\pi\right)^{z-1} \quad (23)$$

So, from (A.4) one precisely obtains the functional equation for the Riemann Zeta $2\cos\left(\frac{\pi z}{2}\right)\left(2\pi\right)^{-z}\Gamma(z)\zeta(z) = \zeta(1-z)$, this fact is another empirical support of why Zeta regularization should be takne seriously in order to obtain 'regularizations' of divergent series and how one can use 'formal' method in analysis in order to prove rigorous results.

APPENDIX C: ZETA REGULARIZATION AND THE SERIES $\sum_{n=0}^{\infty} n^k \quad k>0$

In this paper , we have used the zeta regularization algorithm to get a regularization (equivalent to the dimensional regularization by T'Hooft and Veltmann) for the integrals $\int_0^{\infty} x^m dx$, the question could be could we recover the zeta regularization value

$\sum_{n=0}^{\infty} n^k = \zeta(-k)$ from Euler-Maclaurin summation ? , if we set $f(x) = x^m$ for any 'm' and use the cut-off regularization for the sum $\sum_{n=0}^{\Lambda-1} n^k = \zeta(-k) - \zeta_H(-k,\Lambda)$

$$\frac{\Lambda^{m+1}}{m+1} = \frac{\Lambda^m}{2} + \zeta(-m) - \zeta_H(-m,\Lambda) - \sum_{r=1}^{\infty} \frac{B_{2r} m!(\Lambda^{m-2r+1} - 0^{m-2r+1})}{(2r)!(m-2r+1)!} \qquad (24)$$

Howerver inside (21) if m-2r+1=0 then we can formally put $0^{m-2r+1} = 1$, so after the cancellation of the powers of Λ we are left with the identity $\zeta(-m) = \frac{-B_{m+1}}{m+1}$, which is the usual zeta regularization results , this way of reasoning was known to Ramanujan [3] who also applied to the Harmonic series to get the finite result $\sum_{n=0}^{\infty} \frac{1}{n+a} = -\frac{\Gamma'}{\Gamma}(a)$ for every positive 'a' , this result can be obtained imposing the condition that the measured quantities should not depend on the value of Λ so $\zeta_H(-s+1,\Lambda) = 0$, using this point of view, Ramanujan resummation could be considered as the 'inverse' procedure of our method to regularize divergent integrals $\int_0^{\infty} x^m dx$. Also if we think about Zeta regularization as the limit , valid for every 'm', $m \neq -1$, $\lim_{\Lambda\to\infty} \left(\sum_{i=1}^{\Lambda-1} i^m + \zeta_H(-m,\Lambda) \right) = \zeta(-m) = reg\left(\sum_{i=1}^{\infty} i^m \right)$ perhaps we can think of our zeta-regularization algorithm to get finite results as the limit

$$\lim_{\Lambda\to\infty} \left(\int_0^{\Lambda} x^m dx + \sum_{i=0}^{m} c_i^{\,m} \zeta_H(-i,\Lambda) \right) = \sum_{i=0}^{m} c_i^{\,m} \zeta(-i) = reg\left(\int_0^{\infty} x^m dx \right) \qquad c_i^{\,m} \in R \qquad (25)$$

Where the coefficients $\{c_i^{\,m}\}$ can be obtained from solving the recurrence given by the Euler-Maclaurin formula in (1) , in both cases the idea is to 'substract' the point of infinity on the expressions $\sum_{i=1}^{\infty} i^m$ and $\int_0^{\infty} x^m dx$

References:

[1] Apostol T. " *Introduction to Analytic Number Theory"*, (1976) Springer-Verlag, New York. ISBN 0-387-90163-9

[2] Berndt, Bruce C. "*Ramanujan's Notebooks, Part I"*. New York: Springer, 1985. ISBN 0-387-96110-0.

[3] Delabaere E., "*Ramanujan's Summation, Algorithms Seminar" 2001–2002*, F. Chyzak (ed.), INRIA, (2003), pp. 83–88.

[4] Estrada R. Kanwal R. *"A distributional approach to asymptotics "* Boston Birkhäuser Birkhäuser (2002) **ISBN: 0817641424**

[5] Elizalde E. ; "*Zeta-function regularization is well-defined"*, Journal of Physics **A 27** (1994), L299-304.

[6] Garcia J.J ; *"Zeta Regularization applied to the problem of Riemann Hypothesis and the Calculation of divergent integrals "* e-print avaliable at http://www.wbabin.net/science/moreta23.pdf

[7] Hardy, G. H. (1949), "*Divergent Series"* , Oxford: Clarendon Press.

[8] Milton A. And Stegun I.A *"Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables"*, ISBN 978-0-486-61272-0

[9] Saharian, A. A. *"The Generalized Abel-Plana Formula. Applications to Bessel Functions and Casimir Effect*." http://www.ictp.trieste.it/~pub_off/preprints-sources/2000/IC2000014P.pdf.

[10] Zeidler E. *"Quantum Field theory Vol .1;A Bridge between Mathematicians and Physicists"* Springer (2009) ISBN: 978-3-540-34762-0

[11] Watanabe Y. "*Notes on the Generalized derivative of Riemann-Liouville and its applications to Leibniz's formula II"* Tokhu general Mathematics Journal 34