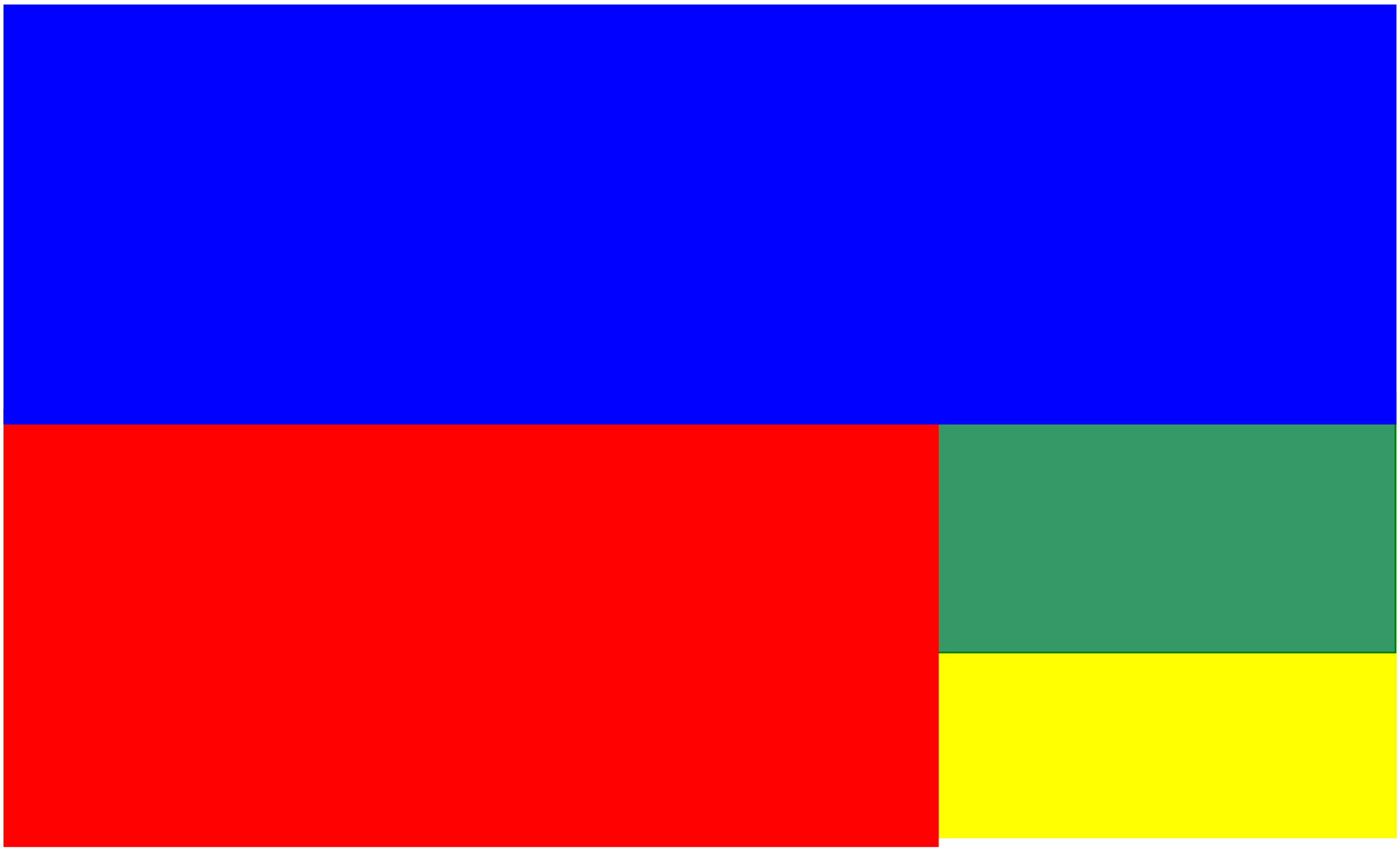
Generalized Partitions and New Ideas On Number Theory and Smarandache Sequences



Smarandache Repeatable Reciprocal Partition of Unity
{2, 3, 10, 15}
$1/2 + 1/3 + 1/10 + 1/15 = 1$

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GENERALIZED PARTITIONS AND SOME NEW IDEAS ON NUMBER THEORY AND SMARANDACHE SEQUENCES

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Editor's Note

This book arose out of a collection of papers written by Amarnath Murthy. The papers deal with mathematical ideas derived from the work of Florentin Smarandache, a man who seems to have no end of ideas. Most of the papers were published in Smarandache Notions Journal and there was a great deal of overlap. My intent in transforming the papers into a coherent book was to remove the duplications, organize the material based on topic and clean up some of the most obvious errors. However, I made no attempt to verify every statement, so the mathematical work is almost exclusively that of Murthy.

I would also like to thank Tyler Brogla, who created the image that appears on the front cover.

Charles Ashbacher

Table of Contents

Chapter 1
Smarandache Partition Functions

Section 1
Smarandache Partition Sets, Sequences and Functions

Unit fractions are fractions where the numerator is 1 and the denominator is a natural number. Our first point of interest is in determining all sets of unit fractions of a certain size where the sum of the elements in the set is 1.

Definition: For n > 0, the Smarandache Repeatable Reciprocal partition of unity for n (SRRPS(n)) is the set of all sets of n natural numbers such that the sum of the reciprocals is 1. More formally,

$$SRRPS(n) = \{ x \mid x = (a_1, a_2, \ldots, a_n) \text{ where } \sum_{r=1}^{n} 1/a_r = 1 \}.$$

$f_{RP}(n)$ = order of the set SRRPS(n).

For example,

SRRPS(1) = { (1) }, $f_{RP}(1) = 1$.
SRRPS(2) = { (2,2)}, $f_{RP}(2) = 1$. (1/2 + 1/2 = 1).
SRRPS(3) = { (3,3,3), (2,3,6), (2,4,4)}, $f_{RP}(3) = 3$.
SRRPS(4) = { (4,4,4,4), (2,4,6,12),(2,3,7,42),(2,4,5,20),(2,6,6,6),(2,4,8,8), (2,3,12,12),(4,4,3,6),(3,3,6,6),(2,3,10,15),(2,3,9,18)}, $f_{RP}(4) = 14$.

Definition: The Smarandache Repeatable Reciprocal Partition of Unity Sequence is the sequence of numbers

SRRPS(1), SRRPS(2), SRRPS(3), SRRPS(4), SRRPS(5), . . .

Definition: For n > 0, the Smarandache Distinct Reciprocal Partition of Unity Set (SDRPS(n)) is SRRPS(n) where the elements of each set of size n must be unique. More formally,

$$SRRPS(n) = \{ x \mid x = (a_1, a_2, \ldots, a_n) \text{ where } \sum_{r=1}^{n} 1/a_r = 1 \text{ and } a_i = a_j \Leftrightarrow i = j \}.$$

$f_{DP}(n)$ = order of SDRPS(n).

For example:

SRRPS(1) = { (1) }, $f_{RP}(1) = 1$.
SRRPS(2) = { }, $f_{RP}(2) = 0$.

$SRRPS(3) = \{ (2,3,6)\}$, $f_{RP}(3) = 1$.
$SRRPS(4) = \{ (2,4,6,12),(2,3,7,42),(2,4,5,20), (2,3,10,15),(2,3,9,18)\}$, $f_{RP}(4) = 5$.

Definition: The Smarandache Distinct Reciprocal partition of unity sequence is the sequence of numbers $f_{DP}(n)$.

Theorem:

$$f_{DP}(n) \geq \sum_{k=3}^{n-1} f_{DP}(k) + (n^2 - 5n + 8)/2, \quad n > 3.$$

Proof:
The inequality will be established in two steps.

Proposition A
For every n, there exists a set of n distinct natural numbers, the sum of whose reciprocals is 1.

Proof of proposition A:
The proof is by induction on n.

Basis step:
$n = 1$, $1/1 = 1$.

Inductive step:
Assume that the proposition is true for r. Then there is a set of distinct natural numbers

$a_1 < a_2 < \ldots < a_r$

such that

$1/a_1 + 1/a_2 + \ldots + 1/a_r = 1$.

Since $1/k = 1/(k+1) + 1/(k(k+1))$, we can modify the original sequence to get

$a_1 < a_2 < \ldots < a_{r-1} < a_{r-1} + 1 < a_{r-1}(a_{r-1} + 1) = a_{r+1}$

where the sum of the reciprocals is still one. Therefore, the proposition is also true for r+1.

Therefore, by the principle of mathematical induction, the expression is true for all n.

Note: If $a_1, a_2, \ldots, a_{n-1}$ are n-1 distinct natural numbers given by the relation

$a_1 = 2$
$a_2 = a_1 + 1$

$a_3 = a_1a_2 + 1$
.
.
.
$a_t = a_1a_2 \ldots a_{t-1} + 1 = a_{t-1}(a_{t-1} - 1) + 1$
.
.
.
$a_{n-2} = a_1a_2 \ldots a_{n-3} + 1$
$a_{n-1} = a_1a_2 \ldots a_{n-2}$

then the numbers form a set of (n-1) distinct natural numbers such that the sum of the reciprocals is one.

Definition: A set of numbers defined in the manner of the sequence above is called a Principle Reciprocal Partition.

Note: It is easy to prove that the elements of a Principle Reciprocal Partition satisfy the congruences

$a_{2t} \equiv 3 \bmod(10)$ and $a_{2t+1} \equiv 7 \bmod(10)$, for $t \geq 1$.

Consider the principle reciprocal partition for n-1 numbers. Each a_t contributes one to f_{DP} (n) if broken into $a_t + 1$, $a_t (a_t + 1)$ except for $t = 1$. (since 2, if broken into 3 and 6, yields $1/2 = 1/3 + 1/6$, the number 3 is repeated and the condition of all numbers being distinct is not fulfilled). There is a contribution of n - 2 from the principle set to f_{DP} (n). The remaining f_{DP} (n- 1) -1 members (excluding the principle partition) of SDRPS(n-1) would contribute at least one each to f_{DP} (n) (breaking the largest number in each such set into two parts) . The contribution to f_{DP} (n) is therefore at least

$$n-2 + f_{DP}(n-1) - 1 = f_{DP}(n-1) + n - 3$$

$$f_{DP}(n) \geq f_{DP}(n-1) + n - 3.$$

Also for each member $(b_1, b_2, \ldots, b_{n-1})$ of SDRPS(n-1) there exists a member of SDRPS(n) i.e. $(2, 2b_1, 2b_2, \ldots, 2b_{n-1})$ as we can see that

$$1 = (1/2)(1+ 1/b_1 + 1/b_2 + \ldots + 1/b_{n-1}) = 1/2 + 1/2b_1 + \ldots + 1/2b_{n-1}.$$

In this way, there is a contribution of $f_{DP}(n-1)$ to $f_{DP}(n)$.

Taking into account all these contributions to $f_{DP}(n)$, we get

$$f_{DP}(n) \geq f_{DP}(n-1) + n - 3 + f_{DP}(n-1)$$

$$f_{DP}(n) \geq 2f_{DP}(n-1) + n - 3$$

$f_{DP}(n) - f_{DP}(n\text{-}1) \geq f_{DP}(n\text{-}1) + n - 3.$

Replacing n in the last formula by n-1, n-2 and so forth, we get

$f_{DP}(n\text{-}1) - f_{DP}(n\text{-}2) \geq f_{DP}(n\text{-}2) + n - 4$

$f_{DP}(n\text{-}2) - f_{DP}(n\text{-}3) \geq f_{DP}(n\text{-}3) + n - 5$

.

.

.

$f_{DP}(4) - f_{DP}(3) \geq f_{DP}(3) + 1.$

Summing up all the above inequalities, we have

$$f_{DP}(n) - f_{DP}(3) \geq \sum_{k=3}^{n-1} f_{DP}(k) + \sum_{r=1}^{n-1} r$$

$$f_{DP}(n) \geq \sum_{k=3}^{n-1} f_{DP}(k) + ((n-3)(n-2))/2 + 1$$

$$f_{DP}(n) \geq \sum_{k=3}^{n-1} f_{DP}(k) + (n^2 - 5n + 8)/2,\ n > 3.$$

Remark: It should be possible to come up with a stronger result, as I believe that there should be more terms on the right. The reason for this belief will be clear from the following theorem.

Theorem: Let m be a member of an element of SRRPS(n), say $m = a_k$, from $(a_1, a_2, \ldots, a_n)$ and by definition

$$\sum_{k=1}^{n} 1/a_k = 1.$$

The m contributes [(d(m) + 1)/2] elements to SRRPS(n+1), where [ ] represents the integer value and d(m) the number of divisors of m.

Proof: For each divisor d of m, there is a corresponding divisor m/d = d'.

Case-I: m is not a perfect square. Then d(m) is even and there are d(m)/2 pairs of the type (d,d') such that dd' = m.

Consider the following identity

$1/(p*q) = 1/(p(p+q)) + 1/(q(p+q))$

for each divisor pair (d,d') of m we have the following breakup

$1/(d*d') = 1/(d(d+d')) + 1/(d'(d+d'))$.

Hence, the contribution of m to SRRPS(n+1) is d(m)/2. As d(m) is even
d(m)/2 = [(d(m)+1)/2] as well.

Case-II m is a perfect square. In this case d(m) is odd and there is a divisor pair
$d=d' = m^{1/2}$. This will contribute one to SRRPS(n+1).The remaining (d(m) –1)/2 pairs of distinct divisors would each contribute one, making the total contribution ((d(m) –1)/2). Therefore, the total number in this case would be

$(d(m) -1)/2 + 1 = (d(m) +1)/2 =[(d(m) +1)/2]$.

Hence m contributes [(d(m) + 1)/2] elements to SRRPS(n+1) and the proof is complete.

Remarks:

1) The total contribution to SRRPS(n+1) by any element of SRRPS(n) is

$\sum [(d(a_k) + 1)/2]$

where each a_k is considered only once irrespective of its repeated occurrence.

2) For SDRPS(n+1), the contribution by any element of SDRPS(n) is given by

$\sum [d(a_k)/2]$

because the divisor pair d = d' = a_k does not contribute.

Hence, the total contribution of SDRP(n) to generate SDRPS(n+1) is the summation over all the elements of SDRPS(n).

$$\sum_{f_{DP}(n)} [\sum_{k=1}^{n} (d(a_k)/2)].$$

Generalization:
It is possible to generalize these results by considering the following identity

$$\frac{1}{pqr} = \frac{1}{pq(p+q+r)} + \frac{1}{qr(p+q+r)} + \frac{1}{rp(p+q+r)}$$

which also suggests

$$\frac{1}{b_1 b_2 \ldots b_r} = \sum_{k=1}^{r} \left[\left(\prod_{t=1,t\neq k}^{r} b_t \right) \left(\sum_{s=1}^{r} b_s \right) \right]^{-1}$$

It is easy to establish this identity by summing up the elements on the right side. From this formula, the contribution of the elements of SDRPS(n) to SDRPS(n+r) can be evaluated if answers to the following open problems can be found.

Open problems:

(1) In how many ways can a number be expressed as the product of 3 of its divisors?

(2) In general in how many ways can a number be expressed as the product of r of its' divisors?

(3) In how many ways can a number be expressed as the product of its divisors?

Any attempt to find answers to the above questions leads to the need for the generalization of the theory of the partition function.

Section 2

A Program to Determine the Number of Smarandache Distinct Reciprocal Partitions of Unity of a Given Length

The previous section introduced the Smarandache distinct reciprocal partition of unity and demonstrated some properties. In this section, a computer program written in the C language will be presented.

```
/* This is a program for finding number of distinct reciprocal partitions of unity of a
given length written by  K Suresh, Software expert, IKOS , NOIDA , INDIA. */

#include<stdio.h>
#include<math.h>
unsigned long TOTAL;

FILE* f;
long double array[100];
```

```
unsigned long count = 0;

void try(long double prod, long double sum, unsigned long pos)
{
        if( pos == TOTAL - 1 )
        {
                // last element..
                long double diff = prod - sum;
                if( diff == 0 ) return;

                array[pos] = floorl(prod / diff);
                if( array[pos] > array[pos-1] && array[pos] * diff == prod )
                {
                  fprintf(f, "(%ld) %ld", ++count,(unsigned long)array[0]);
                  int i;
                  for(i = 1; i < TOTAL; i++) fprintf(f,", %ld", (unsigned long)array[i]);
                  fprintf(f, "\n");
                  fflush(f);
                }
                return;
        }
        long double i;
        if( pos == 0)
          i = 1;
        else
          i = array[pos-1];

        while(1) {
         i++;
         long double new_prod = prod * pow(i, TOTAL-pos);
         long double new_sum = (TOTAL-pos) * (new_prod / i);
         unsigned long j;
         for(j = 0; j < pos; j++) new_sum += new_prod / array[j];
         if( new_sum < new_prod )
                 break;

         new_prod = prod * i;
         array[pos] = i;
         new_sum = prod + sum * i;
         if( new_sum >= new_prod ) continue;

         try(new_prod, new_sum, pos+1);
        }
        return;
}
```

```
main()
{
	printf("Enter no of elements ?");
	scanf("%ld", &TOTAL);
	char fname[256];
	sprintf(fname, "rec%ld.out", TOTAL);
	f = fopen(fname, "w");
	fprintf(f, "No of elements = %ld.\n", TOTAL);

	try(1, 0, 0);
	fflush(f);
	fclose(f);
	printf("Total %ld solutions found.\n", count);

	return 0;
}
```

Using this program, the following table of data was accumulated.

Length	Number of Distinct Reciprocal Primes
1	1
2	0
3	1
4	6
5	72
6	2320
7	245765

Section 3

A Note On Maohua Le's Proof Of Murthy's Conjecture On Reciprocal Partition Theory

In [4], Maohua Le attempted to prove the conjecture that there are infinitely many disjoint sets of positive integers the sum of whose reciprocals is equal to one. He misunderstood the conjecture, perhaps due to inadequate wording. What he actually proved was the proposition that for every n there exists a set of n distinct natural numbers the sum of whose reciprocals is one.

I would like to clarify and restate the conjecture using the following example.

Let A = { $a_1, a_2, \dots, a_r$ } and B = { $b_1, b_2, \dots, b_s$ } be two sets such that

$$\sum_{k=1}^{r} 1/a_k = 1 = \sum_{k=1}^{s} 1/b_k$$

with $A \cap B = \emptyset$.

The conjecture is that there are infinitely many disjoint sets of the type A or B.

Example:

A = { 2, 3, 7, 42 }, B = { 4, 5, 6, 8, 9, 12, 20, 72}.

Section 4

Generalization of Partition Function, Introduction of the Smarandache Factor Partition

The partition function P(n) is defined as the number of ways that a positive integer can be expressed as the sum of positive integers. Two partitions are not considered different if they differ only in the order of their summands. Many results concerning the partition function were discovered using analytic functions by Euler, Jacobi, Hardy, Ramanujan and others. Other properties of the function involving congruences are also known.

In the previous sections, the concept of the Smarandache Reciprocal Partitions of unity was introduced. One of the problems considered was the number of ways in which a number can be expressed as the product of its divisors. In this section, we will examine some generalizations of the concept of partitioning a number.

Definition: Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be a set of natural numbers and $p_1, p_2, \ldots, p_r$ a set of arbitrary primes. The Smarandache Factor Partition (SFP) of $(\alpha_1, \alpha_2, \ldots, \alpha_r)$, $F(\alpha_1, \alpha_2, \ldots, \alpha_r)$ is defined as the number of ways in which the number

$$N = p_1^{\alpha 1}\ p_2^{\alpha 2} \ldots p_r^{\alpha r}$$

can be expressed as the product of its' divisors.

Example: With the set of primes, (2,3), F(1,2) = 4 , as

$N = 2^1 * 3^2 = 18$

(1) N = 18 (2) N = 2*9 (3) N = 3*6 (4) N = 2*3*3.

It is a consequence of the definition of SFP and factors that $F(\alpha_1,\alpha_2) = F(\alpha_2,\alpha_1)$ and in general the order of the α_i in $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ is immaterial. Also, the primes $p_1, p_2, \ldots, p_r$ can be chosen arbitrarily.

Theorem: $F(m) = P(m)$, where $P(m)$ is the number of addition partitions of m.

Proof: Let p be any prime, $N = p^m$ and $m = x_1 + x_2 + \ldots + x_n$ be an addition partition of m. Then,

$N = (p^{x1})(p^{x2}) \ldots (p^{xn})$ is a SFP of N, i.e. each partition of m contributes one SFP.

Also, let one of the SFP of N be

$N = (N_1)(N_2) \ldots (N_k)$. Each N_i has to be of the form $N_i = p^{ai}$.

Let $N_1 = p^{a1}$, $N_2 = p^{a2}$, $\ldots$, p^{ak}. Then

$N = (p^{a1})(p^{a2}) \ldots (p^{ak}) = p^{(a1+a2+\ldots+ak)} \Rightarrow m = a_1 + a_2 + \ldots + a_k$

which gives a partition of m. Obviously, each SFP of N gives one unique partition of m. Therefore, since each approach yields one SFP,

$F(m) = P(m)$.

Theorem: $F(\alpha,1) = \sum_{k=0}^{\alpha} P(k)$

Proof: Let $N = p_1^{\alpha} p_2$, where p_1, p_2 are arbitrarily chosen primes.

Case 1: Writing $N = (p_2)\, p_1^{\alpha}$, keeping p_2 as a separate entity, (one of the factors in the factor partition of N), by the previous theorem would yield $P(\alpha)$ Smarandache factor partitions.

Case 2: Writing $N = (p_1p_2)p_1^{\alpha-1}$ keeping (p_1p_2) as a separate entity (one of the factors of SFP of N) would yield $P(\alpha-1)$ SFPs.

.

.

.

Case r: In general, writing $N = (p_1^r\, p_2)p^{\alpha-r}$ and keeping $(p_1^r\, p_2)$ as a separate entity would yield $P(\alpha-r)$ SFPs.

Contributions towards F(N) in each of the cases are mutually disjoint as $p_1^r p_2$ is unique for a given r, which ranges from 0 to α, which is exhaustive.

Therefore,

$$F(\alpha,1) = \sum_{r=0}^{\alpha} P(\alpha-r)$$

Let $\alpha - r = k$, $r = 0 \Rightarrow k = \alpha$, $r = \alpha \Rightarrow k = 0$.

$$F(\alpha,1) = \sum_{k=\alpha}^{0} P(k)$$

$$F(\alpha,1) = \sum_{k=0}^{\alpha} P(k)$$

which completes the proof.

Examples:

I. $F(3) = P(3) = 3$, Let $p = 2$, $N = 2^3 = 8$.

(1) N = 8, (2) N = 4*2, (3) N = 2*2*2.

II. $F(4,1) = \sum_{k=0}^{4} P(k) = P(0) + P(1) + P(2) + P(3) + P(4)$

$= 1 + 1 + 2 + 3 + 5 = 12.$

Let $N = 2^4 * 3 = 48$, where $p_1 = 2$, $p_2 = 3$.

The Smarandache factor partitions of 48 are

(1) N = 48
(2) N = 24 * 2
(3) N = 16 * 3
(4) N = 12 * 4
(5) N = 12 * 2 * 2
(6) N = 8 * 6
(7) N = 8 * 3 * 2
(8) N = 6 * 4 * 2
(9) N = 6 * 2 * 2 * 2
(10) N = 4 * 4 * 3
(11) N = 4 * 3 * 2 * 2
(12) N = 3 * 2 * 2 * 2 * 2

Definitions: For simplicity, we will use the following abbreviations:

1) $F(\alpha_1, \alpha_2, \ldots, \alpha_r) = F'(N)$, where

$N = p_1^{\alpha 1} p_2^{\alpha 2} \ldots p_r^{\alpha r} \ldots p_n^{\alpha n}$

and p_r is the rth prime. In other words, $p_1 = 2$, $p_2 = 3$, and so forth.

2) For the case where N is a square-free number,

$$\underbrace{F(1, 1, 1, \ldots, 1)}_{\text{n ones}} = F(1\#n).$$

Examples:
$F(1\#2) = F(1,1) = F'(6) = 2$, $6 = 2*3 = p_1*p_2$.
$F(1\#3) = F(1,1,1) = F'(2*3*5) = F'(30) = 5$.

Definition: The Smarandache Star Function $F'^{*}(N)$ is defined as

$$F'^{*}(N) = \sum_{d|N} F'(d_r), \text{ where } d_r \mid N.$$

In other words, F'^{*} = sum of $F'(d_r)$ over all the divisors of N.

Example: N = 12, the divisors are 1, 2, 3, 4, 6 and 12.

$F'^{*}(12) = F'(1) + F'(2) + F'(3) + F'(4) + F'(6) + F'(12) = 1 + 1 + 1 + 2 + 2 + 4 = 11$.

Theorem:

$$F'^{*}(N) = F'(Np),\ (p,N) = 1, \text{ p is prime.}$$

Proof: By definition

$$F'^{*}(N) = \sum_{d|N} F'(d_r), \text{ where } d_r \mid N.$$

Consider d_r, a divisor of N, clearly $Np = d_r(Np/d_r)$. Let $(Np/d_r) = g(d_r)$, then
$N = d_r*g(d_r)$ for any divisor d_r of N, $g(d_r)$ is unique, i. e.

$d_i = d_j \iff g(d_i) = g(d_j)$.

Considering $g(d_r)$ as a single term (an entity, not further split into factors) in the SFP of N*p, one gets $F'(d_r)$ SFPs. Each $g(d_r)$ contributes $F'(d_r)$ factor partitions.

The condition p does not divide N implies that $g(d_i) \neq d_j$ for any divisor, because p divides $g(d_i)$ and p does not divide d_j.

This ensures that the contribution towards F'(Np) from each $g(d_r)$ is distinct and there is no repetition. Summing over all $g(d_r)$'s we get

$$F'(Np) = \sum_{d|N} F'(d_r)$$

or

$$F'^{*}(N) = F'(Np)$$

which completes the proof of the theorem.

The result that $F'^{*}(N) = F'(Np)$, $(p,N) = 1$, p is prime can be used to prove that

$$F(\alpha,1) = \sum_{k=0}^{\alpha} P(k).$$

To see this, start with $N = p^{\alpha}p^{1}$ then $F(\alpha,1) = F'(p^{\alpha} * p_1)$ and from the previous theorem

$$F'(p^{\alpha} * p_1) = F'^{*}(p^{\alpha}) = \sum_{d|p^{\alpha}} F'(d_r).$$

The divisors of p^{α} are $p^0, p^1, \ldots, p^{\alpha}$, so

$$F'(p^{\alpha} p_1) = F'(p^0) + F'(p^1) + \ldots + F'(p^{\alpha}) =$$

$$P(0) + P(1) + P(2) + \ldots + P(\alpha-1) + P(\alpha) \text{ or}$$

$$F(\alpha,1) = \sum_{k=0}^{\alpha} P(k).$$

Theorem: $F(1\#(n+1)) = \sum_{r=0}^{n} {}^{n}C_r \, F(1\#r)$

where ${}^{n}C_r$ is the number of ways r objects can be selected from a set of n objects without regard to order.

Proof: By the previous theorem, $F'(Np) = F'^{*}(N)$, where p does not divide N. Consider the case $N = p_1p_2 \ldots p_n$. We have $F'(N) = F(1\#n)$ and $F'(Np) = F(1\#(n+1))$ as p does not divide N. Combining these expressions, we have

$$F(1\#(n+1)) = F'^{*}(N). \quad (*)$$

The number of divisors of N of the form $p_1p_2 \ldots p_r$, (containing exactly r primes) is ${}^nC_r$. Each of the ${}^nC_r$ divisors of the type $p_1p_2 \ldots p_r$ has the same number of SFPs, namely F(1#r). Hence

$$F'^*(N) = \sum_{r=0}^{n} {}^nC_r\, F(1\#r) \quad (**)$$

From (*) and (**), we have

$$F(1\#(n+1)) = \sum_{r=0}^{n} {}^nC_r\, F(1\#r)$$ and the proof is complete.

Note that F(1#n) is the nth Bell number.

Examples:
F(1#0) = F'(1) = 1
F(1#1) = F'(p_1) = 1
F(1#2) = F'(p_1p_2) = 2
F(1#3) = F'($p_1p_2p_3$) = 5

(i) $p_1\, p_2\, p_3$
(ii) $(p_1\, p_2) * p_3$
(iii) $(p_1p_3) * p_2$
(iv) $(p_2p_3) * p_1$
(v) $p_1 * p_2 * p_3$

Applying the previous theorem to F(1#4)

$$F(1\#4) = \sum_{r=0}^{3} {}^nC_r\, F(1\#r)$$

F(1#4) = ${}^3C_0$ F(1#0) + ${}^3C_1$ F(1#1) + ${}^3C_2$ F(1#2) + ${}^3C_3$ F(1#3)

= 1*1 + 3 * 1 + 1 * 5 = 15

F(1#4) = F' (2*3*5*7) = F'(210) = 15.

(i) 210
(ii) 105 * 2
(iii) 70 * 3
(iv) 42 * 5
(v) 35 * 6
(vi) 35 * 3 * 2
(vii) 30 * 7

(viii) 21 * 10
(ix) 21 * 5 * 2
(x) 15 * 14
(xi) 15 * 7 * 2
(xii) 14 * 5 * 3
(xiii) 10 * 7 * 3
(ixv) 7 * 6 * 5
(xv) 7 * 5 * 3 * 2.

Along similar lines, one can obtain

F(1#5) = 52, F(1#6) = 203, F(1#7) = 877, F(1#8) = 4140, F(1#9) = 21,147.

Definition:

$$F'^{**}(N) = \sum_{d_r|N} F'^{*}(d_r)$$

where d_r ranges over all the divisors of N.

If N is a square-free number with n prime factors, we will use the notation

$$F'^{**}(N) = F^{**}(1\#N).$$

Examples:

$$F'^{**}(p_1p_2p_3) = F^{**}(1\#3) = \sum_{d_r|N} F'(d_r)$$

$$= {}^3C_0\, F'^{*}(1) + {}^3C_1\, F'^{*}(p_1) + {}^3C_2\, F'^{*}(p_1p_2) + {}^3C_3\, F'^{*}(p_1p_2p_3)$$

$$F^{**}(1\#3) = 1 + [3F'(1) + F'(p_1)] + 3[F'(1) + 2F'(p_1) + F'(p_1p_2)] + [F'(1) + 3F'(p_1)\ 3F'(p_1p_2) + F'(p_1p_2p_3)]$$

$$F^{**}(1\#3) = 1 + 6 + 15 + 15 = 37.$$

An interesting observation is

(1) F**(1#0) + F(1#1) = F(1#2)
or
F**(1#0) + F*(1#0) = F(1#2)

(2) F**(1#1) + F(1#2) = F(1#3)
or
F**(1#1) + F*(1#1) = F(1#3)

(3) F**(1#5) + F(1#6) = F(1#7)

or

$$F^{**}(1\#5) + F^{*}(1\#5) = F(1\#7).$$

which suggests the possibility that

$$F^{**}(1\#n) + F^{*}(1\#n) = F(1\#(n+2)).$$

A stronger proposition

$$F'(Np_1p_2) = F'^{*}(N) + F'^{**}(N)$$

is established in the next theorem.

Definition:

$$F'^{n*}(N) = \sum_{d_r|N} F'^{(n-1)*}(d_r) \qquad n > 1$$

where $F'^{*}(N) = \sum_{d_r|N} F'(d_r)$

and d_r ranges over all the divisors of N.

Theorem:

$$F'(Np_1p_2) = F'(N) + F'^{**}(N).$$

Proof: By previous theorem, we know that

$$F'(Np_1p_2) = F'^{*}(Np_1).$$

Let $d_1, d_2, d_3, \ldots, d_n$ be all the divisors of N. The divisors of Np_1 would be

$d_1, d_2, d_3, \ldots, d_n$
$d_1p_1, d_2p_1, d_3p_1, \ldots, d_np_1$

$$F'^{*}(Np_1) = [F'(d_1) + F'(d_2) + \ldots F'(d_n)] +[F'(d_1p_1) + F'(d_2p_1) + \ldots + F'(d_np_1)]$$

$$= F'^{*}(N) + [F'^{*}(d_1) + F'^{*}(d_2) + \ldots + F'^{*}(d_n)].$$

$$F'^{*}(Np_1) = F'^{*}(N) + F'^{**}(N) \text{ (by definition)}$$

$$= F'^{*}(N) + F'^{2*}(N).$$

Which completes the proof.

Theorem:

$$F'(Np_1p_2p_3) = F'^*(N) + 3F'^{2*}(N) + F'^{3*}(N).$$

Proof: By a previous theorem, we have $F'(Np1p2p3) = F'^*(Np1p2)$.
Let $d_1, d_2, d_3, \ldots, d_n$ be all the divisors of N. The divisors of Np_1p_2 would be

$d_1, d_2, d_3, \ldots, d_n$
$d_1p_1, d_2p_1, d_3p_1, \ldots, d_np_1$
$d_1p_2, d_2p_2, d_3p_2, \ldots, d_np_2$
$d_1p_1p_2, d_2p_1p_2, d_3p_1p_2, \ldots, d_np_1p_2$.

Therefore,

$$\begin{aligned} F'^*(Np_1p_2) = {} & [F'(d_1) + F'(d_2) + F'(d_3) + \ldots + F'(d_n)] + \\ & [F'(d_1p_1) + F'(d_2p_1) + \ldots + F'(d_np_1)] + \\ & [F'(d_1p_2) + F'(d_2p_2) + \ldots + F'(d_np_2)] + \\ & [F'(d_1p_1p_2) + F'(d_2p_1p_2) + \ldots + F'(d_np_1p_2)] \end{aligned}$$

$$= F'^*(N) + 2[F'^*(d_1) + F'^*(d_2) + \ldots + F'^*(d_n)] + S$$

where $S = [F'(d_1p_1p_2) + F'(d_2p_1p_2) + \ldots + F'(d_np_1p_2)]$.

Applying the previous theorem, we get

$F'(d_1p_1p_2) = F'^*(d_1) + F'^{**}(d_1)$
$F'(d_2p_1p_2) = F'^*(d_2) + F'^{**}(d_2)$
.
.
.
$F'(d_np_1p_2) = F'^*(d_n) + F'^{**}(d_n)$.

Summing up these expressions, we have

$$S = F'^{2*}(N) + F'^{3*}(N).$$

Substituting this value of S and also taking

$$F'^*(d_1) + F'^*(d_2) + \ldots + F'^*(d_n) = F'^{2*}(N)$$

we get,

$$F'(Np_1p_2p_3) = F'^*(N) + 2F'^{2*}(N) + F'^{2*}(N) + F'^{3*}(N)$$

$F'(Np_1p_2p_3) = F'^{*}(N) + 3F'^{2*}(N) + F'^{3*}(n)$.

This completes the proof.

This result, which is a beautiful pattern, can be further generalized.

Section 5

Open Problems and Conjectures On the Factor/Reciprocal Partition Theory

In this chapter, we present some open problems and conjectures related to the Smarandache Factor Partition function.

Problems:

1) To derive a formula for SFPs of a given length m for $N = p^a q^a$ for

any value of a.

2) To derive a formula for SFPs of

$$N = p_1^2 p_2^2 p_3^2 \ \ldots \ p_r^2$$

3) To derive a formula for SFPs of a given length m of

$$N = p_1^a p_2^a p_3^a \ \ldots \ p_r^a .$$

4) To derive a reduction formula for $p_a q_a$ as a linear

combination of $p^{a-r} q^{a-r}$ for $r = 0$ to $a-1$.

Derive similar reduction formulae for (2) and (3) as well.

5) In general, in how many ways can a number be expressed as the product of its divisors?

6) Every positive integer can be expressed as the sum of the reciprocal of a finite number of distinct natural numbers. (in infinitely many ways.).

Define a function $R_m(n)$ as the minimum number of natural numbers required for such an expression.

7) Determine if every natural number can be expressed as the sum of the reciprocals of a set of natural numbers that are in Arithmetic Progression.

(8). Let $\Sigma\, 1/r \le n \le \sum 1/(r+1)$

where $\sum 1/r$ stands for the sum of the reciprocals of the first r natural numbers and let

$S_1 = \sum 1/r$

$S_2 = S_1 + 1/(r+k_1)$ such that $S_2 + 1/(r+k_1+1) > n \ge S_2$

$S_3 = S_2 + 1/(r+k_2)$ such that $S_3 + 1/(r+k_2+1) > n \ge S_3$

and so on.

Continuing this sequence, after a finite number of iterations m,

$S_{m+1} + 1/(r+k_m) = n$.

Remark: The validity of problem (6) is deducible from problem (8) .

9). (a) There are infinitely many disjoint sets of natural numbers the sum of whose reciprocals is unity.

(b) Among the sets mentioned in (a), there are sets which can be organized in an order such that the largest element of any set is smaller than the smallest element of the next set.

Definition: The Smarandache Factor Partition Sequence is defined in the following way:

T_n = factor partition of $n = F'(n)$.

For example,

$T_1 = 1$, $T_8 = 3$, $T_{12} = 4$ etc.

SFPS is given by

1,1,1,2,1,2,1,3,2,2,1,4,1,2,2,5, 1,4,1,4,2,2,1,7,2, . . .,

Definition: For n a natural number, let S be the smallest number such that $F'(S) = n$. These numbers will be called Vedam Numbers and the sequence formed by the Vedam numbers is the Smarandache Vedam Sequence.

The Smarandache Vedam Sequence is given as follows: $Tn = F'(S)$

1, 4, 8, 12, 16, -?- , 24 , . . .

Note: There is no number whose factor partition is 6, hence the question mark in that position. We will call such numbers Dull numbers. The reader is encouraged to explore the distribution, frequency and other properties of Dull numbers.

Definition: A number is said to be a Balu number if it satisfies the relation $d(n) = F'(n) = r$ and is the smallest such number.

Examples: 1, 16, 36 are all Balu numbers.

$d(1) = F'(1) = 1$, $d(16) = F'(16) = 5$, $d(36) = F'(36) = 9$.

Each Balu number ≥ 16, generates a Balu Class $C_B(n)$ of numbers having the same canonical form satisfying the equation

$d(m) = F'(m)$.

For example:

$C_B(16) = \{ x \mid x = p^4 , p \text{ is a prime.}\} = \{ 16, 81, 256, \ldots\}$. Similarly
$C_B(36) = \{ x \mid x = p^2 q^2 , p \text{ and } q \text{ are primes.}\}$

Conjecture:
10): There are only a finite number of Balu Classes. If this is true, find the largest Balu number.

Section 6

A General Result on the Smarandache Star Function

Theorem:

$$F'(N@1\#n) = F'(Np_1p_2 \ldots p_n) = \sum_{m=0}^{n} [a_{(n,m)} F'^{m*}(N)]$$

where

$$a_{(m,n)} = (1/m!) \sum_{k=1}^{m} (-1)^{m-k} \, {}^mC_k \, k^n$$

Proof:
Let the divisors of N be $d_1, d_2, \ldots, d_k$. Take the divisors of $(Np_1p_2\ldots p_n)$ and arrange them as follows:

$d_1, d_2, \ldots, d_k$ call these type 0
$d_1p_i, d_2p_i, \ldots, d_kp_i$ call these type 1

$d_1p_ip_j$, $d_2p_ip_j$, $d_3p_ip_j$, . . . , $d_kp_ip_j$ call these type 2

. . . .

$d_1p_i\ \ p_j$, $d_2p_i\ \ p_j$, $d_3p_i\ \ p_j$, . . . , $d_kp_i\ \ p_j$ where each term has t primes. Call these type t.

. . . .

$d_1p_1 \ldots p_n$, $d_2p_1 \ldots p_n$, . . . , $d_kp_1 \ldots p_n$, call these type n.

There are ${}^nC_0$ divisors of type 0.
There are ${}^nC_1$ divisors of type 1.
. . .
There are ${}^nC_t$ divisors of type t.
…
There are ${}^nC_n$ divisors of type n.

Let $Np_1p_2 \ldots p_n = M$. Then

$F^*(M) = {}^nC_0$[Sum of the factor partitions of all the divisors of row 0] +
${}^nC_1$[Sum of the factor partitions of all the divisors of row 1] +
${}^nC_2$[Sum of the factor partitions of all the divisors of row 2] +
. . . +
${}^nC_t$[Sum of the factor partitions of all the divisors of row t] +
. . . +
${}^nC_n$[Sum of the factor partitions of all the divisors of row n].

Consider the contributions of the divisor sets one by one.

Row 0 contributes

$F'(d_1) + F'(d_2) + \ldots + F'(d_n) = F'^*(N)$.

Row 1 contributes

$[F'(d_1p_1) + F'(d_2p_1) + F'(d_3p_1) + \ldots + F'(d_kp_1)]$

$= [F'^*(d_1) + F'^*(d_2) + F'^*(d_3) + \ldots + F'^*(d_k)]$

$= F'^{2*}(N)$.

Row 2 contributes

$[F'(d_1p_1p_2) + F'(d_2p_1p_2) + \ldots + F'(d_kp_1p_2)]$.

Applying the theorem $F'(Np_1p_2p_3) = F'^*(N) + 3F'^{2*}(N) + F'^{3*}(N)$ on each of the terms,

$F'(d_1p_1p_2) = F'^*(d_1) + F'^{**}(d_1)$
$F'(d_2p_1p_2) = F'^*(d_2) + F'^{**}(d_2)$
. . .

$F'(d_k p_1 p_2) = F'^{*}(d_k) + F'^{**}(d_k)$.

After summing these rows, we have

$F'^{2*}(N) + F'^{3*}(N)$.

At this point, we will write the coefficients in the form $a_{(n,r)}$, for example

$F'(N@1\#r) = a_{(r,1)}F'^{*}(N) + a_{(r,2)}F'^{2*}(N) + \ldots + a_{(r,t)}F'^{t*}(N) + \ldots + a_{(r,r)}F'^{r*}(N)$.

Consider row t, one divisor set

$d_1p_1p_2\ldots p_t,\ d_2p_1p_2\ldots p_t,\ \ldots,\ d_kp_1p_2\ldots p_t$

and we have

$F'(d_1@1\#t) = a_{(t,1)}F'^{*}(d_1) + a_{(t,2)}F'^{2*}(d_1) + \ldots + a_{(t,t)}F'^{t*}(d_1)$

$F'(d_2@1\#t) = a_{(t,1)}F'^{*}(d_2) + a_{(t,2)}F'^{2*}(d_2) + \ldots + a_{(t,t)}F'^{t*}(d_2)$

. . .

$F'(d_k@1\#t) = a_{(t,1)}F'^{*}(d_k) + a_{(t,2)}F'^{2*}(d_k) + \ldots + a_{(t,t)}F'^{t*}(d_k)$.

Summing up both sides column wise, we get for row t or divisors of type t one of the ${}^{n}C_{t}$ divisor sets contributes

$a_{(t,1)}F'^{2*}(N) + a_{(t,2)}F'^{3*}(N) + \ldots + a_{(t,t)}F'^{(t+1)*}(N)$.

Similarly for row n we have

$a_{(n,1)}F'^{2*}(N) + a_{(n,2)}F'^{3*}(N) + \ldots + a_{(n,n)}F'^{(t+1)*}(N)$.

All the divisor sets of type 0 contribute

${}^{n}C_{0}\ a_{(0,0)}\ F'^{*}(N)$ factor partitions.

All the divisor sets of type 1 contribute

${}^{n}C_{1}\ a_{(1,1)}\ F'^{*}(N)$ factor partitions.

All the divisor sets of type 2 contribute

${}^{n}C_{2}\ [\ a_{(2,1)}\ F'^{2*}(N) + a_{(2,2)}\ F'^{3*}(N)]$ factor partitions.

All the divisor sets of type 3 contribute

${}^{n}C_{3}\ [a_{(3,1)}\ F'^{2*}(N) + a_{(3,2)}\ F'^{3*}(N) + \ldots + a_{(3,3)}\ F'^{4*}(N)\ ]$

All of the divisor sets of type t contribute

$^nC_t\,[a_{(t,1)}\,F'^{2*}(N)\ +\ a_{(t,2)}\,F'^{3*}(N)\ +\ldots+\ a_{(t,t)}\,F'^{(t+1)*}(N)]$

All of the divisor sets of type n contribute

$^nC_n\,[a_{(n,1)}\,F'^{2*}(N)\ +\ a_{(n,2)}\,F'^{3*}(N)\ +\ldots+\ a_{(n,n)}\,F'^{(n+1)*}(N)]$

Summing up the contributions from the divisor sets of all types and considering the coefficient of $F'^{m*}(N)$ for m=1 to (n+1), we get the coefficient of

$F'^{*}(N) = a_{(0,0)} = a_{(n+1,1)}$.

The coefficient of $F'^{2*}(N)$

$= \ ^nC_1\,a_{(1,1)} + {}^nC_2\,a_{(2,1)}\ + {}^nC_3\,a_{(3,1)} + \ldots\ {}^nC_t\,a_{(t,1)}\ + \ldots + {}^nC_n\,a_{(n,1)}$

$= \ a_{(n+1,2)}$.

The coefficient of $F'^{3*}(N)$ is

$= \ ^nC_2\,a_{(2,2)} + {}^nC_3\,a_{(3,2)}\ + {}^nC_4\,a_{(4,2)} + \ldots\ {}^nC_t\,a_{(t,2)}\ + \ldots + {}^nC_n\,a_{(n,2)}$

$= a_{(n+1,3)}$.

The coefficient of $F'^{m*}(N)$ is

$a_{(n+1,m)} = \ ^nC_{m-1}\,a_{(m-1,m-1)} + {}^nC_m\,a_{(m,m-1)}\ + {}^nC_4\,a_{(4,2)} + \ldots + {}^nC_n\,a_{(n,m-1)}$.

The coefficient of $F'^{n+1*}(N)$ is

$a_{(n+1,m+1)} = \ ^nC_n\,a_{(n,n)} = {}^nC_n * {}^{n-1}C_{n-1}\,a_{(n-1,n-1)} = \ ^nC_n * {}^{n-1}C_{n-1} \ldots {}^2C_2\,a_{(1,1)} =$

1.

Consider $a_{(n+1,2)} =$

$= \ ^nC_1a_{(1,1)}\ + {}^nC_2a_{(2,1)}\ + \ldots + {}^nC_ta_{(t,1)}\ + \ldots + {}^nC_na_{(n,1)}$

$= {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$

$= 2^n - 1 = (2^{n+1} - 2)/2$.

Consider $a_{(n+1,3)} =$

$= {}^nC_2a_{(2,2)} + {}^nC_3a_{(3,2)} + \ldots + {}^nC_ta_{(t,2)} + \ldots + {}^nC_na_{(n,2)}$

$= {}^nC_2(2^1 – 1) + {}^nC_3(2^2 – 1) + {}^nC_4(2^3 – 1) + \ldots + {}^nC_n(2^{n-1} – 1)$

$= {}^nC_2(2^1) + {}^nC_3(2^2) + \ldots + {}^nC_n(2^{n-1}) – [{}^nC_2 + {}^nC_3 + \ldots + {}^nC_n]$

$= (1/2)[{}^nC_2(2^2) + {}^nC_3(2^3) + \ldots + {}^nC_n(2^n) - \sum_{r=0}^{n} {}^nC_r - {}^nC_1 - {}^nC_0$

$= (1/2) [\sum_{r=0}^{n} {}^nC_r 2^r - {}^nC_1 2^1 - {}^nC_0 2^0] – [2^n – n –1]$

$= (1/2) [3^n – 2n – 1] – 2^n + n + 1$

$= (1/2) [3^n – 2^{n+1} – 1]$

$= (1/3!)[1*3^{n+1} – 3*2^{n+1} + 3*1^{n+1} – (1) 0^{n+1}].$

Evaluating $a_{(n+1,4)}$

$a_{(n+1,4)} = {}^nC_3a_{(3,3)} + {}^nC_4a_{(4,3)} + \ldots + {}^nC_na_{(n,3)}$

$= {}^nC_3(3^2 + 1 – 2^3)/2 + {}^nC_4(3^3 + 1 – 2^4)/2 + \ldots + {}^nC_n(3^{n-1} + 1 – 2^n)/2$

$= (1/2)[3^2 * {}^nC_3 + 3^3 * {}^nC_4 + \ldots + 3^{n-1} * {}^nC_n] + ({}^nC_3 + {}^nC_4 + \ldots + {}^nC_n)$

$- \quad ({}^nC_3 2^3 + {}^nC_4 2^4 + \ldots + {}^nC_n 2^n]$

$= (1/2) [(1/3) [\sum_{r=0}^{n} {}^nC_r * 3^r – 3^2 * {}^nC_2 – 3\ {}^nC_1 - {}^nC_0] +$

$(\sum_{r=0}^{n} {}^nC_r - {}^nC_2 - {}^nC_1 - {}^nC_0) - (\sum {}^nC_r * 2^r – 2^2 * {}^nC_2 – 2\ {}^nC_1 - {}^nC_0)]$

$= (1/2) [(1/3) (4^n – 9n(n-1)/2 – 3n – 1) + (2^n – n(n-1)/2 – n – 1) –$

$(3^n – 4n(n-1)/2 – 2n – 1)]$

$a_{(n+1,4)} = (1/4!) [(1) 4^{n+1} – (4) 3^{n+1} + (6) 2^{n+1} – (4) 1^{n+1} + 1(0)^{n+1}].$

From the pattern that we observe, it appears that the general formula is

$$a_{(n,r)} = (1/r!) \sum_{k=0}^{r} (-1)^{r-k} * {}^rC_k k^n$$

which we will proceed to establish by induction.

Assume that the following expression is true for r and all n > r

$$a_{(n,r)} = (1/r!) \sum_{k=1}^{r+1} (-1)^{r-k} * {}^{r}C_k\, k^{n+1}.$$

From this, the goal is to derive

$$a_{(n+1,r+1)} = (1/(r+1)!) \sum_{k=0}^{r} (-1)^{r+1-k} * {}^{r+1}C_k\, k^{n+1}.$$

We have

$$a_{(n+1,r+1)} = {}^{n}C_r\, a_{(r,r)} + {}^{n}C_{r+1}\, a_{(r+1,r)} + {}^{n}C_r\, a_{(r+2,r)} + \ldots + {}^{n}C_n\, a_{(n,r)}$$

$$= {}^{n}C_r\, [(1/r!) \sum_{k=0}^{r} (-1)^{r-k}\ {}^{r}C_k * k^r\,] + {}^{n}C_{r+1}\, [(1/r!) \sum_{k=0}^{r} (-1)^{r-k}\ {}^{r}C_k\, k^{r+1}\,] + \ldots$$

$$+\ {}^{n}C_n\, [\,(1/r!) \sum_{k=0}^{r} (-1)^{r-k}\ {}^{r}C_k\, k^n\,]$$

$$= (1/r!) \sum_{k=0}^{r}\ [\,(-1)^{r-k} * {}^{r}C_k\, (\ {}^{n}C_r\, k^r +\ {}^{n}C_{r+1}\, k^{r+1} + \ldots + (\ {}^{n}C_n\, k^n\,)]$$

$$= (1/r!) \sum_{k=0}^{r} [\,(-1)^{r-k} * {}^{r}C_k\, (\sum_{q=0}^{n} {}^{n}C_q * k^q\ - \sum_{q=0}^{r-1} {}^{n}C_q * k^q\,)\,]$$

$$= (1/r!) \sum_{k=0}^{r} [\,(-1)^{r-k} * {}^{r}C_k\, (1+k)^n\,]\ -\ (1/r!) \sum_{k=0}^{n} [(-1)^{r-k}\ {}^{r}C_k\, (\sum_{q=0}^{r-1}\ {}^{n}C_q\, k^q\,)].$$

If we denote the first and second term as T_1 and T_2, we have

$$a_{(n+1,r+1)} = T_1 - T_2.$$

Consider

$$T_1 = (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} *\ {}^{r}C_k\, (1+k)^n]$$

$$= (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} (r!/((k!)(r-k)!)) (1+k)^{n}]$$

$$= (1/(r+1)!) \sum_{k=0}^{r} [(-1)^{r-k} ((r+1)!/((k+1)!(r-k)!)) (1+k)^{n+1}]$$

$$= (1/(r+1)!) \sum_{k=0}^{r} [(-1)^{r-k} * {}^{r+1}C_{k+1} (1+k)^{n+1}]$$

$$= (1/(r+1)!) \sum_{k=0}^{r} [(-1)^{(r+1)-(k+1)} * {}^{r+1}C_{k+1} (1+k)^{n+1}]$$

Let k+1 = s, we get for s= 1 and k=0 and s = r+1 at k = r

$$= (1/(r+1)!) \sum_{s=1}^{r+1} (-1)^{(r+1)-2} * {}^{r+1}C_{s} (s)^{n+1}].$$

Replacing s by k, we get

$$= (1/(r+1)!) \sum_{k=1}^{r+1} [(-1)^{(r+1)-k} * {}^{r+1}C_{k} (k)^{n+1}].$$

If we include the k = 0 case, we get

$$T_1 = (1/(r+1)!) \sum_{k=0}^{r+1} [(-1)^{(r+1)-k} * {}^{r+1}C_{k} (k)^{n+1}]$$

T_1 is the right hand side of the r+1 formula that we need to derive from the inductive hypothesis. To complete the formula, we need to show that

$a_{(n+1,r+1)} = T_1$.

From the expression

$a_{(n+1,r+1)} = T_1 – T_2$

we have to prove that $T_2 = 0$.

$$T_2 = (1/r!) \sum_{k=0}^{r} [(-1)^{r-k}\ {}^rC_k (\sum_{q=0}^{r-1} {}^nC_q\, k^q)].$$

$$= \quad (1/r!) \sum_{k=0}^{r} [(-1)^{r-k}\ {}^rC_k ({}^nC_0\, k^0 + {}^nC_1\, k^1 + {}^nC_2\, k^2 + \ldots + {}^nC_{r-1}\, k^{r-1})].$$

$$= \quad (1/r!) \sum_{k=0}^{r} [(-1)^{r-k}\ {}^rC_k] + {}^nC_1 [(1/r!) \sum_{k=0}^{r} ((-1)^{r-k} *\ {}^rC_k\, k)] +$$

$${}^nC_2 [(1/r!) \sum_{k=0}^{r} ((-1)^{r-k}\ {}^rC_k\, k^2)] + \ldots + {}^nC_{r-1} [(1/r!) \sum_{k=0}^{r} ((-1)^{r-k}\ {}^rC_k\, k^{r-1})]$$

$$= (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} *\ {}^rC_k] + {}^nC_1 [(1/r!) \sum_{k=0}^{r} ((-1)^{r-k} *\ {}^rC_k\, k)] +$$

$$[{}^nC_2 * a_{(2,r)} + {}^nC_3 * a_{(3,r)} + \ldots + {}^nC_{r-1} * a_{(r-1,r)}]$$

= X + Y + Z, where

$$X = (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} *\ {}^rC_k]$$

$$Y = {}^nC_1 [(1/r!) \sum_{k=0}^{r} ((-1)^{r-k} *\ {}^rC_k\, k)]$$

$$Z = [{}^nC_2 * a_{(2,r)} + {}^nC_3 * a_{(3,r)} + \ldots + {}^nC_{r-1} * a_{(r-1,r)}].$$

We shall prove that X = 0, Y = 0 and Z = 0.

$$X = (1/r!) \sum_{k=0}^{r} [(-1)^{r-k} *\ {}^rC_k]$$

$$= (1/r!) \sum^{r} [(-1)^{r-k} *\ {}^rC_{r-k}].$$

$$k=0$$

With the change of variables $r - k = w$, we get $k = 0$, $w = r$ and $k = r$, $w = 0$.

$$= (1/r!) \sum_{w=r}^{0} [(-1)^{w} * {}^{r}C_{w}]$$

$$= (1/r!) \sum_{w=0}^{r} [(-1)^{w} * {}^{r}C_{w}]$$

$$= (1 - 1)^{r}/r! = 0.$$

Therefore, $X = 0$.

$$Y = {}^{n}C_{1} [(1/r!) \sum_{k=0}^{r} ((-1)^{r-k} * {}^{r}C_{k}\, k)]$$

$$= {}^{n}C_{1} [(1/(r-1)!) \sum_{k=1}^{r} ((-1)^{r-1-(k-1)} * {}^{r-1}C_{k-1})]$$

$$= {}^{n}C_{1} [(1/(r-1)!) \sum_{k-1=0}^{r-1} ((-1)^{r-1-(k-1)} * {}^{r-1}C_{k-1})]$$

$$= {}^{n}C_{1}[1/(r-1)!)(1-1)^{r-1}$$

$$= 0.$$

Therefore, we have proven that $Y = 0$.

To prove that $Z = 0$ we begin with

$$Z = [{}^{n}C_{2} * a_{(2,r)} + {}^{n}C_{3} * a_{(3,r)} + . . . + {}^{n}C_{r-1} * a_{(r-1,r)}]$$

and consider the matrix

$$
\begin{array}{ccccccc}
\underline{a_{(1,1)}} & \mathbf{a_{(1,2)}} & \mathbf{a_{(1,3)}} & \mathbf{a_{(1,4)}} & \cdots & & \mathbf{a_{(1,r)}} \\
a_{(2,1)} & \underline{a_{(2,2)}} & \mathbf{a_{(2,3)}} & \mathbf{a_{(2,4)}} & \cdots & & \mathbf{a_{(2,r)}} \\
a_{(3,1)} & a_{(3,2)} & \underline{a_{(3,3)}} & \mathbf{a_{(3,4)}} & \cdots & & \mathbf{a_{(3,r)}} \\
a_{(4,1)} & a_{(4,2)} & a_{(4,3)} & \underline{a_{(4,4)}} & \mathbf{a_{(4,5)}} & \cdots & \mathbf{a_{(4,r)}} \\
 & & & & \cdots & & \\
\cdots & \cdots & \cdots & \cdots & \underline{a_{(r-1,r-1)}} & \mathbf{a_{(r-1,r)}} & \cdots \\
a_{(r,1)} & a_{(r,2)} & a_{(r,3)} & \cdots & a_{(r,r-1)} & \underline{a_{(r,r)}} &
\end{array}
$$

The diagonal elements are underlined and the elements above the main diagonal are in bold.

We have

$$a_{(1,r)} = [\ (1/r!) \sum_{k=0}^{r} ((-1)^{r-k} * {}^{r}C_k * k)] = Y/\ {}^{n}C_1 = 0 \text{ for } r > 1.$$

All of the elements of the first row other than $a_{(1,1)}$ are zero. Also,

$$a_{(n+1,r)} = a_{(n,r-1)} + r * a_{(n,r)}$$

which can easily be established by simplifying the right hand side.

This gives us

$$a_{(2,r)} = a_{(1,r-1)} + r * a_{(1,r)} = 0 \text{ for } r > 2,$$

i.e. $a_{(2,r)}$ can be expressed as a linear combination of two elements of the first row, (other than the one on the main diagonal). This implies that

$$a_{(2,r)} = 0 \text{ for } r > 2.$$

Similarly, $a_{(3,r)}$ can be expressed as a linear combination of two elements of the second row of the type $a_{(2,r)}$ with $r > 3$. This implies that

$$a_{(3,r)} = 0 \text{ for } r > 3.$$

Repeating this process, we get

$$a_{(r-1,r)} = 0.$$

Since each of the coefficients $a_{(i,j)}$ in the formula for Z are zero, it follows that Z = 0.

Therefore, with $X = Y = Z = 0$, giving $T_2 = 0$.

From this,

$a_{(n+1,r+1)} = T_1 - T_2 = T_1$

Since

$$T_1 = (1/(r+1)!) \sum_{k=0}^{r+1} [(-1)^{(r+1)-k} * {}^{r+1}C_k (k)^{n+1}]$$

it follows that

$$a_{(n+1,r+1)} = (1/(r+1)!) \sum_{k=0}^{r+1} [(-1)^{(r+1)-k} * {}^{r+1}C_k (k)^{n+1}]$$

which is the r+1 expression of the inductive hypothesis. Therefore, the proof of the theorem is complete.

Remark: This proof is very lengthy and involves a great deal of algebra. Readers are encouraged to search for a more concise approach.

Section 7

More Results and Applications of the Generalized Smarandache Star Function

Theorem:

$$F'^{n*}(p^\alpha) = \sum_{k=0}^{\alpha} {}^{n+k+1}C_{n-1}\, P(\alpha - k)$$

The following identity will be used in the proof.

$$\sum_{k=0}^{\alpha} {}^{r+k-1}C_{r-1} = {}^{\alpha+r}C_r. \quad (*)$$

Proof: By induction.

Basis step

The case where n = 1 has already been proven.

$$F'^{1*}(p^{\alpha}) = \sum_{k=0}^{\alpha} {}^{1+k+1}C_{1-1}\, P(\alpha - k)$$

Inductive step:
Assume that the formula is true for all $n \leq r$.

$$F'^{r*}(p^{\alpha}) = \sum_{k=0}^{\alpha} {}^{r+k-1}C_{r-1}\, P(\alpha - k).$$

Starting with

$$F'^{r+1*}(p^{\alpha}) = \sum_{k=0}^{\alpha} F'^{r*}\ (p^{t})$$

RHS = $F'^{r}*(p^{\alpha}) + F'^{r}*(p^{\alpha-1}) + F'^{r}*(p^{\alpha-2}) + \ldots + F'^{r}*(p) + F'^{r}*(1)$.

From the inductive hypothesis, we have

$$F'^{r*}(p^{\alpha}) = \sum_{k=0}^{\alpha} {}^{r+k+1}C_{r-1}\, P(\alpha - k).$$

Expanding the RHS from k=0 to α

$F'^{r*}(p^{\alpha}) = {}^{r+\alpha-1}C_{r-1}\, P(0) + {}^{r+\alpha-2}C_{r-1}\, P(1) + \ldots + {}^{r-1}C_{r-1}\, P(\alpha)$

$F'^{r*}(p^{\alpha-1}) = {}^{r+\alpha-2}C_{r-1}\, P(0) + {}^{r+\alpha-3}C_{r-1}\, P(1) + \ldots + {}^{r-1}C_{r-1}\, P(\alpha-1)$

$F'^{r*}(p^{\alpha-2}) = {}^{r+\alpha-3}C_{r-1}\, P(0) + {}^{r+\alpha-4}C_{r-1}\, P(1) + \ldots + {}^{r-1}C_{r-1}\, P(\alpha-2)$

. . .

$F'^{r*}(p) = {}^{r}C_{r-1}\, P(0) + {}^{r-1}C_{r-1}\, P(1)$

$F'^{r*}(1) = {}^{r-1}C_{r-1}\, P(0)$.

Summing up the left and right sides separately, we find that

LHS = $F'^{(r+1)*}\ (p^{\alpha})$.

The RHS contains α + 1 terms in which P(0) occurs, α terms in which P(1) occurs and so on. Expressing this as a series of summations:

$$RHS = [\sum_{k=0}^{\alpha} {}^{r+k-1}C_{r-1}]^* P(0) + \sum_{k=0}^{\alpha-1} {}^{r+k-1}C_{r-1} P(1) + \ldots + \sum_{k=0}^{1} {}^{r+k-1}C_{r-1} P(\alpha-1)$$

$$+ \sum_{k=0}^{0} {}^{r+k-1}C_{r-1} P(\alpha)$$

Applying the (*) identity to each of the summations, we have

$$RHS = {}^{r+\alpha}C_r P(0) + {}^{r+\alpha-1}C_r P(1) + {}^{r+\alpha-2}C_r P(2) + \ldots + {}^{r}C_r P(\alpha)$$

$$= \sum_{k=0}^{\alpha} {}^{r+k}C_r P(\alpha).$$

Or,

$$F'^{(r+1)*}(p^{\alpha}) = \sum_{k=0}^{\alpha} {}^{r+k}C_r P(\alpha).$$

The proposition is true for n = r+1, as we have

$$F'^*(p^{\alpha}) = \sum_{k=0}^{\alpha} P(\alpha-k) = \sum_{k=0}^{\alpha} {}^{k}C_0 P(\alpha-k) = \sum_{k=0}^{\alpha} {}^{k+1-1}C_{1-1} P(\alpha-k).$$

Hence by induction the proposition is true for all n, and the proof is complete.

Theorem:

$$\sum_{k=0}^{n-r} {}^{n}C_{r+k} \, {}^{r+k}C_r \, m^k = {}^{n}C_r (1+m)^{(n-r)}.$$

Proof:

$$LHS = \sum_{k=0}^{n-r} {}^{n}C_{r+k} \, {}^{r+k}C_r \, m^k$$

$$= \sum_{k=0}^{n-r} (n!)/((r+k)!*(n-r-k)!) \ * (r+k)!/((k)!*(r)!) * m^k$$

$$= \sum_{k=0}^{n-r} (n!)/((r)!*(n-r)!) \ * (n-r)!/((k)!*(n-r-k)!) * m^k$$

$$= {}^nC_r \sum_{k=0}^{n-r} {}^{n-r}C_k \, m^k$$

$$= {}^nC_r \, (1+m)^{(n-r)}.$$

Which completes the proof of the theorem.

Theorem:

$$F^m*(1\#n) = \sum_{r=0}^{n} {}^nC_r \, m^{n-r} \, F(1\#r)$$

Proof: By induction.

Basis step.
From section four, we have

$$F*(1\#n) = F\,(\,1\#\,(n+1)) = \sum_{r=0}^{n} {}^nC_r \, F(1\#r) = \sum_{r=0}^{n} {}^nC_r \, (1)^{n-r} \ F(1\#r)$$

so the formula is true for m=1.

Inductive step.
Assume that the formula is true for m=s, which is

$$F^s*(1\#n) = \sum_{r=0}^{n} {}^nC_r \, s^{n-r} \, F(1\#r).$$

Or

$$F^s*(1\#0) = \sum_{r=0}^{0} {}^nC_0 \, s^{0-r} \, F(1\#0) \qquad\qquad F^s*(1\#1) = \sum_{r=0}^{1} {}^nC_1 \, s^{1-r} \, F(1\#1)$$

$$F^s*(1\#2) = \sum_{r=0}^{2} {}^nC_2\, s^{2-r}\, F(1\#1) \qquad\qquad F^s*(1\#3) = \sum_{r=0}^{3} {}^nC_1\, s^{3-r}\, F(1\#3)$$

$F^s*(1\#0)= {}^0C_0\, F(1\#0)$ ----(0)

$F^s*(1\#1) = {}^1C_0\, s^1\, F(1\#0) + {}^1C_1\, s^0 F(1\#1)$ ----(1)

$F^s*(1\#2) = {}^2C_0\, s^2\, F(1\#0) + {}^2C_1\, s^1 F(1\#1) + {}^2C_2\, s^0 F(1\#2)$ ----(2)

.

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$F^s*(1\#r) = {}^rC_0\, s^r\, F(1\#0) + {}^rC_1\, s^1 F(1\#1) + \ldots + {}^rC_r\, s^0 F(1\#r)$ ----(r)

.

.

.

$F^s*(1\#n) = {}^nC_0\, s^r\, F(1\#0) + {}^nC_1\, s^1 F(1\#1) + \ldots + {}^nC_n\, s^0 F(1\#r)$ ----(n)

Multiplying the rth equation by ${}^nC_r$ and then summing, the RHS side becomes

$[{}^nC_0{}^0C_0\, s^0 + {}^nC_1{}^1C_0\, s^1 + {}^nC_2{}^2C_0\, s^2 + \ldots + {}^nC_k{}^kC_0\, s^k + \ldots + {}^nC_n{}^nC_0\, s^n]F(1\#0)$

$[{}^nC_1{}^1C_1\, s^0 + {}^nC_2{}^2C_1\, s^1 + {}^nC_3{}^3C_1\, s^2 + \ldots + {}^nC_k{}^kC_1\, s^k + \ldots + {}^nC_n{}^nC_1\, s^n]F(1\#1)$

. . .

$[{}^nC_r{}^rC_r\, s^0 + {}^nC_{r+1}{}^{r+1}C_r\, s^1 + \ldots + {}^nC_{r+k}{}^{r+k}C_r\, s^k + \ldots + {}^nC_n{}^nC_r\, s^n]F(1\#r)$

$+[{}^nC_n{}^nC_n\, s^0]F(1\#n)$

$$= \sum_{r=0}^{n} \{ \sum_{k=0}^{n-r} {}^nC_{r+k}\, {}^{r+k}C_r\, s^k \}\, F(1\#r)$$

$$= \sum_{r=0}^{n} {}^nC_r\, (1+s)^{n-r}\, F(1\#n) \text{ by the previous theorem.}$$

$$\text{LHS} = \sum_{r=0}^{n} {}^nC_r\, F^s*(1\#r)$$

Let $N = p_1p_2p_3 \ldots p_n$. Then there are ${}^nC_r$ divisors of N containing exactly r primes. Then LHS = the sum of the s^{th} Smarandache star functions of all the divisors of N, or $F'^{(s+1)}*(N) = F^{(s+1)}*(1\#n)$.

Therefore, we have

$$F^{(s+1)}*(1\#n) = \sum_{r=0}^{n} {}^nC_r\,(1+s)^{n-r}\, F(1\#n)$$

which is the induction expression for s+1. Therefore, the formula is true for all n and the proof is complete.

Note: From the previous section, we have

$$F'(N@1\#n) = F'(Np_1p_2 \ldots p_n) = \sum_{m=0}^{n} [a_{(n,m)}\, F'^{m*}(N)]$$

where

$$a_{(m,n)} = (1/m!) \sum_{k=1}^{m} (-1)^{m-k}\ {}^mC_k\, k^n.$$

If $N = p_1p_2 \ldots p_k$, we have

$$F(1\#(k+n)) = \sum_{m=0}^{n} [\ a_{(n,m)} \sum_{t=0}^{k} {}^kC_t\, m^{k-t}\, F(1\#t)\ ].$$

This formula provides a way to express B_n in terms of smaller Bell numbers. In a way, it is a generalization of the theorem

$$F(1\#(n+1)) = \sum_{r=0}^{n} {}^nC_r\, F(1\#r)$$

that was proven in section 4.

Theorem:

$$F(\alpha,1\#(n+1)) = \sum_{k=0}^{\alpha} \sum_{r=0}^{n} {}^nC_r\, F(k,1\#r)$$

Proof:
LHS = $F(\alpha,1\#(n+1)) = F'(p^{\alpha}\ p_1p_2p_3 \ldots p_{n+1}) =$

$F'*(p^{\alpha}p_1p_2p_3...p_n) + \sum F'$ (all the divisors containing only p^0) +

$\sum F'$ (all the divisors containing only p^1) +

$\sum F'$ (all the divisors containing only p^2) +. .

.

$\sum F'$ (all the divisors containing only p^r) +. . .

$\sum F'$ (all the divisors containing only p^{α}).

$$= \sum_{r=0}^{n} {}^{n}C_r\, F(0,1\#r) + \sum_{r=0}^{n} {}^{n}C_r\, F(1,1\#r) + \sum_{r=0}^{n} {}^{n}C_r\, F(2,1\#r) + \sum_{r=0}^{n} {}^{n}C_r\, F(3,1\#r)$$

$$+ \ldots + \sum_{r=0}^{n} {}^{n}C_r\, F(k,1\#r) + \ldots + \sum_{r=0}^{n} {}^{n}C_r\, F(\alpha,1\#r)$$

$$= \sum_{k=0}^{\alpha} \sum_{r=0}^{n} {}^{n}C_r\, F(k,1\#r).$$

This is a reduction formula for $F(\alpha,1\#(n+1))$.

A Result of Significance

From the first theorem of section 4

$$F'(p^{\alpha}@1\#(n+1)) = F(\alpha,1\#(n+1)) = \sum_{m=0}^{n} a_{(n+1,m)}\, F'^{m}*(N)$$

where

$$a_{(n+1,m)} = (1/m!) \sum_{k=1}^{m} (-1)^{m-k} * {}^{m}C_k * k^{n+1}$$

and

$$F'^{m}*(p^{\alpha}) = \sum_{k=0}^{\alpha} {}^{m+k-1}C_{m-1}\, P(\alpha-k).$$

This is the first result of some substance, giving a formula for evaluating the number of Smarandache Factor Partitions of numbers that can be represented in a (one of the most simple) particular canonical form. The complexity is evident. The challenging task posed to the reader is to derive similar expressions for other canonical forms.

Section 8

Properties of the Smarandache Star Triangle

Definition: The following expression was established in section 6.

$$a_{(n,m)} = (1/m!) \sum_{k=1}^{m} (-1)^{m-k} * {}^{m}C_k * k^n$$

we have $a_{(n,n)} = a_{(n,1)} = 1$ and $a_{(n,m)} = 0$ for $m > n$.

Now if the terms are arranged in the following way

$a_{(1,1)}$

$a_{(2,1)}$ $a_{(2,2)}$

$a_{(3,1)}$ $a_{(3,2)}$ $a_{(3,3)}$

.

.

.

$a_{(n,1)}$ $a_{(n,2)}$ ... $a_{(n,n-1)}$ $a_{(n,n)}$

we get a triangle of numbers that we will call the Smarandache Star Triangle (SST).

The first few rows of the SST are

1

1 1

1 3 1

1 7 6 1

1 15 25 10 1

Some properties of the SST.

1) The first and last element of each row is 1.
2) The elements of the second column are $2^{n-1} - 1$, where n is the row number.
3) The sum of all the elements in the nth row is the nth Bell number.

Justification of property 3:
From section 6, we have the formula

$$F'(N@1\#n) = F'(Np_1p_2 \ldots p_n) = \sum_{m=0}^{n} [a_{(n,m)} F'^{m*}(N)].$$

If N = 1, we have $F'^{m}*(1) = F'^{(m-1)}*(1) = F'^{(m-2)}*(1) =\ldots= F'(1) = 1$. Therefore,

$$F'(p_1 p_2 \ldots p_n) = \sum_{r=0}^{n} a(n,m)$$

4) The elements of a row can be obtained by the following reduction formula

$$a_{(n+1,m+1)} = a_{(n,m)} + (m+1) * a_{(n+1,m+1)}.$$

5) If N = p in the theorem of section 6, we have $F'^{m}*(p) = m + 1$. Therefore,

$$F'(pp_1p_2 \ldots p_n) = \sum_{m=1}^{n} a_{(n,m)} F'^{m}*(N)$$

or $$B_{n+1} = \sum_{m=1}^{n} (m+1)\ a_{(n,m)}.$$

6) The elements of the second leading diagonal are triangular numbers in their natural order.
7) If p is a prime, p divides all the elements of the pth row except the first and the last, which are unity. This is established in the following theorem.

Theorem:

$$a_{(p,r)} \equiv 0 \pmod p \text{ if } p \text{ is a prime and } 1 < r < p.$$

Proof:
By definition

$$a_{(p,r)} = (1/r!) \sum_{k=1}^{r} (-1)^{r-k} *{}^{r}C_k *k^p$$

which by rearrangement, is equivalent to

r-1

$$a_{(p,r)} = (1/(r-1)!) \sum_{k=0} (-1)^{r-1-k} * {}^{r-1}C_k * (k+1)^{p-1}$$

$$= (1/(r-1)!) \sum_{k=0}^{r-1} [(-1)^{r-1-k} . {}^{r-1}C_k * ((k+1)^{p-1} - 1)] +$$

$$(1/(r-1)!) \sum_{k=0}^{r-1} (-1)^{r-1-k} * {}^{r-1}C_k.$$

Applying Fermat's Little Theorem, we get

$a_{(p,r)}$ = a multiple of p + 0 => $a_{(p,r)} \equiv 0 (\mathrm{mod}\ p)$

and the proof is complete.

Corollary:

$$F(1\#p) \equiv 2 \ (\mathrm{mod}\ p)$$

Proof:

$$a_{(p,1)} = a_{(p,p)} = 1$$

$$F(1\#p) = \sum_{k=0}^{p} a_{(p,k)} = \sum_{k=2}^{p-1} a_{(p,k)} + 2$$

$F(1\#p) \equiv 2 \ (\mathrm{mod}\ p)$,

since the summation is evenly divisible by p.

8) The coefficient of the rth term, $b_{(n,r)}$ in the expansion of x^n in the form

$$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) + \ldots + b_{(n,r)}\, {}^{x}P_r + \ldots + b_{(n,n)}\, {}^{x}P_n$$

is equal to $a_{(n,r)}$.

Theorem: B_{3n+2} is even and all other B_k is odd.

Proof: From section 4, we have

$F'(Nq_1q_2) = F'^*(N) + F'^{**}(N)$ when q_1 and q_2 are prime and $(N,q_1) = (N,q_2) = 1$.

With $N = p_1p_2p_3\ldots p_n$, one can write

$F'(p_1p_2p_3\ldots p_nq_1q_2) = F'^*(p_1p_2p_3\ldots p_n) + F'^{**}(p_1p_2p_3\ldots p_n)$

or

$F(1\#(n+2)) = F(1\#(n+1)) + F^{**}(1\#n)$

However,

$$F^{**}(1\#n) = \sum_{r=0}^{n} {}^{n}C_r\, 2^{n-r} F(1\#r)$$

$$F^{**}(1\#n) = \sum_{r=0}^{n-1} ({}^{n}C_r\, 2^{n-r} F(1\#r)) + F(1\#n).$$

The first term is an even number and we will call it E. This gives us

$F(1\#(n+2)) - F(1\#(n+1)) - F(1\#n) = E$.

Case 1: $F(1\#n)$ is even and $F(1\#(n+1))$ is also even $\Rightarrow F(1\#(n+2))$ is even.

Case II: $F(1\#n)$ is even and $F(1\#(n+1))$ is odd $\Rightarrow F(1\#(n+2))$ is odd.

By the previous theorem,

$F(1\#(n+3)) - F(1\#(n+2)) - F(1\#(n+1)) = E$, $\Rightarrow$ $F(1\#(n+3))$ is even.

Finally we get

$F(1\#n)$ is even $\Leftrightarrow$ $F(1\#(n+3))$ is even.

We know that $F(1\#2) = 2$ Þ $F(1\#2)$, $F(1\#5)$, $F(1\#8)$, . . .are even

$\Rightarrow B_{3n+2}$ is even otherwise B_k is odd.

Section 9

Smarandache Factor Partitions Of a Typical Canonical Form

In previous sections, some of the properties of the Smarandache Factor Partition function were demonstrated. In this section, we will derive a formula for the case where

$N = p_1^{\alpha 1}\, p_2^{\alpha 2}$.

Theorem:

$$F'(p_1^{\alpha} p_2^{2}) = F(\alpha,2) = \sum_{k=0}^{\alpha} P(k) + \sum_{j=0}^{r} \sum_{i=0}^{\alpha-2j} P(i)$$

where $r = [\alpha/2]$, $\alpha = 2r$ or $\alpha = 2r + 1$.

Proof: The proof will be by examining a set of mutually exclusive and exhaustive cases. Only the numbers in the brackets [ ] are to be further decomposed.

Case I:

$$(p_2)\,[p_1^{\alpha} p_2^{2}] \text{ gives } F'^{*}(p_1^{\alpha}) = \sum_{i=0}^{\alpha} P(i).$$

Case II:

$\{A_1\} \rightarrow \quad (p_2^{2})\,[p_1^{\alpha}] \quad \text{-------}\rightarrow P(\alpha)$

$\{A_2\} \rightarrow \quad (p_2^{2}\, p_1)\,[p_1^{\alpha-1}] \quad \text{-------}\rightarrow P(\alpha-1)$

.

.

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$\{A_\alpha\} \rightarrow \quad (p_2^{2}\, p_1^{\alpha})\,[p_1^{\alpha-\alpha}] \quad \text{-------}\rightarrow P(\alpha-\alpha) = P(0)$.

Therefore, case II contributes

$$\sum_{i=0}^{\alpha} P(i).$$

Case III:

$\{B_1\} \rightarrow (p_1p_2)(p_1p_2)\,[p_1^{\alpha-2}] \quad \text{-------}\rightarrow P(\alpha-2)$

$\{B_2\} \rightarrow (p_1p_2)\,(p_1^{2}\, p_2)\,[p_1^{\alpha-3}] \quad \text{-------}\rightarrow P(\alpha-3)$

.

.

.

$\{B_{\alpha-2}\} \rightarrow (p_1p_2)(p_1^{\alpha-1}\, p_2)\,[p_1^{\alpha-\alpha}] \quad \text{-------}\rightarrow P(\alpha-\alpha) = P(0)$.

Therefore, case III contributes

α-2

$$\sum_{i=0} P(i).$$

Case IV:

$\{C_1\} \rightarrow (p_1^2\, p_2)\ (p_1^2\, p_2\ )\ [p_1^{\alpha-4}]$ ------→ $P(\alpha-4)$
$\{C_2\} \rightarrow (p_1^2\, p_2)(p_1^3\, p_2\ )\ [p_1^{\alpha-5}]$ ------→ $P(\alpha-5)$
.
.
.
$\{C_{\alpha-4}\} \rightarrow (p_1^2\, p_2)(p_1^{\alpha-2}\, p_2\ )\ [p_1^{\alpha-\alpha}]$ ------→ $P(\alpha-\alpha) = P(0)$

Therefore, case IV contributes

$$\sum_{i=0}^{\alpha-4} P(i).$$

Note: The factor partition $(p_1^2\, p_2)\ (p_1\, p_2\ )\ [p_1^{\alpha-3}]$ was covered in case III. Therefore, it does not appear in case IV.

Case V:

$\{D_1\} \rightarrow (p_1^3\, p_2)\ (p_1^3\, p_2\ )\ [p_1^{\alpha-6}]$ ------→ $P(\alpha-4)$
$\{D_2\} \rightarrow (p_1^3\, p_2)(p_1^4\, p_2\ )\ [p_1^{\alpha-7}]$ ------→ $P(\alpha-5)$
.
.
.
$\{D_{\alpha-4}\} \rightarrow (p_1^3\, p_2)(p_1^{\alpha-3}\, p_2\ )\ [p_1^{\alpha-\alpha}]$ ------→ $P(\alpha-\alpha) = P(0)$

Therefore, case V contributes

$$\sum_{i=0}^{\alpha-6} P(i)..$$

Using a similar line of reasoning, case VI contributes

$$\sum_{i=0}^{\alpha-8} P(i).$$

and we get contributions up to

$\alpha-2r$

$$\sum_{i=0} P(i).$$

where $2r < \alpha < 2r + 1$ or $r = [\alpha/2]$.

Summing up all the cases, the total is

$$F'(p_1^{\alpha}p_2^{2}) = F(\alpha,2) = \sum_{k=0}^{\alpha} P(k) + \sum_{j=0}^{r} \sum_{i=0}^{\alpha-2j} P(i)$$

where $r = [\alpha/2]$, $\alpha = 2r$ or $\alpha = 2r + 1$. This completes the proof.

Corollary:

$$F'(p_1^{\alpha}p_2^{2}) = \sum_{k=0}^{r} (k+2) [P(\alpha-2k) + P(\alpha-2k-1)].$$

Proof: In the previous theorem, consider the case where $\alpha = 2r$.

$$F'(p_1^{2r}p_2^{2}) = F(\alpha,2) = \sum_{k=0}^{2r} P(k) + \sum_{j=0}^{r} \sum_{i=0}^{\alpha-2j} P(i).$$

The second term on the RHS can be expanded in the following way

$$
\begin{array}{l}
P(\alpha) + P(\alpha-1) + P(\alpha-2) + P(\alpha-3) +\ldots+ P(2) + P(1) + P(0) \\
\qquad\qquad P(\alpha-2) + P(\alpha-3) +\ldots+ P(2) + P(1) + P(0) \\
. \qquad\qquad\qquad\qquad P(\alpha-4) +\ldots P(2) + P(1) + P(0) \\
. \\
. \\
\qquad\qquad\qquad\qquad\qquad\qquad P(2) + P(1) + P(0) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P(0).
\end{array}
$$

Summing the terms column by column,

$$= [P(\alpha) + P(\alpha-1)] +2 [P(\alpha-2) + P(\alpha-3)]+ 3 [P(\alpha-4) + P(\alpha-5)]+\ldots$$

$$+ (r-1) [P(2) + P(1)] + r P(0).$$

$$= \sum^{r} (k+1) [P(\alpha-2k) + P(\alpha-2k-1)].$$

$k=0$

Note that P(-1) has been defined to be zero.

Therefore,

$$F'(p_1^{\alpha}p_2^{2}) = \sum_{k=0}^{r} P(k) + \sum_{k=0}^{r} (k+1) [P(\alpha-2k) + P(\alpha-2k-1)]$$

or

$$F'(p_1^{\alpha}p_2^{2})= \sum_{k=0}^{r} (k+2) [P(\alpha-2k) + P(\alpha-2k-1)].$$

In the case where $\alpha = 2r+1$, the second term in the expression of the corollary can be expanded as

$$\begin{array}{l}
P(\alpha) + P(\alpha-1) + P(\alpha-2) + P(\alpha-3) +\ldots+ P(2) + P(1) + P(0) \\
\qquad\qquad P(\alpha-2) + P(\alpha-3) +\ldots+ P(2) + P(1) + P(0) \\
. \qquad\qquad\qquad\qquad P(\alpha-4) +\ldots P(2) + P(1) + P(0) \\
. \\
. \\
\qquad\qquad\qquad\qquad\qquad P(3)+ P(2) + P(1) + P(0) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad P(1)+ P(0)
\end{array}$$

Summing the terms on a column-by-column basis, we have

$$[P(\alpha) + P(\alpha-1)] +2 [P(\alpha-2) + P(\alpha-3)]+ 3 [P(\alpha-4) + P(\alpha-5)]+\ldots$$

$$+ (r-1) [P(3) + P(2)] + r[P(1) + P(0)]$$

$$= \sum_{k=0}^{r} (k+1) [P(\alpha-2k) + P(\alpha-2k-1)] , \alpha =2r+1.$$

Adding on the first term, we get

$$F'(p_1^{\alpha}p_2^{2}) = \sum_{k=0}^{r} (k+2) [P(\alpha-2k) + P(\alpha-2k-1)]$$

Therefore, for all values of α, we have

$$F'(p_1^{\alpha} p_2^{2}) = \sum_{k=0}^{[\alpha/2]} (k+2) \, [\, P(\alpha-2k) + P(\alpha-2k-1)]$$

and the proof of the corollary is complete.

Section 10

Length/Extent of Smarandache Factor Partitions

Definition: If we denote each Smaradanche Factor Partition (SFP) of N, say $F_1, F_2, \ldots,$ F_r arbitrarily and the F_k be the SFP representation of N as the product of its divisors in the following way:

F_k ---- $N = (h_1)(h_2)\,(h_3)(h_4)\ldots(h_t)$, where each h_i $(1 < i < t)$ is an entity in the SFP ' F_k ' of N.

Then, $T(F_k) = t$ is defined as the length of the factor partition F_k.

For example, 60 = 15 x 2 x 2 is a factor partition (F_k) of 60. Then $T(F_k) = 3$.

$T(F_k)$ can also be defined as one more than the number of multiplication signs in the factor partition.

Definition: The extent of a number is defined as the sum of the lengths of all the SFP's of the number.

Example:
Consider F(1#3)

$N = p_1p_2p_3 = 2 \times 3 \times 5 = 30$.

with the table of partitions

SN	Factor Partition	Length
1	30	1
2	15 * 2	2
3	10 * 3	2
4	6 * 5	2
5	5 * 3 * 2	3

The extent of 30 is 1 + 2 + 2 + 2 + 3 = 10.

We observe that

F(1#4) – F(1#3) = 10 = Extent{F(1#14)}.

Consider F(1#4)

N = 2 x 3 x 5 x 7 = 210
with the table of partitions

SN	Factor Partition	Length
1	210	1
2	105 * 2	2
3	70 * 3	2
4	42 * 5	2
5	35 * 6	2
6	35 * 3 * 2	3
7	30 * 7	2
8	21 * 10	2
9	21 * 5 * 2	3
10	15 * 14	2
11	15 * 7 * 2	3
12	14 * 5 * 2	3
13	10 * 7 * 3	3
14	7 * 6 * 5	3
15	7 * 5 * 3 * 2	4

Extent(210) = Σ length = 37.

We observe that

F(1#5) – F(1#4) = 37 = Extent{F(1#4)}.

Furthermore, it has been verified that

F(1#6) – F(1#5) = Extent{F(1#5)}

which leads to the following conjectures.

Conjecture 1:

F(1#(n+1)) - F(1#n) = Extent { F(1#n)}

Conjecture 2:

n

$$F(1\#(n+1)) = \sum_{r=0} \text{Extent } \{F(1\#r)$$

If conjecture 1 is true, then we would have

$$F(1\#2) - F(1\#1) = \text{Extent } \{ F(1\#1)\}$$

$$F(1\#3) - F(1\#2) = \text{Extent } \{ F(1\#2)\}$$

$$F(1\#4) - F(1\#3) = \text{Extent } \{ F(1\#3)\}$$

.

.

.

$$F(1\#(n+1)) - F(1\#n) = \text{Extent } \{ F(1\#n)\}$$

Summing up the terms, we would have

$$F(1\#(n+1)) - F(1\#1) = \sum_{r=1}^{n} \text{Extent } \{F(1\#r)\}.$$

$F(1\#1) = 1 = \text{Extent}\{F(1\#0)\}$ can be taken, hence we have

$$F(1\#(n+1)) = \sum_{r=0}^{n} \text{Extent } \{F(1\#r)\}.$$

An Interesting Observation

The following entries form a chart of r versus w, where w is the number of SFPs having the length r.

$F(1\#0) = 1$, $\sum r * w = 1$

r	1
w	1

$F(1\#1) = 1$, $\sum r *w = 1$

r	1
w	1

$F(1\#2) = 2$, $\sum r * w = 3$

R	1	2

w	1	1

F(1#3) = 5 , $\sum$ r *w = 10

R	1	2	3
w	1	3	1

F(1#4) = 15 , $\sum$ r * w = 37

r	1	2	3	4
w	1	7	6	1

F(1#5) = 52, $\sum$ r *w = 151

r	1	2	3	4	5
w	1	15	25	10	1

The interesting observation is that the entries of row w are the same as those of the nth row of the Smarandache Star Triangle introduced in section 8.

Conjecture 3:

$$w_r = a_{(n,r)} = (1/r!) \sum_{k=0}^{r} (-1)^{r-k} .{}^{r}C_k .k^n$$

where w_r is the number of SFPs of F(1#n) having length r.

Further study: Readers are encouraged to the length and contents of other cases and search for any interesting patterns.

Section 11

More Ideas On Smarandache Factor Partitions

Let

$$N = p_1^{\alpha_1} \; p_2^{\alpha_2} \; p_3^{\alpha_3} \; . \; . \; . \; p_r^{\alpha_r}$$

Definition:

1) L(N) is the length of the factor partition of N which has the maximum number of terms. In the case of the prime factorization above,

$$L(N) = \sum^{r} \alpha_i.$$

2) $A_{L(N)}$ = A set of L(N) distinct primes.

3) B(N) = { p: p | N , p is a prime },

$B(N) = \{ p_1 , p_2 , \ldots p_r \}$.

4) $\Psi[N, A_{L(N)}] = \{ x \mid d(x) = N \text{ and } B(x) \subseteq A_{L(N)} \}$, where d(x) is the number of divisors of x.

Our next step will be to derive an expression for the order of the set $\Psi[N, A_{L(N)}]$ defined above.

There are F'(N) factor partitions of N. Let F_1 be one of them. Then

$F_1 \longrightarrow N = s_1 * s_2 * s_3 * \ldots * s_t$.

If

$$\theta = p_1^{s1-1} p_2^{s2-1} p_3^{s3-1} \ldots p_t^{st-1} p_{t+1}^{0} p_{t+2}^{0} \ldots p_{L(N)}^{0}$$

where $p_t \in A_{L(N)}$, then $\theta \in \Psi[N, A_{L(N)}]$ for

$$d(\theta) = s_1 * s_2 * s_3 * \ldots * s_t * 1 * 1 * 1 \ldots = N.$$

Therefore, each factor partition of N generates a few elements of Ψ.

Let $E(F_1)$ denote the number of elements generated by F_1

$$F_1 \rightarrow N = s_1 * s_2 * s_3 * \ldots * s_t.$$

Multiplying the right side by one as many times as is required to make the number of terms in the product equal to L(N).

$$N = \prod_{k=1}^{L(N)} s_k$$

where $s_{t+1} = s_{t+2} = s_{t+3} = . . . = s_{L(N)} = 1$.

Let x_1 s's are equal
x_2 s's are equal
.
.
.
x_m s's are equal

such that $x_1 + x_2 + x_3 + . . . + x_m = L(N)$.

Then we have

$E(F_1) = \{L(N)\}! / \{(x_1)!(x_2)!(x_3)! . . . (x_m)!\}$.

Summing over all the factor partitions, we have

$$O(\Psi[N, A_{L(N)}]) = \sum_{k=1}^{F'(N)} E(F_k).$$

Example:

$N = 12 = 2^2*3, L(N) = 3 , F'(N) = 4$

Let $A_{L(N)} = \{2,3,5\}$

$F_1 \text{-----}\longrightarrow N = 12 = 12 * 1 * 1, x_1 = 2, x_2 = 1$

$E(F_1) = 3! / \{(2!)(1!)\} = 3$

$F_2 \text{-----}\longrightarrow N = 12 = 6 * 2 * 1, x_1 = 1, x_2 = 1, x_3 = 1$

$E(F_2) = 3! / \{(1!) (1!)(1!)\} = 6$

$F_3 \text{-----}\longrightarrow N = 12 = 4 * 3 * 1, x_1 = 1, x_2 = 1, x_3 = 1$

$E(F_3) = 3! / \{(1!) (1!)(1!)\} = 6$

$F_4 \text{-----}\longrightarrow N = 12 = 3 * 2 * 2, x_1 = 1, x_2 = 2$

$E(F_4) = 3! / \{(2!)(1!)\} = 3$

$$O(\Psi[N, A_{L(N)}]) = \sum^{F'(N)} E(F_k) = 3 + 6 + 6 + 3 = 18.$$

k=1

To verify, we have

Ψ[N, $A_{L(N)}$] = { 2^{11}, 3^{11}, 5^{11}, $2^5 * 3$, $2^5 * 3$, $3^5 * 2$, $3^5 * 5$, $5^5 * 2$,

$5^5 * 3$, $2^3 * 3^2$, $2^3 * 5^2$, $3^3 * 2^2$, $3^3 * 5^2$, $5^3 * 2^2$, $5^3 * 3^2$, $2^2 * 3 * 5$,

$3^2 * 2 * 5$, $5^2 * 2 * 3$}.

Section 12

A Note On the Smarandache Divisor Sequences

Definition: The Smarandache Divisor Sequences are defined in the following way:
P_n ={ x | d(x) = n }, d(x) = number of divisors of n.

Examples:

P_1 = {1}
P_2 = { x | x is a prime }
P_3 = { x | $x = p^2$, p is a prime }
P_4 = { x | $x = p^3$ or $x = p_1p_2$, p ,p_1 , p_2 are primes }.

Let F_1 be an SFP of N. Let Ψ_{F1} = {y| d(y) = N }, generated by the SFP F_1 of N. It has been shown in that each SFP generates elements of Y or P_n. Here each SFP generates infinitely many elements of P_n. Similarly, Ψ_{F1}, Ψ_{F2}, Ψ_{F3},. . . $\Psi_{F'(N)}$, are defined. It is evident that all these F_k's are disjoint and also $P_N = \cup \Psi_{Fk} \quad 1 \le k \le F'(N)$.

Theorem: There are F'(N) disjoint and exhaustive subsets into which P_N can be decomposed.

Proof: Let $\theta \in P_N$ and express it in canonical form

$$\theta = \quad p_1^{\alpha 1} \; p_2^{\alpha 2} \; p_3^{\alpha 3} \; . . . \quad p_r^{\alpha r}.$$

Then $d(\theta) = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1) . . .(\alpha_r+1)$ and it follows that $\theta \in \Psi_{Fk}$ for some k where F_k is given by

$$N = (\alpha_1+1)(\alpha_2+1)(\alpha_3+1) . . .(\alpha_r+1).$$

Finally, if $\theta \in \Psi_{Fs}$, and $\theta \in \Psi_{Ft}$ then from unique factorization theorem F_s and F_t are identical SFPs of N. This completes the proof.

Section 13

An Algorithm for Listing the Smarandache Factor Partitions

Definition: $F_x'(y)$ is the number of the Smarandache Factor Partitions (SFPs) of y which involve terms not greater than x.

If F_1 is a factor partition of y:

$F_1 \rightarrow x_1 * x_2 * x_3 * \ldots x_r$, then F_1 is included in $F_x'(y)$ iff $x_i \leq x$ for $1 \leq i \leq r$.

Clearly, $F'_x(y) \leq F'(y)$, and equality holds iff $x \geq y$.

Example: $F'_8(24) = 5$. Out of 7 only the last 5 are included in $F'_8(24)$.

(1) 24
(2) 12 * 2
(3) 8 * 3
(4) 6 * 4
(5) 6 * 2 * 2
(6) 4 * 3 * 2
(7) 3 * 2 * 2 * 2.

ALGORITHM: Let $d_1, d_2, d_3, \ldots d_r$ be the divisors of N in descending order. To list the factor partitions of N, execute the following sequence of steps:

(A) (1) Start with $d_1 = N$ as the first partition.

(2) For index i = 2 to r
(2)(a) Write all the factor partitions involving d_i .

Note: When listing the factor partitions, care should be taken that the terms from left to right are written in descending order.

At $d_k \geq N^{1/2} \geq d_{k+1}$, and above, this will ensure that there is no repetition.

Example: N = 36, Divisors are 36, 18, 12, 9, 6, 4, 3, 2, 1.

$36 \rightarrow 36$
$18 \rightarrow 18 * 2$
$12 \rightarrow 12 * 3$

9 $\rightarrow$ 9 * 4

9 * 2 * 2

6 $\rightarrow$ 6 * 6

6 $\rightarrow$ 6 * 3 * 2

----------------------------------$d_k = N^{1/2}$

4 $\rightarrow$ 4 * 3 * 3

3 $\rightarrow$3 * 3 * 2 * 2

2 $\rightarrow$ NIL

1 $\rightarrow$ NIL

Formula for F'(N)

$$F'(N) = \sum_{d_r/N} F'_{dr}(N/d_r).$$

Example:

$$N = 216 = 2^3 3^3$$

(1)	216	$\rightarrow F_{216}(1) = 1$
(2)	108 * 2	$\rightarrow F_{108}(2) = 1$
(3)	72 * 3	$\rightarrow F_{72}(3) = 1$
(4)	54 * 4	$\rightarrow F_{54}(4) = 2$
(5)	54 * 2 * 2	
(6)	36 * 6	$\rightarrow F_{36}(6) = 2$
(7)	36 * 3 * 2	
(8)	27 * 8	$\rightarrow F_{27}(8) = 3$
(9)	27 * 4 * 2	
(10)	27 * 2 * 2 * 2	
(11)	24 * 9	$\rightarrow F_{24}(9) = 2$
(12)	24 * 3 * 3	
(13)	18 *12	$\rightarrow F_{18}(12) = 4$
(14)	18 * 6 * 2	
(15)	18 * 4 * 3	
(16)	18 * 3 * 2 * 2	
(17)	12 * 9 * 2	$\rightarrow F_{12}(18) = 3$
(18)	12 * 6 * 3	
(19)	12 * 3 * 3 * 2	
(20)	9 * 8 * 3	$\rightarrow F_9(24) = 5$
(21)	9 * 6 * 4	
(22)	9 * 6 * 2 * 2	
(23)	9 * 4 * 3 * 2	
(24)	9 * 3 * 2 * 2	
(25)	8 * 3 * 3 * 3	$\rightarrow F_8(27) = 1$
(26)	6 * 6 * 6	$\rightarrow F_6(36) = 4$
(27)	6 * 6 * 3 * 2	
(28)	6 * 4 * 3 * 3	

(29) 6 * 3 * 3 * 2 * 2
(30) 4 * 3 * 3 * 3 * 2 * 2 $\rightarrow F_4(54) = 1$
(31) 3 * 3 * 3 * 2 * 2 * 2 $\rightarrow F_3(72) = 1$
$\rightarrow F_2(108) = 0$
$\rightarrow F_1(216) = 0.$

$$F'(216) = \sum_{d_r/N} F'_{dr}(216/d_r) = 31.$$

Remarks: This algorithm would be quite helpful in the development of a computer program for the listing of SFPs.

Section 14

A Program For Determining the Number of SFPs

Section 13 ended with a comment about how the algorithm would be helpful in constructing a computer program to list the SFP's. In this section, such a program in the 'C' language that lists the SFPs will be presented.

```
#include<stdio.h>
/*This is a program for finding the number of factor partitions of a given number written
by K.Suresh, Software expert, IKOS, NOIDA , INDIA. */
FILE* f;
unsigned long np = 0;
unsigned long try_arr[1000];
unsigned long n = 0;
unsigned long num_div = 0;
unsigned long div_arr[10000];
unsigned long max_length = 0;
unsigned long width_arr[1000];
unsigned long extent = 0;

void find_partitions(unsigned long pos, unsigned long div_idx, unsigned long prod)
{
        unsigned long i;
        for(i = div_idx; i < num_div; i++)
        {
                try_arr[pos] = div_arr[i];
                unsigned long new_prod = prod * div_arr[i];
                if( new_prod == n )
                {
                        if( max_length < pos + 1 )
                                max_length = pos + 1;
```

```
                width_arr[pos+1]++;
                extent += pos + 1;

                fprintf(f, "(%ld)\t%ld = %ld ", ++np, n, try_arr[0]);
                unsigned long k;
                for(k = 1;k <= pos;k++)
                {
                  fprintf(f, "X %ld ", try_arr[k]);
                }

                //printf(".\n");
                fprintf(f, ".\n");
                break;
            }
            else if( new_prod < n )
                find_partitions(pos+1, i, new_prod);
        }
        return;
}

main()
{
        f = fopen("fp.out", "w");
        while(1)
        {
        // initialize..
        n = 0;
        np = 0;
        num_div = 0;
        unsigned long i;
        for(i = 0; i < 1000; i++) width_arr[i] = 0;

        // take input..
        printf("Enter number..:\n");

        scanf("%ld", &n);
        if( n == 0 ) break;
        fprintf(f, "----------------------------------------------------------------\n");
        fprintf(f, "Number = %ld\n", n);

        // populate divisor array.
        for(i = 2; i <= n; i++)
        {
            if( n == i * (n / i) )
            {
                div_arr[num_div++] = i;
```

```
		}
	}
	// start recursion..
	find_partitions(0,0,1);

	// output..
	printf("Total number of factor partitions = %ld.\n", np);
	printf("----------------------------------------------\n");
	printf("\tr\t\tw\n");
	printf("----------------------------------------------\n");
	for(i = 1;i <= max_length; i++)
		printf("\t%ld\t\t%ld\n", i, width_arr[i]);
	printf("----------------------------------------------\n");
	printf("Extent = %ld\n", extent);

	fprintf(f,"Total number of factor partitions = %ld.\n", np);
	fprintf(f,"   ------------------------------------------------------\n");
	fprintf(f,"\tr\t\tw\n");
	fprintf(f,"   ------------------------------------------------------\n");
	for(i = 1;i <= max_length; i++)
		fprintf(f,"\t%ld\t\t%ld\n", i, width_arr[i]);
	fprintf(f,"   ------------------------------------------------------\n");
	fprintf(f,"Extent = %ld\n", extent);
	fprintf(f, "-----------------------------------------------------------------------------\n");

	}
	fflush(f);
	fclose(f);
	return 0;
}
```

Based on this program, tables for several different canonical forms are given.

Table -I

Canonical form	Number	SFPs
p^2	$2^2 = 4$	2
p^2q^2	$2^23^2 = 36$	9
$p^2q^2r^2$	$2^23^25^2 = 900$	66
$p^2q^2r^2s^2t^2$	$2^23^25^27^2 =44100$	712
$p^2q^2r^2s^2t^2u^2$	$2^23^25^27^211^2 =5336100$	10457

Table -II

Canonical form	Number	SFPs
p^2	3^2 =9	1
Pq^2	2 .3^2 =18	4
p^2q^2	$2^2 3^2$ = 36	9
p^3q^2	$2^3 3^2$ =72	16
p^4q^2	$2^4 3^2$ = 144	29
p^5q^2	$2^5 3^2$= 288	47
p^6q^2	$2^6 3^2$ = 576	77
p^7q^2	$2^7 3^2$ =1152	118
p^8q^2	$2^8 3^2$ = 2304	181
p^9q^2	$2^9 3^2$ = 4608	267
$p^{10}q^2$	$2^{10} 3^2$ = 9216	392
$p^{11}q^2$	$2^{11} 3^2$ = 18432	560
$p^{12}q^2$	$2^{12} 3^2$ =36864	797
$p^{13}q^2$	$2^{13} 3^2$ =73728	1111
$p^{14}q^2$	$2^{14} 3^2$ =147456	1541

Table-III

Canonical Form	Number	SFPs
p^3	$3^3 = 27$	3
$p^3 q$	$3^3 2 = 54$	7
$p^3 q^2$	$3^3 2^2 = 108$	16
$p^3 q^3$	$3^3 2^3 = 216$	31
$p^3 q^4$	$3^3 2^4 = 432$	57
$p^3 q^5$	$3^3 2^5 = 864$	97
$p^3 q^6$	$3^3 2^6 = 1728$	162
$p^3 q^7$	$3^3 2^7 = 3456$	257
$p^3 q^8$	$3^3 2^8 = 6912$	401
$p^3 q^9$	$3^3 2^9 = 13824$	608
$p^3 q^{10}$	$3^3 2^{10} = 27648$	907
$p^3 q^{11}$	$3^3 2^{11} = 55296$	1325
$p^3 q^{12}$	$3^3 2^{13} = 110592$	1914
$p^3 q^{13}$	$3^3 2^{13} = 221184$	2719

Table-IV

Canonical Form	Number	SFPs
p^4	34 = 81	5
p^4q	$3^4.2 = 162$	12
$p^4 q^2$	$3^4 2^2 = 324$	29
$p^4 q^3$	$3^4 2^3 = 648$	57
$p^4 q^4$	$3^4 2^4 = 1296$	109
$p^4 q^5$	$3^4 2^5 = 2592$	189
$p^4 q^6$	$3^4 2^6 = 5184$	323
$p^4 q^7$	$3^4 2^7 = 10368$	522
$p^4 q^8$	$3^4 2^8 = 20736$	831
$p^4 q^9$	$3^4 2^9 = 41472$	1279
$p^4 q^{10}$	$3^4 2^{10} = 82944$	1941
$p^4 q^{11}$	$3^4 2^{11} = 165888$	2876
$p^4 q^{12}$	$3^4 2^{12} = 331776$	4215

Table –V

	P^0	P^1	P^2	P^3	P^4	P^5	P^6	P^7	P^8
Q^0	1	1	2	3	5	7	11	15	22
Q^1	1	2	4	7	12	19	30	45	67
Q^2	2	4	9	16	29	47	77	118	181
Q^3	3	7	16	31	57	97	162	257	401
Q^4	5	12	29	57	109	189	323	522	831
Q^5	7	19	47	97	189	339	589	975	1576
Q^6	11	30	77	162	323	589	1043	1752	2876
Q^7	15	45	118	257	522	975	1752	2998	4987
Q^8	22	67	181	401	831	1576	2876	4987	8406

In Table V, the SFPs have been arranged for $\mathbf{q^i\, p^j}$ as the number in the i^{th} row and j^{th} column, as can be seen for $p^4\, q^7$, where the number of SFPs is 522. This obviously forms a symmetric matrix.

Readers are encouraged to look for any interesting patterns in this data.

Section 15

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Chapter 2

Smarandache Sequences

Section 1

On The Largest Balu Number And Some SFP Equations

Definition: For a positive integer N let d(N) and F'(N) be the number of distinct divisors and the Smarandache Factor Partitions (SFP) respectively. If N is the smallest number satisfying d(N) = F'(N) = r for some r , then N is called a Balu number.

In [1] Maohua Le proves Murthy's conjecture that there are only a finite number of Balu numbers. In this section, it will be proved that 36 is the largest Balu number.

It is well-known that if $N = p_1^{a1} * p_2^{a2} \ldots * p_k^{ak}$, then

$d(N) = (a_1 + 1)(a_2 + 1)(a_3 + 1) \ldots (a_k + 1)$.

Proposition 1:

$F'(N) > P(a_1) P(a_2) Pa_3) \ldots P(a_k)$,

for k > 1 and where $P(a_i)$ is the additive partition of a_i.

Proof:
From [2] we have that $F'(p^a) = P(a)$ for $k = 1$.

We proceed by induction on k.

Basis step:
For k = 2, let $N = p_1^{a1} p_2^{a2}$.

Consider those SFPs of N in which no element is a multiple of $p_1 p_2$. In other words the SFPs in which the factors of p_1^{a1} and p_2^{a2} are isolated. It is quite evident that each SFP of p_1^{a1}, when combined with each SFP of p_2^{a2} gives one SFP of N $(=p_1^{a1} p_2^{a2})$. Therefore, they contribute $P(a_1) P(a_2)$ SFPs of N. There are more SFPs like $(p_1 * p_2) * (N/ p_1 * p_2)$ which are not counted. Hence we have $F'(N) > P(a_1) P(a_2)$. The proposition has been established for k =2.

Inductive step:

Let the proposition be true for k = m. Then we have

$F'(N) > P(a_1)\, P(a_2)\, Pa_3) \ldots P(a_m)$.

Let $M = N * p_{m+1}{}^{am+1}$. Then, with $F'(p_{m+1}{}^{am+1}) = P(p_{m+1}{}^{am+1})$ and applying similar arguments it is evident that

$F'(M) > P(a_1)\, P(a_2)\, Pa_3) \ldots P(a_m)\, P(a_{m+1})$.

Therefore, by the principle of mathematical induction, the proof is complete.

We also have

$P(1) = 1, P(2) = 2$ and $P(3) = 3$. $P(4) = 5$, $P(5) = 7$

and we have $P(a) > a + 1$ for $a > 4$. This gives us

$P(a_1)\, P(a_2)\, Pa_3) \ldots P(a_k) > (a_1 +1)(a_2 + 1)(a_3 + 1) \ldots (a_k +1) = d(N)$

for $a_i > 4$.

From the proposition, we have

$F'(N) > (a_1 +1)(a_2 + 1)(a_3 + 1) \ldots (a_k +1) = d(N)$ for $a_i > 4$.

The next step will be to prove a stronger proposition.

Proposition 2:

$F(a, b) > F(a) * F(b)$ for $a > 2$ and $b > 2$

or in other words

$F'(p_1{}^a p_2{}^b) > F'(p_1{}^a) * F'(p_2{}^b)$

Proof: The proof is similar to that for the first proposition.

Proposition 3: If $N = p_1{}^{a1} p_2{}^{a2} p_3$, then $F'(N) > d(N)$.

Proof: Let $M = p_1{}^{a1} p_2{}^{a2}$. Then, from [2] we have

$$F'(N) = F'(Mp_3) = F'^*(M) = \sum_{d/M} F'(d_r)$$

where $d_r \mid M$. This equals

$$F'(p_1{}^{a1} p_2{}^{a2}) + F'(p_1{}^{a1} p_2{}^{a2-1}) + F'(p_1{}^{a1-1} p_2{}^{a2}) + \{d(p_1{}^{a1} p_2{}^{a2}) - 3\}$$

where each term in the expression is more than one.

From proposition 2, this is greater than

$F'(p_1^{a1})F'(p_2^{a2}) + F'(p_1^{a1}).F'(p_2^{a2-1}) + F'(p_1^{a1-1}).F'(p_2^{a2}) + \{d(p_1^{a1} p_2^{a2}) - 3\}$

where each term is more than one. This is greater than

$P(p_1^{a1})P(p_2^{a2}) + P(p_1^{a1}).P(p_2^{a2-1}) + P(p_1^{a1-1}).P(p_2^{a2}) + \{d(p_1^{a1} p_2^{a2}) - 3\}$

where each term is more than one.

We have $P(a) > (a+1)$ for $a > 4$, from which it follows that this is greater than

$(a_1 + 1)(a_2 + 1) + a_1(a_2 + 1) + a_2(a_1 + 1) + (a_1 + 1)(a_2 + 1) - 3.$

$> 2(a_1 + 1)(a_2 + 1),\ a_1,\ a_2 > 4$

$> 2d(M) = d(M p_3) = d(N).$

Therefore, we have proven that

$F(`p_1^{a1} p_2^{a2} p_3.) > d(p_1^{a1} p_2^{a2} p_3.)$ for $a_1, a_2 > 4$

and to determine the largest Balu number, we need to examine only the cases where $k < 4$.

When these cases are examined, it is easy to see that 1, 16, 36 is the complete list of Balu numbers.

We also note that

1) The equation d(N) = F'(N) + 1 has five solutions: 2, 4, 8, 24 and 60.
2) The equation F'(N) = 2*F'(M) has three solutions

F'(6480) = 2* F'(2160)= 424 , F'(360) = 2* F'(180) = 52,

F'(72) = 2* F'(30) =16.

Readers are also invited to find solutions to:

(a). F'(N) = k* F'(M), for different values of k.

(b) F'(N) /F'M) = N/M, (one solution is N=360. M = 180.), another is N= 210 , M = 70 F'(N) =15. and F'(M) = 5.

The following table contains some solutions, which are highlighted in bold.

Canonical form	Number	SFP F'(N)	d(N)
p^4q^4r	$2^4 3^4 5 = 6480$	424	50
p^4q^3r	$2^4 3^3 5 = 2160$	212	40
p^4q^2r	$2^4 3^2 5 = 720$	98	30
p^4qr	$2^4 *3* 5 = 240$	38	20
p^4q	$2^4*3 = 144$	29	10
p^3q^3r	$2^3 *3^3 *5 = 1080$	109	32
p^3q^2r	$2^3*3^2*5 = 360$	52	24
p^3qr	$2^3*3*5= 120$	21	16
p^3q	$2^3*3 =24$	7	8
p^2q^2r	$2^2*3^2*5 =180$	26	18
p^2qr	$2^2*3*5 = 60$	11	12
p^2q	$2^2*3 = 12$	4	6
Pqr	$2*3*5 = 30$	5	8
p^2q^2	**$2^2*3^2 = 36$**	**9**	**9**
pq	$2*3 =6$	2	4
p^4	**$2^4 = 16$**	**5**	**5**
p^3	$2^3 = 8$	3	4
p^2	$2^2 = 4$	2	3
p	2	1	2
p^0	**1**	**1**	**1**

Section 2

Smarandache Pascal Derived Sequences

Definition: Start with any sequence S_b, which we will call the base sequence. A Smarandache Pascal derived sequence S_d is defined as follows:

$$T_{n+1} = \sum_{k=0}^{n} {}^{r}C_k * t_{k+1},$$ where t_k is the kth term of the base sequence.

Let the terms of the base sequence S_b be

$b_1, b_2, b_3, b_4, \ldots$

Then the Smarandache Pascal derived sequence S_d: $d_1, d_2, d_3, d_4, \ldots$ is defined as follows:

$d_1 = b_1$
$d_2 = b_1 + b_2$
$d_3 = b_1 + 2b_2 + b_3$
$d_4 = b_1 + 3b_2 + 3b_3 + b_4$

…

$$d_{n+1} = \sum_{k=0}^{n} {}^{n}C_k\, b_{k+1}$$

These derived sequences exhibit interesting properties for some base sequences.

Examples:
1) S_b : 1, 2, 3, 4, . . . (natural numbers).

S_d : 1, 3, 8, 20, 48, . . .

which can be written in the form

$2x2^{-1}, 3x2^{0}, 4x2^{1}, 5x2^{2}, 6x2^{3}, \ldots$

It is also easy to prove that

$T_n = 4(T_{n-1} - T_{n-2})$ for $n > 2$

and

$T_n = (n+1) * 2^{n-2}$.

2) S_b : 1, 3, 5, 7, . . . (odd numbers).
S_d : 1, 4, 12, 32, 80, . . .

The first differences are 1, 3, 8, 20, 48, . . . , which is the same as the S_d sequence for the natural numbers. The S_d sequence can also be written in the form

$1x2^0, 2x2^1, 3x2^2, 4x2^3, 5x2^4, \ldots$

Again, it is easy to prove that

$T_n = 4(T_{n-1} - T_{n-2})$ for $n > 2$ and $T_n = n * 2^{n-1}$.

3) Smarandache Pascal Derived Bell Sequence
Start with the Smarandache Factor Partitions (SFP) sequence for the square-free numbers, which is the same as the Bell number sequence.

S_b: 1, 1, 2, 5, 15, 52, 203, 877, 4140, . . .

The derived sequence is

S_d: 1, 1, 2, 5, 15, 52, 203, 877, 4140, . . .

which is the same sequence. Because of this property, we will call it the Pascal Self Derived Sequence.

4) Start with the Fibonacci sequence as the base sequence

S_b: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 114, 233, . . .

The derived sequence is

S_d: 1, 2, 5, 13, 34, 89, 233, . . .

which is every other element of the Fibonacci sequence. From this, it follows that

$$F_{2n} = \sum_{k=0}^{n} {}^{n}C_k * F_k$$

, where F_k is the k^{th} term of the base Fibonacci sequence.

If we take the previous derived sequence as the base sequence we get the following derived sequence S_{dd}

S_{dd}: 1, 3, 10, 35, 125, 450, 1625, 5875, 21250, . . .

It is interesting to note that the first two terms are divisible by 5^0, the next two terms by 5^1, the next two by 5^2, the next two by 5^3 and so forth. Expressed as a formula

$$T_{2n} \equiv T_{2n-1} \equiv 0 \pmod{5^n}.$$

If we carry out the division, we have

1, 3, 2, 7, 5, 18, 13, 47, 34, 123, 89, . . . (*)

and the sequence formed by the odd numbered terms is

1, 2, 5, 13, 34, 89, . .

which is the original sequence S_d that was used as the base.

Another interesting characteristic of the (*) sequence is that every even numbered term is the sum of the two adjacent odd numbered terms.

(3 = 1+2, 7 = 2 +5, 18 = 5 + 13 etc.).

Conjecture: F_n is the nth Fibonacci number.

$$F_{2m+1} = (1/5^m) \sum_{r=0}^{2m+1} \{ {}^{2m+1}C_r (\sum_{k=0}^{r} {}^{r}C_k F_k) \}.$$

which would be yet another beautiful result involving the Fibonacci numbers if it could be proven.

Note: A companion formula where the Fibonacci numbers are replaced with the Lucas numbers could also be considered.

The next operation with be the Pascalisation of the Fibonacci sequence with the indexes in arithmetic progression.

Consider the following sequence formed by the Fibonacci numbers whose indexes are in arithmetic progression.

F_1, F_{d+1}, F_{2d+1}, F_{3d+1}, . . . on Pascalisation gives the following sequence

1, $d *F_2$, $d^2 *F_4$, $d^3 *F_6$, $d^4 *F_8$, . . ., $d^n *F_{2n}$, . . .

For d = 5, we have the sequences

Base sequence: F_1, F_6, F_{11}, F_{16}, . . .

1, 13, 233, 4181, 46368, . . .

Derived sequence:

1, 14, 260, 4920, 93200, . . .

in which we notice that

260= 20 *(14- 1), 4920 = 20 *(260 - 14) , 93200 = 4920 - 260),

which suggests the following conjecture.

Conjecture: The terms of the Pascal derived sequence for d = 5 are given by

$T_n = 20.(T_{n-1} - T_{n-2})$ for $n > 2$.

For d = 8 we have

Base sequence: $F_1, F_9, F_{17}, F_{25}, \ldots$

S_b ---- 1, 34, 1597, 75025, . . .
S_d ---- 1, 35, 1666, 79919, . . .

$= 1, 35, (35-1) * 7^2, (1666 - 35) * 7^2, \ldots$ etc.

which suggests the following conjecture.

Conjecture: The terms of the Pascal derived sequence for d = 8 are given by

$T_n = 49 * (T_{n-1} - T_{n-2})$, $n > 2$.

We also put forward the following similar conjectures.

For d = 10, $T_n = 90 *(T_{n-1} - T_{n-2})$, $(n > 2)$.
For d = 12, $T_n = 18^2 *(T_{n-1} - T_{n-2})$, $(n > 2)$.

Note: There seems to be a direct relation between d and the coefficient of $(T_{n-1} - T_{n-2})$ (or the common factor) of each term.

5) Smarandache Pascal Derived Square Sequence
Start with the sequence of perfect squares

S_b: 1, 4, 9, 16, 25, . . .

S_d: 1, 5, 18, 56, 160, 432, . . .

Which can be expressed in the form

1, 5 *1, 6 * 3, 7 * 8, 8 *20, 9 *48 , . . . , ($T_n = (n+3)t_{n-1}$) ,

where t_r is the r^{th} term of Pascal derived natural number sequence.

It is also possible to derive $T_n = 2^{n-2} * (n+3)(n)/ 2$.

6) Smarandache Pascal Derived Cube Sequence
Start with the sequence of perfect squares
S_b: 1, 8, 27, 64, 125, . . .

S_d: 1, 9, 44, 170, 576, 1792, . . .

In this case, we have $T_n \equiv 0(\mathrm{mod}\ (n+1))$.

Similarly we can derive sequences for higher powers, which can be analyzed for patterns.

7) Smarandache Pascal Derived Triangular Number Sequence
Start with the sequence of triangular numbers

S_b: 1, 3, 6, 10, 15, 21, . . .

S_d: 1, 4, 13, 38, 104, 272, . . .

8) Smarandache Pascal Derived Factorial Sequence
Start with the sequence of factorial numbers

S_b: 1, 2, 6, 24, 120, 720, 5040, . . .

S_d: 1, 3, 11, 49, 261, 1631, . . .

We can verify that $T_n = n *T_{n-1} + T_{n-2} + 1$.

Open problem: Are there infinitely many primes in the previous S_d sequence?

Start with the natural number sequence again

S_b: 1, 2, 3, 4, 5, . . .

The corresponding derived sequence is

S_d: $2 *2^{-1}, 3 *2^{0}, 4 *2^{1}, 5 *2^{2}, 6 *2^{3}, \ldots$

Using this as the base sequence, we can get another derived sequence, which we denote by

S_{dd} or S_{d2}: 1, 4, 15, 54, 189, 648, . . .

Which can be rewritten as

$1, 4 *3^0 , 5 *3^1, 6 *3^2 , 7 *3^3 \ldots$

Similarly, we can use this as the base sequence to get the new derived sequence

S_{d3}: $1, 5 *4^{0,} 6 *4^1, 7 *4^2 , 8 *4^3 , \ldots$

The pattern of the first few derived sequences suggests the pattern

S_{dk}: $1, (k+2) *(k+1)^0, (k+3) *(k+1)^1 , (k+4) *(k+1)^2, \ldots, (k+r) *(k+1)^{r-2}$

which can be proven by induction.

Generalization:

We can take any arithmetic progression with the first term a and the common difference b as the base sequence and get the derived k^{th} order sequences to generalize the above results.

Section 3

Depascalization of Smarandache Pascal Derived Sequences and Backward Extended Fibonacci Sequence

In the previous sequence, we started with a base sequence S_b

$b_1, b_2, b_3, \ldots$

and then used the base sequence to create the derived sequence S_d.

$d_1 = b_1$
$d_2 = b_1 + b_2$
$d_3 = b_1 + 2b_2 + b_3$
$d_4 = b_1 + 3b_2 +3 b_3 + b_4$
…

$$d_{n+1} = \sum_{k=0}^{n} {}^nC_k \, b_{k+1}$$

Definition: Given a derived sequence S_d, the process of extracting the base sequence S_b will be called Depascalization. The interesting observation is that this will involve Pascal's triangle, although with a difference.

It is clear that

$b_1 = d_1$

$b_2 = -d_1 + d_2$

$b_3 = d_1 - 2d_2 + d_3$

$b_4 = -d_1 + 3d_2 - 3d_3 + d_4$

. . .

Which suggests the general formula

$$b_{n+1} = \sum_{k=0}^{n} (-1)^{n+k} * {}^{n}C_k * d_{k+1}$$

which can be established by induction.

In the examples to be given, we will see that depascalized sequences exhibit some interesting patterns.

To begin with we define the Backward Extended Fibonacci Sequence (BEFS) in the following way.

We start with the Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, . . .

where $F_1 = 1$, $F_2 = 1$, and $F_{n-2} = F_n - F_{n-1}$, $n > 2$.

If we allow n to take negative values 0, -1, -2, . . . and we subtract the terms,

$F_0 = F_2 - F_1 = 0$, $F_{-1} = F_1 - F_0 = 1$, $F_{-2} = F_0 - F_{-1} = -1$, . . .

the Fibonacci sequence can be extended backwards

. . . F_{-6} F_{-5}, F_{-4}, F_{-3}, F_{-2}, F_{-1}, $\underline{\mathbf{F_0}}$, F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, . . .

. . . -8, 5, -3, 2 , -1, 1 **<u>0</u>**, 1, 1, 2 , 3, 5 8, 13, 21, 34, . . .

This sequence will be called the Fibonacci Extended Backwards Sequence (FEBS).

Depascalization of the Fibonacci Sequence

The Fibonacci sequence is

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, . . .

The corresponding depascalised sequence $S_{d(-1)}$ comes out to be

$S_{d(-1)}$ ----   1, 0, 1, -1, 2, -3, 5, -8, . . .

Note that this sequence is nothing more than the extended Fibonacci sequence rotated 180^0 about F_1 and then the left terms omitted.

If we depascalize one more time to get the sequence $S_{d(-2)}$

1, -1, 2, -5, 13, -34, 89, -233, …

This sequence can be obtained from the Fibonacci sequence by:

1) Removing even numbered terms.
2) Multiplying the alternate terms by –1.

Conjecture 1: If the first r terms of the Fibonacci Sequence are removed and the remaining sequence is Pascalised , the resulting derived sequence is

F_{2r+2} , F_{2r+4} , F_{2r+6} , F_{2r+8} , . . .

where F_r is the rth term of the Fibonacci Sequence.

Conjecture 2: In the FEBS, if we take T_r as the first term and depascalize the right side of it, then we get the resulting sequence as the left side of it (looking rightwards) with T_r as the first term.

As an example, let r = 7, T_7 = 13

. . . T_{-6} T_{-5}, T_{-4}, T_{-3}, T_{-2} , T_{-1} , T_0, T_1, T_2, T_3, T_4, T_5, T_6, $\underline{\mathbf{T_7}}$, T_8, T_9, . . .
. . . -8, 5, -3, 2 , -1, 1 0, 1, 1, 2 , 3, 5 8, <u>**13**</u>, 21, 34, 55, 89, . . .

→→→→→→→→→→→

depascalisation

The depascalized sequence is

13, 8, 5, 3, 2, 1, 1, 0, 1, -1, 2, -3, 5, -8 . .

which is obtained by rotating the FEBS around 13 (T_7) by 180 and then removing the terms on the left side of 13.

Readers are encouraged to search for more fascinating results.

Section 4

Proof of the Depascalization Theorem

In the previous section, the operation of extracting a base sequence from a derived sequence was defined. Known as depascalization, it was stated that the general term of the base sequence (b_i) was given by the formula

$$b_{n+1} = \sum_{k=0}^{n} (-1)^{n+k} * {}^{n}C_k * d_{k+1}.$$

In this section, it will be proven that this is indeed the case.

Theorem: Given the terms of the derived sequence

$d_1 = b_1$
$d_2 = b_1 + b_2$
$d_3 = b_1 + 2b_2 + b_3$
$d_4 = b_1 + 3b_2 + 3 b_3 + b_4$
…

$$d_{n+1} = \sum_{k=0}^{n} {}^{n}C_k \, b_{k+1}$$

The value of the elements of the base sequence can be computed using the formula

$$b_{n+1} = \sum_{k=0}^{n} (-1)^{n+k} * {}^{n}C_k * d_{k+1}.$$

Proof: By induction on the subscript of b.

Basis step: If n = 0, then $(-1)^0 = 1$, ${}^0C_0 = 1$, so $b_1 = d_1$.

Inductive step: Let the formula be true for all subscripts less than or equal to k+1. Then we have

$$b_{k+1} = {}^{k}C_0 (-1)^{k+2} d_1 + {}^{k}C_1 (-1)^{k+1} d_2 + \ldots + {}^{k}C_k (-1)^2.$$

We also have

$$d_{k+2} = {}^{k+1}C_0 \, b_1 + {}^{k+1}C_1 \, b_2 + \ldots + {}^{k+1}C_r \, b_{r+1} + \ldots + {}^{k+1}C_{k+1} \, b_{k+2}$$

which gives

$b_{k+2} = (-1)\ {}^{k+1}C_0\, b_1 - {}^{k+1}C_1\, b_2 - \ldots - {}^{k+1}C_r\, b_{r+1} - \ldots + d_{k+2}$.

Substituting the values of $b_1, b_2, \ldots$ in terms of $d_1, d_2, \ldots$, the coefficient of d_1 is

$$(-1)\ {}^{k+1}C_0 + (-{}^{k+1}C_1)(-{}^{1}C_0) + (-{}^{k+1}C_2)({}^{2}C_0) + \ldots + (-1)^r * {}^{k+1}C_r)({}^{r}C_0) + \ldots +$$

$$(-1)\ {}^{k+1}({}^{k+1}C_k)({}^{k}C_0) - {}^{k+1}C_0 + {}^{k+1}C_1 .\ {}^{1}C_0 - {}^{k+1}C_2 * {}^{2}C_0 + \ldots + (-1)^r * {}^{k+1}C_r * {}^{r}C_0$$

$$+ \ldots + (-1)^{k+1} * {}^{k+1}C_k * {}^{k}C_0.$$

Similarly, the coefficient of d_2 is

$${}^{k+1}C_1 * {}^{1}C_1 + {}^{k+1}C_2 * {}^{2}C_1 + \ldots + (-1)^{r+1} * {}^{k+1}C_r * {}^{r}C_1 + \ldots + (-1)^{k+1} * {}^{k+1}C_k * {}^{k}C_1.$$

Repeating the process, the coefficient of d_{m+1} is

$${}^{k+1}C_m * {}^{m}C_m + {}^{k+1}C_{m+1} * {}^{m+1}C_m - \ldots + (-1)^{r+m} * {}^{k+1}C_{r+m} * {}^{r+m}C_m + \ldots +$$

$$(-1)^{k+m} * {}^{k+1}C_k * {}^{k}C_m$$

$$= \sum_{h=0}^{k-m} (-1)^{h+1}\ {}^{k+1}C_{m+h} * {}^{m+h}C_m .$$

Which is equal to

$$\sum_{h=0}^{(k+1)-m} (-1)^{h+1}\ {}^{k+1}C_{m+h} * {}^{m+h}C_m + (-1)^{k+m} * {}^{k+1}C_{k+1} * {}^{k+1}C_m .$$

By a theorem in section 7 of chapter 1,

$$\sum_{k=0}^{n-r} {}^{n}C_{r+k}\ {}^{r+k}C_r\, m^k = {}^{n}C_r\, (1+m)^{(n-r)}.$$

Applying this theorem, we have

$$= {}^{k+1}C_m \{ 1 + (-1)\}^{k+1-m} + (-1)^{k+m} * {}^{k+1}C_m$$
$$= (-1)^{k+m} .\ {}^{k+1}C_m.$$

which shows that the proposition is true for (k+2) as well. The proposition has already been verified for the index equal to 1, hence by induction the proof is complete.

In matrix notation if we write

$[\, b_1, b_2, .. b_n \,]_{1xn} * [p_{i,j}\,]'_{nxn} = [\, d_1, d_2, .. d_n \,]_{1xn}$

where $[p_{i,j}\,]'_{nxn}$ = the transpose of $[p_{i,j}\,]_{nxn}$ and $[p_{i,j}\,]_{nxn}$ is given by $p_{i,j} = {}^{i-1}C_{j-1}$ if $i \le j$ else $p_{i,j} = 0$.

Then, we get the result

If $[q_{i,j}]_{nxn}$ is the transpose of the inverse of $[p_{i,j}\,]_{nxn}$, then

$q_{i,j} = (-1)^{j+i} * {}^{i-1}C_{j-1}$.

We also have

$[\, b_1, b_2, .. b_n \,] * [q_{i,j}\,]'_{nxn} = [\, d_1, d_2, .. d_n \,]$

where $[q_{i,j}\,]'_{nxn}$ = the transpose of $[q_{i,j}\,]_{nxn}$.

Section 5

Smarandache Friendly Numbers and A Few More Sequences

Definition: If the sum of any set of consecutive terms of a sequence equals the product of the first and the last number of the set, then this pair is called a Smarandache Friendly Pair with respect to the sequence.

1) Smarandache Friendly Natural Number Pairs:
Consider the natural number sequence

1, 2, 3, 4, 5, 6, 7, . . .

then the Smarandache friendly pairs are

(1,1), (3, 6), (15,35) , (85, 204), . . .

as 3 + 4 + 5 +6 = 18 = 3 * 6

15 + 16 + 17 + . . . + 33 + 34 + 35 = 525 = 15 * 35.

There exist infinitely many such pairs. This is evident from the fact that if (m, n) is a friendly pair then so is the pair (2n+m, 5n +2m -1).

2) Smarandache Friendly Prime Pairs:
Starting with the sequence of prime numbers

2, 3, 5, 7, 11, 13, 17, 23, 29, . . .

we have 2 +3 + 5 = 10 = 2 * 5. Therefore, (2, 5) is a friendly prime pair.

$3 + 5 + 7 + 11 + 13 = 39 = 3 * 13$, so (3,13) is a friendly prime pair.

$5 + 7 + 11 + . . .+ 23 + 29 + 31 = 155 = 5 * 31$, so (5, 31) is a friendly prime pair.

Similarly (7, 53) is also a Smarandache friendly prime pair. In a friendly prime pair (p, q) we define q as the big brother of p.

Open problem: Are there infinitely many friendly prime pairs?
Open problem: Are there big brothers for every prime p?
3) Smarandache Under-Friendly Pair:
If the sum of any set of consecutive terms of a sequence is a divisor of the product of the first and the last number of the set then this pair is called a Smarandache Under- Friendly Pair with respect to the sequence.
4) Smarandache Over-friendly Pair:

If the sum of any set of consecutive terms of a sequence is a multiple of the product of the first and the last number of the set then this pair is called a Smarndache Over-Friendly Pair with respect to the sequence.

5) Smarandache Sigma Divisor Prime Sequence:
The sequence of primes p_n, which satisfy the congruence

$$\sum_{r=1}^{n-1} p_r \equiv 0 (\mathrm{mod}\ p_n).$$

The first few terms of the sequence are

2, 5, 71, …

since 5 divides 10, and 71 divides $568 = 2 + 3 + 5 + . . . + 67$.

Open problem: Is the Smarandache Sigma Prime Sequence infinite?

Conjecture: Every prime divides at least one cumulative sum.

6) Smarandache Smallest Number With ‘n’ Divisors Sequence

1, 2, 4, 6, 16, 12, 64, 24, 36, 48, 1024, . . .

$d(1) = 1, d(2) = 2, d(4) = 3, d(6) = 4, d(16) = 5, d(12) = 6$, . . .

$d(T_n) = n$, where T_n is the smallest number having n divisors. It is clear that $T_p = 2^{p-1}$, if p is prime. The sequence $T_n + 1$ is

2, 3, 5, 7, 17, 13, 65, 25, 37, 49, 1025, . . .

Conjecture: The $T_n + 1$ sequence contains infinitely many primes.

Conjecture: Seven is the only Mersenne prime in the $T_n + 1$ sequence.

Conjecture: The $T_n + 1$ sequence contains infinitely many perfect squares.

7) Smarandache Integer Part k^n Sequences (SIPS)

(i) Smarandache Integer Part π^n Sequence

$[\pi^1], [\pi^2], [\pi^3], \ldots$

where [ ] means the integer part of the expression. The first few terms of the sequence are

3, 9, 31, 97, . . .

(ii) Smarandache Integer Part e^n Sequence

$[e^1], [e^2], [e^3], \ldots$

2, 7, 20, 54, 148, 403, . . .

Conjecture: Every SIPS contains infinitely many primes.

8) Smarandache Summable Divisor Pairs (SSDP)
This is a set of ordered pairs (m,n), where d(m) + d(n) = d(m+n), where d(n) is the number of divisors of n.

For example, we have d(2) + d(10) = d(12) ,

d(3) + d(5) = d(8),

d(4) + d(256) = d(260),

d(8) + d(22) = d(30) , etc.

hence (2, 10) , (3,5) , (4, 256) , (8, 22) are SSPDs.

Conjecture: There are infinitely many SSPDs.

Conjecture: For every integer m there exists and integer m such that (m,n) is an SSDP.

9) Smarandache Product of Digits Sequence
The nth term of this sequence T_n is defined as the product of the digits of n.

1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 2, 4, 6, 8, 10, 12, . . .

10) Smarandache Sigma Product Of Digits Natural Sequence
The nth term of this sequence is defined as the sum of the products of all the numbers from 1 to n.

1, 3, 6, 10, 15, 21, 28, 36, 45, 45, 46, 48, 51, 55, 60, 66, 73, 81, 90, 90, 92, 96, . . .

Here we consider the terms of the sequence for some values of n.

For n = 9 we have $T_n = 45$.

For n = 99 we have $T_n = 2070 = 45^2 + 45$..

Similarly we have

$T_{999} = (T_9)^3 + (T_9)^2 + T_9 = 45^3 + 45^2 + 45 = (45^4 - 1) / (45 - 1) = (45^4 - 1) / 44$

The pattern suggested by the previous sequence can easily be proved.

This can be further generalized for a number system other than base 10.

For a number system with base b the $(b^r - 1)^{th}$ term in the Smarandache sigma product of digits sequence is

$$2[\{b(b-1)/2\}^{r+1} - 1] / \{b^2 - b - 2\}$$

Further exploration: The task ahead is to find the n^{th} term in the above sequence for an arbitrary value of n.

11) Smarandache Sigma Product of Digits Odd Sequence

1, 4, 9, 16, 25, 26, 29, 34, 41, 50, 52, 58, 68, 82, 100, 103, 112, 127, 148, . . .

It can be proved that for $n = 10^r - 1$, T_n is the sum of the r terms of the Geometric progression with the first term as 25 and the common ratio as 45.

12) Smarandache Sigma Product Of Digits Even Sequence

2, 6, 12, 20, 20, 22, 26, 32, 40, 40, 44, 52, 62, 78, 78, 84, 96, 114, 138, . . .

It can again be proven that for $n = 10^r - 1$, T_n is the sum of the r terms of the Geometric progression with the first term as 20 and the common ratio as 45.

Open Problem: Are there infinitely many common members in sequences {11} and {12}?

Section 6

Some New Smarandache Sequences, Functions and Partitions

1) Smarandache LCM Sequence (SLS)
L(n) is the Least Common Multiple (LCM) of the natural numbers from 1 through n.
The first few numbers are

SLS → 1, 2, 6, 12, 60, 60, 420, 840, 2520, 2520, . . .

Smarandache LCM Odd Sequence (SLOS)

$L_O(n)$ is the Least Common Multiple (LCM) of the first n odd natural numbers.

SLOS → 1, 3, 15, 105, 415, 4565, . . .

The Smarandache LCM Even Sequence (SLES) can be defined in a similar way

SLES → 2, 4, 12, 24, 240, 240 . . .

It is easy to see that

T_{2n+1} (SLES) = 2^n T_n (SLOS) ,

which is a direct consequence of the definition .

Additional ideas for exploration:

A) If each term of the SLS is incremented by 1, we get the new sequence.

2 ,3 ,7, 13, 61, 421, 841. 2521, . . .

Does this sequence contain infinitely many primes?

B) Does

$$\sum_{n=1}^{\infty} L(n)/n! \text{ exist? If it does, determine the value.}$$

C) Does

$$\sum_{n=1}^{\infty} 1/L(n) \text{ exist? If it does, determine the value.}$$

2) Divisor Sequences

Define A_n = { x | d(x) = n}.

Then

A_1 = {1}

A_2 = { p | p is a prime }

A_3 = { x | x = p^2 , p is a prime }

A_4 = { x | x = p^3 or x = p_1p_2 , p , p_1 ,p_2 are primes } .

A_4 → 6 , 8 , 10 , 14 , 15 , 21 , 22 , 26 , 27 , . . .

We have

$$\sum 1/T_n = 1 \text{ for } A_1$$

and the limit does not exist for A_2.

$$\sum_{n=1}^{\infty} 1/L(n) \text{ exists and is less than } \pi^2/6 \text{ for } A_3 \text{ as}$$

$$\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6.$$

The above limit exists for A_p, where p is a prime. It is a point for further exploration to determine if the limits exist for A_4, A_6 and so forth.

Divisor Multiple Sequence

SDMS = {n | n = k* d(n) }.

SDMS $\rightarrow$ 1, 2, 8 ,9, 12 , . . .

3) Quad Prime Sequence Generator

SQPSG = { r | 90r+11 , 90r+13 , 90r+17, 90r+ 19 are all primes }

SQPSG $\rightarrow$ 0 , 1, 2 , . . .

Open problem: Are there infinitely many terms in this sequence?

4) Prime Location Sequences

Definition

P_0 = sequence of primes .

P_1 = sequence of primeth primes

$P_1 \rightarrow$ 3, 5, 11, 17, . . .

P_2 = sequence of primeth , primeth prime .

. . .

P_r = sequence of primeth , primeth , . . . r times ,primes

Open problem: If T_n is the n^{th} term of P_r, then what is the minimum value of r for which

$$\sum_{n=1}^{\infty} 1/T_n \text{ exists ?}$$

5) Partition Sequences

(i) Prime partition

The number of partitions of n into prime parts

$Sp_p(n) \rightarrow 0, 1, 1, 1 ,1 2, 2, 3, . . .$

(ii) The number of partitions of n into composite parts

$Sp_c(n) \rightarrow 1, 1, 1, 2, 1, 3, . . .$

(iii) Divisor partitions
The number of partitions into numbers that are divisors of n.

$SP_d(n) \rightarrow 1, ,1 ,1 ,2 ,1, \ldots$

(iv) Co-prime Partitions ($SP_{cp}(n)$)
The number of ways n can be partitioned into co-prime parts.

(v) Non-Co-prime Partitions ($SP_{ncp}(n)$)
The number of ways n can be partitioned into non co-prime parts.

(vi) Prime Square Partitions
The number of ways n can be partitioned into prime square parts.

These ideas could be generalized to define many more such sequences.

6) Combinatorial Sequences
Define a sequence in the following way

$T_1 = 1$, $T_2 = 2$, T_n = sum of all the products of the previous terms taken two at a time (n>2).

This sequence will be abbreviated SCS(2) and the first few terms are

SCS(2) = 1, 2, 2, 8 ,48, . . .

This definition can be generalized in the following way:
Start out with the explicit values of $T_j = j$ being assigned to the first r terms, then define all subsequent terms

T_n = sum of all the products of the previous terms taken r at a time, n >r.

This sequence will be abbreviated SCS(r).

Another, similar sequence can be defined by

Let $T_k = k$ for k = 1 to n.

T_r = sum of all products of (r-1) terms of the sequence taken

(r-2) at a time (r > n) . This sequence will be abbreviated SC_vS.

For n = 2, the first few elements of the sequence are

$T_1 = 1, T_2 = 2, T_3 = 3, T_4 = 17, \ldots$

Open problem: How many of the consecutive terms of SCS(r) are pairwise coprime?

Open problem: How many of the terms of SC_vS are primes?

(ii) Prime product sequences

SPPS(n)

T_n = sum of all the products of primes chosen from the first n primes taking (n-1) primes at a time. The first few terms are

SPPS(n) $\to$ 1, 5, 31, 247, 2927, . . .

$T_1 = 1$, $T_2 = 2 + 3$, $T_3 = 2*3 + 2*5 + 3*5 = 31$.

$T_4 = 2*3*5 + 2*3*7 + 2*5*7 + 3*5*7 = 247$ etc.

Open problem: How many of the numbers in this sequence are prime?

7) ϕ -SEQUENCE

$(S\phi S) = \{n \mid n = k * \phi(n)\}$

$S\phi S \to 1, 2, 4, 6, 8, 12, \ldots$

8) Prime Divisibility Sequence

SPDS = $\{n \mid n \text{ divides } p_n + 1, p_n \text{ is the } n^{th} \text{ prime}\}$

SPDS→ 1, 2, 3, 4, 10, . . .

9) Divisor Product Sequence

$T_n = \Pi d_k$ where d_k is a divisor of n .

The first few terms of this sequence are

1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, . . .

Section 7

Smarandache Reverse Auto Correlated Sequences and Some Fibonacci Derived Smarandache Sequences

Definition: Let $a_1, a_2, a_3, \ldots$, be a sequence. We define a Smarandache Reverse Auto-Correlated Sequence (SRACS) $b_1, b_2, b_3, \ldots$ in the following way:
$b_1 = a^2_1$, $b_2 = 2a_1a_2$, $b_3 = a^2_2 + 2a_1a_3$,

Or by applying the formula

$$b_n = \sum_{k=1}^{n} a_k * a_{n-k+1}.$$

Such a transformation will be called the Smarandache Reverse Auto Correlation Transformation (SRACT).

Example 1:
Starting with the base sequence

1, 2, 3, 4, 5, . . .

or, expressed another way

${}^1C_1$, ${}^2C_1$, ${}^3C_1$, ${}^4C_1$, ${}^5C_1$, . . .

The SRACS is

1, 4, 10, 20, 35, . . ., which can be rewritten as
${}^3C_3$, ${}^4C_3$, ${}^5C_3$, ${}^6C_3$, ${}^7C_3$, . . .

We will call this sequence SRACS(1)

Taking this as the base sequence, we can compute SRACS(2)

1, 8, 36, 120, 330, . . , which can be rewritten as

${}^{7}C_{7}$, ${}^{8}C_{7}$, ${}^{9}C_{7}$, ${}^{10}C_{7}$, ${}^{11}C_{7}$, . . .

Taking this as the base sequence, we have SRACS(3)

1, 16, 136, 816, 3876, . . . or, expressed another way

${}^{15}C_{15}$, ${}^{16}C_{15}$, ${}^{17}C_{15}$, ${}^{18}C_{15}$, ${}^{19}C_{15}$, . . . ,

The expressions of the sequences using the ${}^{n}C_{r}$ notation suggests the following conjecture.

Conjecture: The sequence obtained by performing the Smarandache Reverse Auto Correlation Transformation (SRACT) n times starting with the set of natural numbers is given by the following:

SRACS(n)

${}^{h-1}C_{h-1}$, ${}^{h}C_{h-1}$, ${}^{h+1}C_{h-1}$, ${}^{h+2}C_{h-1}$, ${}^{h+3}C_{h-1}$, . . . where $h = 2^{n+1}$.

Example 2: Using the triangular numbers as the base sequence

1, 3, 6, 10, 15, . . .

or expressed another way

${}^{2}C_{2}$, ${}^{3}C_{2}$, ${}^{4}C_{2}$, ${}^{5}C_{2}$, ${}^{6}C_{2}$, . . .

The SRACS is

1, 6, 21, 56, 126, . . .

which can be rewritten as

${}^{5}C_{5}$, ${}^{6}C_{5}$, ${}^{7}C_{5}$, ${}^{8}C_{5}$, ${}^{9}C_{5}$, . . .

and we can call it SRACS(1).

Using this as the base sequence, we compute SRACS(2)

1, 12 , 78 , 364 , 1365, . . .

$^{11}C_{11}$, $^{12}C_{11}$, $^{13}C_{11}$, $^{14}C_{11}$, $^{15}C_{11}$, . . .

Taking this as the base sequence we get the elements of SRACS(3)
1, 24, 300, 2600, 17550, . . .

$^{23}C_{23}$, $^{24}C_{23}$, $^{25}C_{23}$, $^{26}C_{23}$, $^{27}C_{23}$, . . . ,

This pattern suggests the following conjecture.

Conjecture: The sequence obtained by performing the Smarandache Reverse Auto Correlation transformation (SRACT) n times starting with the set of Triangular numbers is given by SRACS(n).

$^{h-1}C_{h-1}$, $^{h}C_{h-1}$, $^{h+1}C_{h-1}$, $^{h+2}C_{h-1}$, $^{h+3}C_{h-1}$, . . . where $h = 3 * 2^n$.

Conjecture: Given the base sequence as $^{n}C_{n}$, $^{n+1}C_{n}$, $^{n+2}C_{n}$, $^{n+3}C_{n}$, $^{n+4}C_{n}$, . . .
The set of elements of the sequence SRACS(n) is given by

$^{h-1}C_{h-1}$, $^{h}C_{h-1}$, $^{h+1}C_{h-1}$, $^{h+2}C_{h-1}$, $^{h+3}C_{h-1}$, . . . where $h = (n+1)*2^n$.

Some Fibonacci Derived Smarandache Sequences
1. Smarandache Fibonacci Binary Sequence (SFBS)

In the Fibonacci Rabbit problem we start with an immature pair ' I ', which matures after one season to 'M'. In the second season, this mature pair breeds a new immature pair, in the third season the first mature pair breeds another immature pair and the immature pair becomes mature. In the fourth season, the immature pair reach maturity and both mature pairs breed an immature pair. This continues, where for any season, all immature pairs become mature and all mature pairs breed an immature pair. When this is repeated, we get the sequence

I, M, MI, MIM, MIMMI, MIMMIMIM, MIMMIMIMMIMMI,

If we replace I by 0 and M by 1, we get the following binary sequence

0, 1, 10, 101, 10110, 10110101, 1011010110110, …

When this sequence is converted into the equivalent decimal form, we have

0, 1, 2, 5, 22, 181, 5814, . . .

We will call this sequence the SFBS and we will derive a reduction formula for the general terms of the sequence.

From the binary pattern, we observe

$T_n = T_{n-1}\ T_{n-2}$ {the digits of the T_{n-2} placed to the left of the digits of T_{n-1}.}.

Also the number of digits in T_r is nothing but the r^{th} Fibonacci number by definition . Hence we have

$$T_n = T_{n-1} * 2^{F(n-2)} + T_{n-2}\ .$$

Open problem: How many elements of this sequence are prime?

Open problem: How many elements of this sequence are Fibonacci numbers?

2. Smarandache Fibonacci Product Sequence

The Fibonacci sequence is 1, 1, 2, 3, 5, 8, . . .

Starting with $T_1 = 2$, and $T_2 = 3$ and then using the general formula, $T_n = T_{n-1} * T_{n-2}$ we get the following sequence

2, 3, 6, 18, 108, 1944, 209952, … (*)

In this sequence, which is determined by the first two terms, you can find the entire Fibonacci sequence. This is clear if you write the sequence in the following way:

$2^1, 3^1, 2^1\ 3^1, 2^1\ 3^2, 2^2\ 3^3, 2^3\ 3^5, 2^5\ 3^8, \ldots$

From this, it can be seen that $T_n = 2^{Fn-1} * 3^{Fn}$.

This idea can be extended by choosing r terms instead of only two and changing the recursive term to

$$T_n = T_{n-1}\ T_{n-2}\ T_{n-3} \ldots T_{n-r} \text{ for } n > r.$$

Conjecture: The sequence obtained by incrementing the elements of (*) by one contains infinitely many primes.

Conjecture: The sequence obtained by incrementing the elements of (*) by one does not contain any Fibonacci numbers.

Section 8

Smarandache Star (Stirling) Derived Sequences

Definition: Let $b_1, b_2, b_3, \ldots$ be a sequence of numbers S_b, that will be the base sequence. Then the Smarandache Star Derived Sequence S_d is defined by using the star triangle

1				
1	1			
1	3	1		
1	7	6	1	
1	15	25	10	1

in the following way.

$d_1 = b_1$

$d_2 = b_1 + b_2$

$d_3 = b_1 + 3b_2 + b_3$

$d_4 = b_1 + 7b_2 + 6b_3 + b_4$

. . .

$$d_{n+1} = \sum_{k=0}^{n} a_{(m,r)} * b_{k+1}$$

where $a_{(m,r)}$ is given by

$$a_{(m,r)} = (1/r!) \sum_{k=1}^{r} (-1)^{r-k} * {}^{r}C_k * k^n$$

For example:

(1) If the base sequence S_b is 1, 1, 1, . . . then the derived sequence S_d is

1, 2, 5, 15, 52, . . . ,

in other words the sequence of Bell numbers. $T_n = B_n$.

(2) If S_b - 1, 2, 3, 4, . . . then

S_d - 1, 3, 10, 37, . . .

we have $T_n = B_{n+1} - B_n$

The Significance of the above transformation will be clear when we consider the inverse transformation. It is clear that the star triangle is nothing but the Stirling Numbers of the Second kind.

Consider the inverse transformation. Given the Smarandache Star Derived Sequence S_d, we wish to retrieve the original base sequence S_b. We get b_k for k = 1, 2, 3, 4,… in the following way :

$b_1 = d_1$

$b_2 = -d_1 + d_2$

$b_3 = 2d_1 - 3d_2 + d_3$

$b_4 = -6d_1 + 11d_2 - 6d_3 + d_4$

$b_5 = 24d_1 - 50d_2 + 35d_3 - 10d_4 + d_5$

The triangle of coefficients

1				
-1	1			
2	-3	1		
-6	11	-6	1	
24	-50	35	-10	1

is made up of the Stirling numbers of the first kind.

Some of the properties of this triangle are:

1) The first column numbers are $(-1)^{r-1} *(r-1)!$, where r is the row number.
2) The sum of the numbers in each row is zero.
3) Sum of the absolute values of the terms in the rth row is r!.

Additional properties of the triangle can be found in the book by Krishnamurthy.

This provides us with a relationship between the Stirling numbers of the first kind and those of the second kind, which can be better expressed in the form of a matrix.

Let $[b_{1,k}]_{1xn}$ be the row matrix of the base sequence.

Let $[d_{1,k}]_{1xn}$ be the row matrix of the derived sequence.

Let $[S_{j,k}]_{nxn}$ be a square matrix of order n in which $s_{j,k}$ is the k^{th} number in the j^{th} row of the star triangle (array of the Stirling numbers of the second kind).

Then we have

$[T_{j,k}]_{nxn}$ is a square matrix of order n in which $t_{j,k}$ is the k^{th} number in the j^{th} row of the array of the Stirling numbers of the first kind.

From this, we have

$[b_{1,k}]_{1xn} * [S_{j,k}]'_{nxn} = [d_{1,k}]_{1xn}$

$[d_{1,k}]_{1xn} * [T_{j,k}]'_{nxn} = [b_{1,k}]_{1xn}$.

Which suggests that $[T_{j,k}]'_{nxn}$ is the transpose of the inverse of the transpose of the matrix $[S_{j,k}]'_{nxn}$.

Readers are encouraged to construct alternate proofs by using a combinatorial approach or other techniques.

Section 9

Smarandache Strictly Staircase Sequence

Definition: Starting with a number system in the base b, we will define a sequence using the following postulate.

1) Numbers are listed in increasing order.
2) In any number, the kth digit is less than the (k+1)st digit.

For example, if b = 6, we have the sequence

1, 2, 3, 4, 5, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45, 123, 124, 125, 134, 135, 145, 234, …

For convenience, we write the terms row wise with the rth row containing numbers with r digits.

(1) 1, 2, 3, 4, 5. { ${}^5C_1$ = 5 numbers }

(2) 12, 13, 14, 15, 23, 24, 25, 34, 35, 45. { ${}^5C_2$ = 10 numbers }

(3) 123, 124, 125, 134,135,145, 234, 235,245, 345. { ${}^5C_3$ = 10 numbers }

(4) 1234, 1235, 1245, 1345, 2345. { ${}^5C_4$ = 5 numbers }

(5) 12345. { ${}^5C_5$ = 1 number }

The following properties are quite evident and are easy to prove.

** The space is considered a number with zero digits.

(1) There are ${}^{b-1}C_r$ (${}^5C_r$ in this case) numbers having exactly r digits.

(2) There are 2^{b-1} (2^5 =32, in this case) numbers in the finite sequence including the space which is considered as the lone number with zero digits.

3. The sum of the product of the digits of the numbers having exactly r digits is the absolute value of the r^{th} term in the b^{th} row of the array of the Stirling numbers of the first kind.

4. The sum of all the sums considered in (3) = b! - 1 (6! - 1 = 719 in this case) .

Open problem: Derive an expression for the sum of all the r digit numbers and therefore for the sum of the whole sequence.

Open problem: If the nth number in the sequence has index n, derive a formula to determine the index of any number in the sequence.

Section 10

The Sum of the Reciprocals Of the Smarandache Multiplicative Sequence

Definition: The kth term (k > 2) of a multiplicative sequence with initial terms m_1 and m_2 is the smallest number equal to the product of two previous terms.

For example, with $m_1 = 2$ and $m_2 = 3$, the sequence is

2, 3, 6, 12, 18, 24, 36, 48, 54, 72, 96, 108, 144, . . .

In this section, it will be proved that the limit of the sum of the reciprocals of the terms of the multiplicative sequence exists for all initial terms m_1 and m_2.

Theorem: The sum of the reciprocals of the multiplicative sequence with initial terms m_1 and m_2. The sum is

$S = 1 / \{ (m_1 - 1) (m_2 - 1) \} + 1/m_1 + 1/m_2.$

Discussion:

Consider the sequence

2, 3, 6, 12, 18, 24, 36, 48, 54, 72, 96 , 108, 144, . . .

It can be written as

$2, 3, 2*3, 2^2*3, 2*3^2, 2^3*3, 2^2*3^2, 2^4*3, 2*3^3, 2^3*3^2, \ldots$

Or, for every n > 2, $T_n = 2^r 3^s$ for some r and s pair. Also, for every $u \in N$ and $v \in N$ there exists some k > 2 for which $2^u*3^v = T_k$. In a nutshell, every term of the form 2^x*3^y appears in the sequence for all values of x and y. On similar lines considering the general multiplicative sequence with m_1 and m_2 as the first two terms of the sequence we have all the terms of the type $m_1^r * m_2^s$ occurring in the sequence with $r + s > 1$,

$(r \in N, s \in N)$.

Proof of theorem: Let

$$S = \sum_{n=1}^{\infty} 1/T_n .$$

Consider the product

$P = (1/m_1 + 1/m_1^2 + 1/m_1^3 + . . .) (1/m_2 + 1/m_2^2 + 1/m_2^3 + . . .).$

We have the sums of the geometric series with common ratio < 1

$$P = \{ (1/m_1)/ (1 - 1/m_1) \} \{ (1/m_2)/ (1 - 1/m_2) \}$$
$$P = \{ 1/ (m_1 - 1) \} \{ 1/ (m_2 - 1) \}$$
$$P = 1 / \{ (m_1 - 1) (m_2 - 1) \}.$$

These sums can be written

$$P = \sum_{r, s = 1}^{\infty} 1/ (m_1^r * m_2^s)$$

or

$$P = \sum_{n=1}^{\infty} 1/T_n - 1/T_1 - 1/T_2$$

or

$$P = \sum_{n=1}^{\infty} 1/T_n - 1/m_1 - 1/m_2.$$

Replacing the summation by S

$$P = S - 1/m_1 - 1/m_2$$

and then rewriting

$$S = P + 1/m_1 + 1/m_2,$$

which gives

$S = 1 / \{ (m_1 - 1) (m_2 - 1) \} + 1/m_1 + 1/m_2$.

For the case where $m_1 = 2$ and $m_2 = 3$, we have $S = 4/3$.

Generalization: The idea of the multiplicative sequence can be generalized by taking the first r terms as $m_1, m_2, m_3, . . . m_r$, with the $(r+1)^{th}$ term defined as the smallest number equal to the product of r previous distinct terms. It can be proved using reasoning similar to that used previously that the limit of the sum of the reciprocals of the terms of the generalized multiplicative sequence also exists and is given by

$$S = \prod_{k=1}^{r} \{1/ (m_k - 1) \} - \sum_{k=1}^{r} 1/ m_k$$

Section 11

Decomposition of the Divisors of A Natural Number Into Pairwise Co-Prime Sets

With n a natural number, let $d_1, d_2, d_3, d_4, d_5, . . .$ be the divisors of n. Given this, we could ask the question:

In how many ways can we choose a pair of divisors which are co-prime to each other?

Similarly, in how many ways can one choose a triplet, quadruplet and so forth of divisors of n which are pairwise co-prime?

Examples:

Let $N = 48 = 2^4 * 3$. The ten divisors of 48 are 1, 2, 3, 4, 6, 8, 12, 16, 24, and 48. The set of co-prime pairs will be represented by $D_2(48)$ and the set of co-prime triplets by $D_3(48)$.

$D_2 (48) = \{(1,2), (1, 3), (1, 4), (1, 6), (1,8), (1, 12), (1,16), (1, 24), (1, 48), (2, 3), (4,3), (8, 3), (16, 3)\}$.

The order of $D_2 (48) = 13$.

$D_3 (48) = \{(1,2, 3), (1, 3, 4), (1, 3, 8), (1,3, 16)\}$.

The order of $D_3 (48) = 4$.

$D_4 (48) = \{\} = D_5 (48) = ...= D_9 (48) = D_{10} (48)$.

As another example, consider n = 30 = 2 x 3 x 5 (a square free number). The 8 divisors of 30 are

1, 2, 3, 5, 6, 10, 15, 30.

$D_2(30) = \{(1,2), (1, 3), (1, 5), (1, 6), (1, 10), (1, 15), (1, 30), (2, 3), (2, 5), (2, 15), (3, 5), (3, 10), (5, 6)\}$.

The order of $D_2(30) = 13$.

Note that this is easily generalized to $O[D_2(p_1p_2p_3)] = 13$, where the primes are distinct.

$D_3(30) = \{(1,2, 3), (1, 2, 5), (1, 3, 5), (2, 3, 5), (1, 3, 10), (1, 5, 6), (1,2, 15)\}$.

The order of $D_3(30) = 7$.

$D_4(30) = \{(1,2, 3, 5)\}$.

The order of $D_4(30) = 1$.

Open problem: Determine the order of $D_r(N)$.

In this section, we consider the simple case of n being a square-free number for r = 2, 3 and so forth.

(A) r = 2.

A reduction formula will be derived, followed by a direct formula.

Let $N = p_1p_2p_3...p_n$ where p_k is a prime for k = 1 to n. We denote $D_2(N) = D_2(1\#n)$ for convenience. We will derive a reduction formula for

$D_2(1 \# (n+1))$.

Let q be a prime such that (q, N) = 1. Then $D_2(Nq) = D_2(1\#(n+1))$ and by definition

$D_2(1\#n)$ is contained in $D_2(1\#(n+1))$. This provides us with $O[D_2(1\#n)]$ elements of $D_2(1\#(n+1))$.

Consider an arbitrarily chosen element (d_k, d_s) of $D_2(1\#n)$. This element when combined with q yields exactly two elements of $D_2(1\#(n+1))$. i.e. (qd_k, d_s) and (d_k, qd_s).

Hence every element of the set $D_2(1\#n)$ contributes two additional elements when combined with the prime q.

The element (1, q) has not been considered in the previously mentioned cases, therefore the total number of elements of $D_2(1\#(n+1))$ is 3 times the order of $D_2(1\#n) + 1$.

And so it follows that,

$O[D_2(1\#(n+1))] = 3 * O[D_2(1\#n)] + 1$. (*)

Applying this formula for the evaluation of $O[D_2(1\#4)]$.

We know that $O[D_2(p_1p_2p_3)] = O[D_2(1\#3)] = 13$, hence $O[D_2(1\#4)] = 3 *13+ 1 = 40$.

Example:

This can be verified by considering N = 2 *3 *5 *7 = 210, where the divisors are

1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210.

$D_2(210)$ = { (1,2) , (1, 3) , (1, 5) , (1, 6) , (1, 7) , (1, 10), (1, 14) ,(1 ,15), (1,21),

(1, 30), (1, 35), (1, 42), (1, 70), (1, 105), (1, 210), (2, 3), (2, 5), (2, 7), (2, 15), (2, 21),

(2, 35), (2, 105), (3, 5), (3, 7), (3, 10), (3, 14), (3, 35), (3, 70), (5, 6), (5, 7), (5, 14),

(5, 21), (5, 42), (7, 6), (7, 10), (7, 15), (7, 30), (6, 35), (10, 21), (14, 15)}.

Therefore, $O[D_2(210)] = 40$.

The formula (*) can be reduced to a direct formula by applying simple induction to get

$O[D_2(1\#n)] = (3^n - 1) /2$.

(B) r = 3.

For r=3, we derive a reduction formula.

We have $D_3(1\#n)$ is contained in $D_3 (1\#(n+1))$ hence this contributes $O[D_3(1\#n)]$ elements to $D_3 (1\#(n+1))$.

Let us choose an arbitrary element of $D_3(1\#n)$ say (a , b , c). The additional prime q yields (qa, b, c), (a, qb, c), (a, b, qc) or three additional elements. In this way we get

$3 * O[D_3(1\#n)]$ elements.

Let the product of the n primes be N and let $(d_1 , d_2, d_3 , . . . d_{d(N)})$ be all the divisors of N.

Consider $D_2 (1\#n)$ which contains d(N) - 1 elements in which one member is unity = d_1. In other words, $(1, d_2), (1, d_3), . . ., (1, d_{d(N)})$.

If q is placed as the third element with these as the third element we get d(N) - 1 elements of $D_3 (1\#(n+1))$. The remaining elements of $D_2 (1\#n)$ yield elements that are already covered in the previous paragraph.

Considering the exhaustive contributions from all three cases above we get

$O[D_3 (1\#(n+1))] = 4 * O[D_3(1\#n)] + d(N) - 1$

$O[D_3 (1\#(n+1))] = 4 * O[D_3(1\#n)] + 2^n - 1$

$O[D_3(210)] = 4 * O [D_3 (30)] 8 -1$

$O[D_3(210)] = 4 * 7 + 8 -1 = 35$.

To verify, the elements are listed here.

$D_3(210)$ = {(1,2, 3) , (1, 2 , 5) , (1, 3, 5) , (1, 2, 7), (1, 3, 7), (1 ,5 , 7), (1, 2 , 15), (1, 2, 21), (1, 2, 35), (1, 2, 105), (1, 3, 10), (1, 3, 14), (1, 3, 35), (1, 3, 70), (1, 5, 6), (1, 5, 14), (1, 5, 21), (1, 5, 42), (1,7, 6), (1,7, 10), (1, 7, 15), (1, 7, 30), (2, 3, 5), (2, 3, 7), (2, 5, 7), (2, 3, 35), (2, 5, 21), (2, 7,15), (3, 5, 7), (3, 5, 14), (3, 7, 10), (5, 7, 6), (1, 6, 35), (1, 10, 21), (1, 14, 15)}.

Open problem: To obtain a direct formula from the reduction formula

$O[D_3 (1\#(n+1))] = 4 * O[D_3(1\#n)] + 2^n – 1$.

Regarding the general case, $O [D_r (1\#n)]$, we derive an inequality.

Let $(d_1, d_2, d_3, \ldots d_r)$ be an element of $O [D_r (1\#n)]$.

Introducing a new prime q other than the prime factors of N we see that this element in conjunction with q gives r elements of $D_r (1\#(n+1))$. In other words,

$(qd_1 , d_2, d_3 , \ldots d_r), (d_1 , qd_2, d_3 , \ldots d_r), \ldots ,(d_1, d_2, d_3 , \ldots qd_r)$.

Furthermore, $D_r (1\#n)$ is contained in $D_r (1\#(n+1))$. Hence we get

$O[D_r (1\#(n+1))] > (r+1) * O[D_r (1\#n)]$.

Finding a precise formula is a difficult task that is left as a challenge for the reader. The general case is an additional challenge.

Section 12

On the Divisors of the Smarandache Unary Sequence

Definition: The Smarandache Unary Sequence is defined as

u(n) = 11 . . . 1, or the digit '1' repeated p_n times, where p_n is the nth prime.

It is not known if this sequence contains an infinite number of primes.

Let $I(m) = \underbrace{11 \ \ldots \ 1}_{m \text{ times}} = (10^m – 1)/9$.

Then $u(n) = I(p_n)$ and the following proposition will be proven.

Proposition: $I(p-1) \equiv 0 (\mathrm{mod}\ p)$.

Proof: Clearly, 9 divides $10^{p-1} – 1$. From Fermat's little theorem if $p \geq 7$ is a prime then p divides $(10^{p-1} -1) /9$ as $(p, 9) = (p , 10) = 1$. Therefore, p divides I(p-1).

This proposition will be used to prove the main result of this section.

Theorem: If d is a divisor of u(n) then $d \equiv 1 \pmod{p_n}$, for all $n > 3$.

Proof: Let d be a divisor of u(n) and let $d = p^a q^b r^c$. .,where p, q, and r are prime factors of d .

If p divides d, then p divides u(n). Also, p divides I(p-1) from the proposition. In other words,

p divides $(10^{p-1} - 1)/9$ and p divides $(10^p - 1)/9$

p divides $(10^{A(p-1)} - 1)/9$ and p divides $(10^{B.p} - 1)/9$

p divides $(10^{(A(p-1) - B.p}) / 9$

p divides $10^{B.p}\{(10^{A(p-1) - B.p} - 1)/9\}$

p divides $(10^{A(p-1) - B.p} - 1)/9$.

There exist A and B such that

$A(p - 1) - B * p_n = (p - 1, p_n)$. As p_n is a prime there are two possibilities:

(i). $(p - 1, p_n) = 1$ or (ii). $(p - 1, p_n) = p_n$.

In the first case, from (3) we get p divides (10 - 1)/9 or p divides I, which is absurd as p > I. Therefore, $(p-1, p_n) = p_n$ or p_n divides p-1.

$$p \equiv 1 \pmod{p_n}$$

$$\Rightarrow \quad p^a \equiv 1 \pmod{p_n}.$$

Along similar lines

$$q^b \equiv 1 \pmod{p_n}$$

hence $d = p^a q^b r^c \ldots \equiv 1 \pmod{p_n}$.

Which completes the proof.

Corollary: For every prime p, there exists at least one prime q such that

$q \equiv 1 \pmod{p}$.

Proof: Since $u(n) \equiv 1 \pmod{p_n}$, and every divisor of u(n) is $\equiv 1 \pmod{p_n}$, the corollary holds.

Section 13

Smarandache Dual Symmetric Functions and Corresponding Numbers of the Type of Stirling Numbers of the First Kind

It is known that the rising factorial (x+1) (x+2)(x+3). . . (x+n), the coefficients of different powers of x are the absolute values of the Stirling numbers of the first kind.

Let $x_1, x_2, x_3, \ldots x_n$ be the roots of the equation

$(x+1)(x+2)(x+3)\ldots(x+n) = 0$.

Then the elementary symmetric functions are

$x_1 + x_2 + x_3 + , \ldots, + x_n = \Sigma\, x_1$, (sum of all the roots taken one at a time)

$x_1x_2 + x_1x_3 + \ldots x_{n-1}x_n = \Sigma\, x_1x_2$. (sum of all the products of the roots taking two at a time)

$\Sigma\, x_1x_2x_3\ldots x_r$ = (sum of all the products of the roots taking r at a time) .

In the previous expressions, we have summed products. In the following definition, the dual of these expressions will be defined, where the roles of addition and multiplication are interchanged.

Definition: The Smarandache Dual Symmetric functions are formed by taking the product of the sums instead of the sum of the products. As an example, the following is a chart for the four variables x_1, x_2, x_3, x_4.

Elementry symmetric funcions (sum of the products)	Smarandache Dual Symmetric functions (Product of the sums)
$x_1 + x_2 + x_3 + x_4$	$x_1x_2x_3x_4$
$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$	$(x_1 + x_2)(x_1 + x_3)(x_1 + x_4)(x_2 + x_3)(x_2 + x_4)(x_3 + x_4)$
$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$	$(x_1 + x_2 + x_3)(x_1 + x_2 + x_4)(x_1 + x_3 + x_4)(x_2 + x_3 + x_4)$
$x_1x_2x_3x_4$	$x_1 + x_2 + x_3 + x_4$

For convenience, the vacuous case of taking the product of zero sums at a time is defined to be one.

Now if we take $x_r = r$ in the above we get the absolute values of the Stirling numbers of the first kind. For the first column: 24, 50, 35, 10, 1.

The corresponding numbers for the second column are 10, 3026, 12600, 24, 1.

The triangle of the absolute values of the Stirling numbers of the first kind is

1				
1	1			
2	3	1		
6	11	6	1	
24	50	35	10	1

The corresponding Smarandache dual symmetric triangle is

1				
1	1			
3	2	1		
6	60	6	1	
10	3026	12600	24	1

The next row (5^{th}) numbers are

15, 240240, 2874009600, 4233600, 120, 1.

The following properties of the above triangle are evident.

(1) The leading diagonal contains unity.

(2) The r^{th} row element of the second leading diagonal contains r!.

(3) The first column entries are the corresponding triangular numbers.

Readers are encouraged to find additional relationships between the two triangles.

Application: The Smarandache Dual Symmetric functions give us another way of generalizing the Arithmetic-Geometric Mean Inequality. One can easily prove that:

$$(x_1x_2x_3x_4)^{1/4} \le [\{ (x_1 + x_2) (x_1+ x_3)(x_1 + x_4)(x_2+ x_3)(x_2 + x_4)(x_3+ x_4)\}^{1/6}] / 2$$
$$\le [\{(x_1 + x_2 +x_3)(x_1+ x_2 + x_4)(x_1 +x_3 + x_4)(x_2+ x_3 +x_4)\}^{1/4}] / 3$$
$$\le \{x_1 + x_2 + x_3 + x_4\}/ 4.$$

The generalization of the above inequality can also be easily established.

Section 14

On the Infinitude of the Smarandache Additive Square Sequence

Definition: The Smarandache additive square sequence consists of all numbers that are perfect squares and the sum of the digits is also a perfect square. The first few terms are

1, 4, 9, 36, 81, 100, 121, 144, 169, 196, 225, 324, 529, **. . .**

The question whether this sequence is infinite has been open and our next task is to prove that it is in fact infinite.

Theorem: The sum of the digits of the square of the number 'one followed by n three's' is a perfect square if $n = 60t^2 + 76t + 24$, for some t and is equal to $(30t+ 19)^2$.

Proof: Consider the Smarandache Patterned Perfect Square sequence

169, 17689, 1776889, 177768889, . . .

whose root sequence is

13, 133, 1333, 13333 . . .

It is clear that this sequence follows a pattern.

Proposition I:

The square of one followed by n three's is equal to one followed by (n-1) seven's, followed by six, followed by (n-1) eight's followed by a nine.

Proof of proposition I:

The general term of the previous square sequence is $T_n = 10^n + 3*(11\ldots \text{ n times})$

$T_n = 10^n + 3* (10^n -1) /9 = (1/3) (4*10^n -1)$

$(T_n)^2 = (1/9) (16*10^{2n} - 8*10^n + 1)$.

The general term of the root sequence is $T_n = 1\ 777. . .6\ 888\ldots 9$

$= 10^{2n} + 7 *10^{n+1} * (111, (n\text{-}1) \text{ times}) + 6*10^n + 80* (111, (n\text{-}1) \text{ times}) + 9$

$= 10^{2n} + 7 *10^{n+1} * (10^{n-1} -1)/9 + 6*10^n + 80* (10^{n-1} -1)/9 + 9$

$= (1+ 7/9) * 10^{2n} + (-70/9 + 6 + 8/9) *10^n + (-80/9 + 9)$

$= (1/9) (16*10^{2n} - 8*10^n + 1)$

which is the same as $(T_n)^2$.

This completes the proof of proposition I.

The sum of the digits of this type of square number is
$1 + 6 + 9 + (n-1) (7 + 8) = 15n + 1$.

The only thing that remains is to prove that the Diophantine equation $15n + 1 = k^2$ has infinitely many solutions.

Proposition II:

If $n = 60t^2 + 76t + 24$ then $15n + 1$ is a perfect square for all values of t.

Proof of proposition II:

Consider the Diophantine equation $15n + 1 = k^2$
or equivalently $15n = (k-1)(k+1)$.

Let $k-1 = 3r$
$k+1 = 5s$
and $n = r*s$.

Subtracting the first from the second, we have
$5s = 3r + 2$.

Let $r = 10t + 6$, then $s = 6t + 4$
and $n = r*s = (10t + 6)(6t + 4) = 60t^2 + 76t + 24$.

This gives the value of $k^2 = (30t + 19)^2$.

The combination of proposition I and proposition II proves the theorem.

By examining additional formulas, it may be possible to find additional infinite families of numbers in the Smarandache Patterned Perfect Square sequence.

Section 15

On the Infinitude of the Smarandache Multiplicative Square Sequence

In section 14, the infinitude of the Smarandache Additive Square sequence as proven. In this section, the similar sequence with multiplication replacing addition is examined and it is proven that it contains an infinite number of terms.

Definition: The Smarandache multiplicative square sequence are all perfect squares where the product of the digits is also a perfect square. The first few terms of this sequence are

1, 4, 9, 144, 289, . . .

Theorem: The square of the number 8 followed by n three's is a member of the Smarandache multiplicative square sequence if n is an odd number ≥ 3. The product of the digits of the square is equal to $\{2^{(5r+1)}*3^3\}^2$, where $r = (n-1)/2$.

Proof:

Start with the sequence

693889, 69438889, 6944388889, 694443888889,. . .

which has the root sequence

833, 8333, 83333, 833333, . . .

where the pattern is obvious.

Proposition I:

The square of 8 followed by n three's is 69 followed by (n-2) 4's, followed by 3, followed by (n-2) 8's followed by 889.

Proof of proposition I:

The general term of the root sequence is given by $T_n = 8*10^n + 3*$ (111… (n) times)
$T_n = 8*10^{n+1} + 3*(10^{n+1} – 1)/9 = (1/3) (25*10^n - 1)$.

Upon squaring, we have

$(T_n)^2 = (1/9)\{ 625*10^{2n}- 50*10^n + 1\}$.

The general term of the sequence of squares is

T_n = 69 444… 3 888… 889

$=69*10^{2n} + 4*10^{n+2}$ (111… n-2 times) + $3*10^{n+1}$ + 8000* (111… n-2 times) + 889

$= 69*10^{2n} + 4*10^{n+2} *(10^{n-2} -1)/9 + 3*10^{n+1} + 8000*(10^{n-2} -1)/9 + 889$

$= (69 + 4/9)*10^{2n} + (-400/9 + 30 + 80/9)* 10^n + (-8000/9 + 889)$

$= (1/9)\{625*10^{2n}- 50*10^n + 1\}$.

Which is the same as the square of the root sequence, therefore the proof of the proposition is complete.

Proof of the theorem:
Returning to the theorem, the product of the digits of the number

$(1/9)\{625*10^{2n}- 50*10^n + 1\}$ is

$P = 6* 9* 4^{n-2}* 3* 8^{n-2}* 8*8*9 = 2^{(5n-3)}*3^6$.

P is a perfect square when 5n-3 is even or when n is odd. For example, n = 2r+1, in which case $P = 2^{(10r+2)}* 3^6 = \{2^{(5r+1)}*3^3\}^2$.

This completes the proof of the main theorem. Readers are encouraged to find additional infinite families of numbers in the Smarandache Multiplicative Square sequence.

Section 16

Another Classification of the Ocean of Smarandache Sequences

Definition: If a sequence of natural numbers can be used to express every natural number as a sum of distinct numbers in the sequence, then it is said to be a Smarandache Accomodative Sequence (SAS).

Example: The set of powers of 2: 1, 2 , 2^2, 2^3, 2^4, . . . , 2^n, . . . is an SAS sequence. This is the sequence used to represent numbers in digital computers. In general, $n = \sum a_k t_k$, where a_k = 0 or 1 and t_k is a power of 2.

A second example of an SAS sequence is the sequence of Fibonacci numbers. Readers are encouraged to analyze the ocean of Smarandache sequences for additional examples.

Conjecture I: The following sequences from the Smarandache inventory compiled by Henry Ibstedt in [1] are Smarandache Accomodative.

Sequences numbered 6, 9, 10, 11, 14, 15, 93, 94, 95 and 123.

The validity of the conjecture is obvious for sequences 93, 94, 95 and 123.

Definition: If all natural numbers can be expressed as the sum or difference to terms of a sequence, then the sequence is said to be Smarandache Semi-Accomodative.

Example: The set of powers of 3: 1, 3, 3^2, 3^3 , ... ,3^n , . . . is Smarandache Semi-Accomodative.

In general, the formula is $n = \sum a_k t_k$, where a_k = -1, 0 or 1 and t_k is a power of 3.

Example: The set of triangular numbers 1, 3, 6, 10, 15, . . . , m*(m+1)/2, … is

Smarandache Semi-Accomodative, as $n = t_{n+1} - t_n$.

Conjecture II: Prime numbers are Smarandache Semi-Accomodative with a_k = -1 for at most only one value of k.

2, 3, 5, 7,11, 13, 17, . . .

1 = 3-2, 4 = 7- 3, 6 = 13 - 7 = 17 – 11, 8 = 3 + 5 = 13 – 5 = 19 – 11, …

Further Generalization: Given a sequence T, if a finite set of numbers $A = \{ a_1 , a_2 \ . . . a_r \}$ exists such that every natural number n could be expressed as $n = \sum a_k t_k$, $a_k \in A$, $t_k \in T$, then the sequence T is defined to be accommodative w. r. t. A. (the term accommodative is used in the sense that the sequence accommodates all natural numbers as the linear combination of its terms with a finite set of coefficients.)

As an example, for SAS, A = {1, 0} and for SSAS, A = {1, 0, -1}.

Note: A large number of Smarandache sequences for which $t_1 = 2$ are accommodative with the exception of 1.

Open Problem: It is an open problem to find sequences for which the set A exists, and in the event that it exists to find the smallest one.

Section 17

Pouring a Few More Drops in the Ocean of Smarandache Sequences and Series

In this section, some fresh ideas on Smarandache sequences and conjectures are presented.

1) Smarandache Forward Reverse Sum Sequence.

$T_1 = 1$

$T_{n+1} = T_n + R(T_n)$, where $R(T_n)$ is the digits of T_n reversed.

The first few terms of this sequence are:

1, 2, 4, 8, 16, 77, 154, 605, 1111, 2222, 4444, 8888, 17776, 85547, 160165, 661166, 1322332, 3654563, 7309126, 13528163, 49710694, . . .

77 = 16 + 61, 605 = 154 + 451, etc.

Conjecture: There are infinitely many palindromes in this sequence.

Conjecture: 16 is the only square in this sequence.

2) Smarandache Reverse Multiple Sequence.

The sequence of numbers that are multiples of their reversals, palindromes are considered trivial and are not included.

The first few terms of this sequence are

8712, 9801, 87912, 98901, 879912, . . .

8712 = 2178 *4.

Points of note about this sequence.

1) The sequence is infinite.
2) There are two families of numbers, one derived from 8712 and the other from 9801. Each family is constructed by placing 9's in the middle.
3) The concatenation of two terms of this sequence derived from the same family is also a member of that family.

3) Smarandache Factorial Prime Generator.

$T_1 = 1$

$$\prod_{k=1}^{n} T_k + 1$$ is a prime, where T_n is the smallest such number.

The first few terms of this sequence are:

1, 2, 3, 5, 7, . . .

4) Smarandache Prime-Prime Sequence.

If the primes are placed in sequence 2, 3, 5, 7, 11, 13, 17, 19, . . .

$T_1 = 2$, T_{n+1} = prime number T_n

The first few terms are

2, 3, 5, 11, 31, . . .

Since $T_1 = 2$, $T_2 = 3$ the second prime, $T_3 = 5$ the third prime, $T_4 = 11$ the fifth prime.

5) Smarandache Triangular-Triangular Number Sequence.

Using the sequence of triangular numbers 1, 3, 6, 10, 15, 21, 28, 36, 45, . . .

$T_1 = 3$, T_{n+1} = triangular number T_n

The first few terms are

3, 6, 21, 231, 26796, . . .

Since $T_1 = 3$, $T_2 = 6$ the third triangular, $T_3 = 21$ the sixth triangular, $T_4 = 231$ the twenty-first triangular.

6) Smarandache Divisors of Divisors Sequence.

$T_1 = 3$, and $T_{n-1} = d(T_n)$, the number of divisors of T_n, where T_n is smallest such number.

The first few terms of the sequence are

3, 4, 6, 12, 72, $2^8.3^7$, $2^{2186} * 3^{255}$, . . . {where 3^7-1 = 2186 and 2^8-1 = 255}

3, 4, 6, 12, 72, 559872, $2^{2186} * 3^{255}$, . . .

The sequence obtained by incrementing each term in the above sequence by 1 is

4, 5, 7, 13, 73, 559873, $2^{2186} * 3^{255} + 1$, . . .

Conjecture: All of the terms in the previous sequence beyond the first term are prime.

The motivation behind this conjecture: As the neighboring number is highly composite (the smallest number having such a given large number of divisors), the chances of it being a prime are very high.

7) Smarandache Divisor Sum-divisor Sum Sequences (SDSDS).
Consider the following sequences in which each term is the sum of the divisors of the previous term:

a) 1, 1, 1, 1, 1, 1, . . .

b) 2, 3, 4, 7, 8, 15, 23, 24, 52, . . .

c) 5, 6, 12, 28, 56, 120, 240, 744, 1920, . . .

d) 9, 13, 14, 24, . . .

e) 10, 18, 39, 56, . . .

f) 11, 12, 28, . . .

g) 16, 31, 32, 63, 104, . . .

h) 17, 18, . . .

i) 19, 20, 42, . . .

In the above sequences $T_n = \sigma(T_{n-1})$, with T_1 as the generator of the sequence. A number which appears in a previous sequence is not to be used as a generator.

Open problem: How many of the numbers like 12, 18, 24, 28, 56 … are members of two or more sequences?

Open problem: Are there numbers that are members of more than two sequences?

Definition: We define the Smarandache Divisor Sum Generator Sequence (SDSGS) as the sequence formed by (the generators) the first terms of each of the above sequences.

1, 2, 5, 9, 10, 11, 16, 17, 19, . . .

Open problem: Is SDSGS finite or infinite?

8) Smarandache Reduced Divisor Sum Periodicity Sequences.

In the following sequences the sum of the proper divisors only is taken till the sequence terminates at 'one ' or repeats itself.

a) 1, 1, 1, . . .

b) 2, 1, . . .

c) 3, 1, . . .

d) 4, 3, 1, . . .

e) 5, 1,. . .

f) 6, 6, 6, . . .

g) 7, 1, . . .

h) 12, 16, 15, 8, 7, 1 , …

For 220, which is the first of a pair of amicable numbers:

220, 284, 220, 284, . . .

We define the life of a number as the number of terms in the corresponding sequence till a 'one' is arrived at. For example, the life of 2 is 2 and that of 12 is 6. The life of a perfect number like 6 or 28 or that of a amicable number pair like (220, 284) is infinite. The same is true for a sociable number like the five number chain 12496, 14288, 15472, 14536, 14264. We can call them immortal numbers.

Open problem: If n is any arbitrary number, is there another number k whose life is n?

9) T_n = smallest prime of the form $n*k + 1$, $k \geq 1$.

The first terms of this sequence are

2, 3, 7, 5, 11, 13, 29, 17, 19, 11, 23, 13, 53, 29, 31, 17, …

Conjecture: For every n there exists a number k < n such that $n*k +1$ is a prime.

Conjectures: Given any number N, there exists

a) A perfect square in which N appears in some position.
b) A prime in which N appears in some position.
c) A cube in which N appears in some position.
d) A power in which N appears in some position.

Proposition: If N is a perfect square with 2n digits, then there exists at least one perfect square of 2n or 2n+1 digits and infinitely many other perfect squares in which the first n digits are the same as that of N.

Proof: Let $N = r^2$. Then the number of digits in r are n, and let s be the ten's complement of r. Then $r + s = 10^n$.

We have $| r^2 - s^2 | = | r – s | * | r + s | = | r – s | * 10^n$. Therefore, the first n digits of s^2 are the same as that of $r^2 = N$.

Also, all the numbers of the type $k = 10^x + r$, are of the required type for

$x > 4n + 1$. This completes the proof.

Example: $N = 12439729 = 3527^2$, ten's complement of 3527 = 6473.

$6473^2 = 41899729$.

Conjecture: Let N be an n digit number. For every r, there exists a number k such that 10^n divides $k^r - N$, if there is an r^{th} power $\equiv N \pmod{10}$. This is a generalization of the proposition.

Section 18

Smarandache Pythagoras Additive Square Sequence

1) The Smarandache Pythagoras Additive Square Sequence.

The Smarandache Pythagoras Additive Square sequence is the set of numbers that are perfect squares and where the sum of the digits is also a perfect square. The first few terms of this sequence are

1, 4, 9, 100, . . .

This sequence is infinite, as can be seen from the following theorem.

Theorem: The square of the number $6 *(10^n - 1) /9$, is a member of the Smarandache Pythagoras Additive Square Sequence if n is given by $41r^2 -4r$, and the sum of the square of the digits is $(4r -2)^2$.

Proof: Consider the following sequence

6**, 66, 666, 6666, 66666, 666666, . . . (*)**
the squares of the terms in this sequence are

36, 4356, 443556, 44435556, 4444355556, . . (**)

Proposition: The square of the number formed from n sixes is equal to the number (n-1) '4's, followed by a '3' followed by (n-1) '5' s followed by 6.

Proof of the proposition:
The general term of the (*) sequence is given by $T_n = 6 *(10^n – 1) /9$, which, when squared, gives $(T_n)^2 = (4/9)(10^{2n} -2*10^n + 1)$.
The general term of the (**) sequence is given by

$4*10^{n+1}$ (111. . . n-1 , times) + $3*10^n$ +50 (111. . . n-1 , times) + 6
$= 4*10^{n+1} (10^{n-1} – 1)/9 + 3*10^n +50 (10^{n-1} – 1)/9 + 6$
$= (4/9) *10^{2n} + (-40/9 + 3 + 5/9)* 10^n + (-50/9 + 6)$
$= (4/9)(10^{2n} -2*10^n + 1)$.
This is the same as that derived by squaring the terms in the (*) sequence, so the proof of the proposition is complete.
Let S be the sum of the squares of the digits of the general term of the (**) sequence. Then we have
$S = (n-1)* (4^2 + 5^2) + 3^2 + 6^2 = 41*n + 4$.
This will be a member of the Smarandache Pythagoras Additive Square Sequence if 41n + 4 $=k^2$ or equivalently, 41n = (k –2)(k+2).
Let 41r = k+2, n = r (k-2), then we have k = 41r - 2 and
$n = 41r^2 - 4r$. This completes the proof of the theorem.

For r = 1, we have n = 37 and for r = 2, we have n = 148.
Note: It is easy to see that the sum of the digits of the general term is 9n.
Using similar reasoning, it is possible to find additional infinite families of elements of the Smarandache Pythagoras Additive Square Sequence.
Additional sequences:
1) Smarandache (m^{th}) Power Additive Square Sequence

It is defined as a sequence in which the term and the sum of the m^{th} power of the digits are both perfect squares.

(For m = 1 and m =2 we get the Smarandache additive square sequence and Smarandache Pythagoras Additive Square Sequence respectively.)

2) Smarandache (m^{th}) Power Additive m^{th} power Sequence

When m = 3, we get the Smarandache additive cubic sequence.

1, 8, 125, 512, 1000, 1331, …

3) Smarandache (m^{th}) Power Additive n^{th} power Sequence:
When m=2 and n=3 we get

1, 27, 216, 1000, . . ., 798005999, . . . 10973607685048, . . .

A large number of open questions can be formulated by expanding on these results.

Section 19
The Number of Elements the Smarandache Multiplicative Square Sequence and the Smarandache Additive Square Sequence have in Common

In the previous section, the Smarandache Multiplicative Square sequence
1, 4, 9, 144, 289, . . .
and the Smarandache Additive Square sequence

1, 4, 9, 36, 81, 100, 121, 144, 169, 196, 225, 324, 529, . . .
were defined. A natural question to ask is, "How many elements do these two sequences have in common?" In this section, we will prove that the sequences have an infinite number of terms in common.

Theorem: If $n = 4m^2 - 3$, the square of the number '9' followed by n '6's followed by '9' is a member of the Smarandache Multiplicative Square sequence and the Smarandache Additive Square sequence. The sum of the digits of this number is '$36m^2$' and the product of the digits is $108^2 * 20^p$ where $p = 4m^2$.

Proof: Consider the sequence

969, 9669, 96669, 966669, . . . (*)

If we square the terms of this sequence, we get

93896, 93489561, 9344895561, 934448955561, . . . (**)

Proposition: The square of the number '9' followed by n '6' s followed by '9', is equal to 93 followed by (n-1) '4's followed by '89' followed by (n-1) '5's followed by '61'.

Proof: Consider the numbers of the (*) sequence. The number 9 followed by n 6's followed by a 9.

$T_n = 9*10^{n+1} + 60*111\ldots + 9 = 9*10^{n+1} + 60*(10^n - 1)/9 + 9 = (1/3)(290*10^n + 7)$.

$N = (1/3)(29*10^{n+1} + 7)$

$N^2 = (1/9)(841*10^{(2n+2)} + 406*10^{n+1} + 49)$. (***)

Consider the numbers of the (**) sequence, 93 followed by (n-1) 4's, followed by (n-1) 5's followed by 61.

$$T_n = 93*10^{(2n+2)} + 4*10^{n+3}*(111\ldots (n\text{-}1) \text{ times}) + 89*10^{n+1} + 500*(111\ldots (n\text{-}1) \text{ times}) + 61.$$

$$= 93*10^{(2n+2)} + 4*10^{n+3}(10^{n-1} - 1)/9 + 89*10^{n+1} + 500*(10^{n-1} - 1)/9 + 61$$

$$= (93 + 4/9)*10^{2n+2} + (-400/9 + 89 + 5/9)*10^{n+1} + (-500/9 + 61)$$

$$= (1/9)(841*10^{(2n+2)} + 406*10^{n+1} + 49).$$

This is the same as the element of the (***) sequence, which completes the proof.

The next step is to complete the proof of the theorem.

Consider the general term of the (**) sequence

T_n =93 444…89 555…61.

Letting S represent the sum of the digits of T_n
$S = 9 + 3 + 4(n-1) + 8 + 9 + 5(n-1) + 6 + 1$
$= 9(n+3)$.

Letting P represent the product of the digits of T_n

$P = 9*3*4^{n-1}*8*9*5^{n-1}*6*1$
$= 108^2 *20^{n-1}$.

From this formula, if n is odd, then P is a perfect square. Using the formula for the sum of the digits, if $S = 9(n+3) =k^2$ then 9 divides k^2, $k = 3r$, $9(n+3) = k^2 = 9r^2$ or $n+3 = r^2$. Since n is odd, r is even, so letting $r = 2m$, we have $n = 4m^2 - 3$, which completes the proof of the theorem.

Section 20

Smarandache Patterned Perfect Cube Sequences

Consider the sequences

10011, 100011, 1000011, 1000011, . . .

where t_n = 1 followed by 'n' zeros followed by 11, or $10^{n+3} + 11$.

The sequence formed by the cubes of the numbers in this sequence is

1003303631331, 1000330036301331, 1000033000363001331,
1000003300003630001331, . . .

Theorem: The two sequences above form a Smarandache patterned perfect cube sequence.

Proof: The n^{th} term of (1) is given by $T_n = 10^{n+3} + 11$.
Hence the n^{th} term of corresponding cube sequence is

$T_n^3 = 10^{3n+9} + 33 * 10^{3n+6} + 363* 10^{n+3} + 1331$.

Which has the pattern, 1 followed by (n+1) zeros, followed by 33, followed by (n) zeros, followed by 363, followed by (n-1) zeros, followed by 1331.

This completes the proof.

The following sequences are examples of additional cubic root sequences

9, 99, 999, 999, . . .

729, 970299, 997002999, 999700029999, . . .

97, 997, 9997,. . .

912673, 991026973, 999100269973, 999910002699973, …

98, 998, 9998, 99998, . . .

941192, 994011992, 999400119992, 999940001199992, . . .

The proof that these sequences are also Smarandache Patterned Perfect Cube sequences is similar to the one given so they are omitted.

Section 21

The Smarandache Additive Cubic Sequence Is Infinite

Definition: The sequence of numbers that are perfect cubes and where the sum of the digits is also a perfect cube is called the Smarandache additive cubic sequence. The first few numbers in this sequence are

1, 8, 125, 512, 1000, 1331, 8000, 19683, 35937, **. . .**

Theorem: The cube of the number A(n) given by $A(n) = 10^n - 1$, is a member of The Smarandache additive cubic sequence when $n = 12k^3$. Furthermore, the sum of the digits of $\{A(n)\}^3$ is equal to $216k^3 = (6k)^3$.

Proof: Consider the Smarandache patterned perfect cube along with its root sequence

9, 99, 999, 9999, 99999, . . .

729, 970299, 997002999, 999700029999, . . .

The general formulas for the elements of these sequences are

$T_n = 10^n - 1$, $T_n^3 = 10^{3n} - 3*10^{2n} + 3*10^n - 1$.

The cube sequence is then

$t_n = 10^{2n+1} * (10^{n-1} - 1) + 7*10^{2n} + 2*10^n + (10^n - 1)$.

Which, on simplification gives

$t_n = 10^{3n} - 3*10^{2n} + 3*10^n - 1 = T_n^3$.

The sum of the digits for $t_n = 9(n-1) + 7 + 2 + 9n = 18n$.

If $n = 12k^3$, then, sum of digits for $t_n = 216k^3 = (6k)^3$.

This completes the proof of the theorem.

Section 22

More Examples and Results On the Infinitude of Certain Smarandache Sequences

1) Maohua Le has given some examples that prove that the reduced Smarandache Square-Digital subsequence is infinite. Here, we give another example.

The square of the numbers $A(n) = (10^n - 3)$ for all values of n yields terms of a reduced Smarandache Square-Digital subsequence.

A(n): 7 , 97 , 997 , 9997 , 99997 , . . .
$(A(n))^2$: 49, 9409, 994009, 99940009, 9999400009, . . .

2) The Smarandache multiplicative square sequence was defined in section 15 and it was proven that the sequence is infinite. The following is another infinite family of members of this sequence.

A(n): 38, 338, 3338, 33338, . . .
$(A(n))^2$: 1444, 114244 , 11142244 , 1111422244 , . .

The general formulas for the elements of these sequences are

$A(n) = 10*(10^n - 1)/3 + 8$

$\{A(n)\}^2 = 10^{n+2}* (10^n - 1)/9 + 4*10^{n+1} + (200/9)(10^{n-1} - 1) + 44 = \{10*(10^n - 1)/3 + 8)^2$.

The product of the digits of $(A(n))^2 = 2^{n+5}$, which is a perfect square for odd n

(n = 2k+1).

Note 1: It is interesting to see that the reverse of the elements of A(n) exhibits the same property.

83, 833, 8333, 83333,. . .
6889, 693889, 69438889, 6944388889, 694443888889 . . .

Note 2: The sum of the digits of $(A(n))^2$ is given by $12 + n + 2(n-1) = 3n + 10$.

If we look for values of n which make $3n + 10$ a perfect square r^2, we will get an infinite additive square sequence. In other words, if $n = (3k^2 \pm 2k - 3)$ then $3n + 10 = 9k^2 \pm 6k+1 = (3k \pm 1)^2$, is a perfect square. Therefore, for $n = (3k^2 \pm 2k - 3)$, we get an infinite Smarandache additive square sequence. Moreover, if we also have $k = 2m$, an even number, then these numbers are also members of a multiplicative square sequence. We have finally arrived at an infinite sequence of numbers that are simultaneously members of both the Smarandache additive square sequence and the Smarandache Multiplicative square sequence.

The members of the sequence

$$\{A(n)\}^2 = \{10*(10^n - 1)/3 + 8)\}^2$$

are simultaneously members of both the Smarandache additive square sequence as well as Smarandache Multiplicative square sequence for n = 5 , 13 , 37 , 53 , 93, 117 , . . .

Section 23

Smarandache Symmetric (Palindromic) Perfect Power Sequences

Definition: The Smarandache Symmetric Perfect mth Power sequence is the set of numbers which are simultaneously mth powers and palindromic.

The first few terms of the Smarandache symmetric perfect square sequence are:

1, 4, 9, 121, 484, 14641, . . .

The first few terms of the Smarandache Symmetric Perfect cube sequence are:

1, 8, 343, 1331, . . . 1367631, . . .

In this section, we will verify that the sequence is infinite for some values of m.

Theorem: The Smarandache symmetric perfect mth power sequence has infinitely many terms for m = 1, 2, 3 and 4.

Proof: We will start with m = 2.
Consider the following sequences, where the second is the squares of the first.

11, 101, 1001, 10001, 100001, . . .
121, 10201, 1002001, 100020001, . . .

The formulas for the general terms of the sequences are $T_n = 10^n + 1$,and $T_n^2 = 10^{2n} + 2*10^n + 1$. This verifies the theorem for m = 2.

Consider the following sequences, where the second is the cubes of the elements of the first.

11, 101, 1001, 10001, 100001, . . .
1331, 1030301, 1003003001, 1000300030001, . . .

This verifies the theorem for $m = 3$.

Consider the following sequences, where the second is made up of the fourth powers of the first

11, 101, 1001, 10001, 100001, . . .
14641, 104060401, 1004006004001, . . .

This verifies the theorem for $m = 4$.

Note 1: The three sequences of squares, cubes and fourth powers in the previous theorem are also examples of infinite additive square, cubic and fourth power sequences.

Note 2: The general term of the root sequence can also be expressed in the form $T_n = 2(10^n + 1)$.

Note 3: The root sequence can be taken as $10^{2n} + 10^n + 1$ or $10^{3n} + 10^{2n} + 10^n + 1$, for $m = 2$.

Conjecture: The Smarandache Symmetric Perfect m^{th} power sequence has infinitely many terms for all values of m.

Section 24

Some Propositions On the Smarandache n2n Sequence

Definition: The elements of the Smarandache n2n sequence are created by concatenating n and 2n together. The first few terms of the sequence are

12, 24, 36, 48, 510, 612, 714, 816, . . . , 1224, 1326,1428, . . .

The n^{th} term is given by the formula $a(n) = 2*n + n*10^r$, where $r = d(2n)$, the number of digits of n.

It has been conjectured by Russo that the sequence contains infinitely many primes. However, it has been proven that it contains no primes. In this section, some properties of the sequence will be presented.

To compute the final digit sum of a number the digits are summed. If that sum is greater than ten, the digits of the sum are added and this process is repeated until a single digit number is computed.

Consider the final sum of the digits of the elements of the Smarandache n2n sequence. The pattern that we get is

3, 6, 9, 3, 6, 9, 3, 6, 9, 3, 6, 9, . . .

By definition, the following properties are evident.

(i) The sum of digits is divisible by three and hence each term of the sequence is divisible by 3.
(ii) The only valid digits sum occurring is 9, a necessary condition for a perfect square.
(iii) The digit sequence {3, 6, 9}, is repeated periodically.
(iv) The nth term is divisible by 2n.

The final sum of the digits for the nth term is given by

d = 3 , 6 or 9 accordingly as n = 3r+1 , 3r+2 , or 3r.

Let the sequence obtained by dividing the terms of the Smarandache n2n sequence by 3 be called the Smarandache n2n by three sequence, which is

4, 8, 12, 16, 170, 204, 238, 272, . . .

Our next step will be to prove that this sequence contains infinitely many perfect squares.

Theorem: The n^{th} term of the above sequence is a perfect square equal to n^2 itself, if n is given by $n = \{10^k+2\}/3$.

Proof: We have $n = \{10^k+2\}/3$, which has exactly k digits. We get the corresponding term of the Smarandache n2n sequence by

$a(n) = 2*n + n*10^r = 2* \{10^k+2\}/3 + [\{10^k+2\}/3]*\{ 10^k\}$.

Therefore, the corresponding term of the Smarandache n2n by three sequence is given by

$b(n) = a(n)/3 = (1/3)* [2* \{10^k+2\}/3 + \{(10^k+2)/3\}*(10^k)]$
$b(n) = (1/9)*\{ 10^{2k} + 4*10^k + 4 \} = \{ (10^k+2)/3\}^2 = n^2$.

This completes the proof.

The sequence of numbers is

34, 334, 3334, 3334, . . .

and the corresponding terms of the Smarandache n2n by three sequence are

1156, 111556, 11115556, 1111155556, . . .

(Another sequence of patterned perfect squares.)
The corresponding terms of the Smarandache n2n sequence are

3468, 334668, 33346668, 3333466668, . . .

Note: It follows directly from this theorem that the Smarandache n2n sequence contains infinitely many terms of the form $3*n^2$.

It is that term a(n) of the Smarandache n2n sequence is divisible by 6n. We will define the Smarandache n2n by 6n sequence by c(n) = a(n)} /(6n). The first few terms of the sequence are

2, 2, 2, 2, 17, 17, 17,. . . 17, 167, 167, . . . 1667, 1667, . .

And for some specific terms

c(5) = 510/30 = 17, c(49) = 4998/294 = 17, c(50) = 50100/300 = 167 ,
c(499) = 499998/(6*499) = 167 , c(500) = 5001000/3000 = 1667.

Conjecture: The sequence c(n) contains infinitely many primes.

Section 25

The Smarandache Fermat Additive Cubic Sequence

Definition: The Smarandache Fermat Additive Cubic Sequence is constructed from the numbers that are perfect cubes and the sum of the cubes of their digits is also a perfect cube. The name of Fermat is included in the description to relate it with the fact that though the sum of two cubes can not yield a third cube, the sum of more than two cubes can be a third cube ($3^3 + 4^3 + 5^3 = 6^3$).

The first few terms of the sequence are

1, 8, 474552, 27818127, . . .

where

$474552 = 78^3$, and $4^3 + 7^3 + 4^3 + 5^3 + 5^3 + 2^3 = 729 = 9^3$
$27818127 = 303^3$, sum of cubes of digits = $1728 = 12^3$.

Theorem: The Smarandache Fermat Additive Cubic sequence contains an infinite number of terms.

Proof: Consider the following sequence

303^3, 3003^3, 30003^3, . . .

27818127, 27081081027, 27008100810027, . . .

The general term is given by

$T_n = 27*(10^{n+1} +1)^3 = 27* (10^{3(n+1)} + 3* 10^{2(n+1)} + 3*10^{(n+1)} + 1)$.

The sum of the cubes of the digits for every term is

$2* (1^3 + 2^3 + 7^3 + 8^3) = 1728 = 12^3$

and the proof is complete.

Note 1: It is interesting to note that the digits 1, 2, 7, 8 are used twice in every term with the rest of the digits being zero. The sum of the cubes of the digits also is made from the same digits.

Note 2: A permutation of the four digits 2178 has the property that 4* 2178 = 8712, the number obtained by reversing the digits. In fact the numbers obtained by placing it adjacent to itself any number of times also have the same property.

In other words, 4* 21782178 = 87128712. This also holds if any number of nines are placed in the center, 4*21978 =87912, 4*219978 = 879912. The only other such number is 1089. (9*1089 = 9801).

In the previous paragraphs, we have described a sequence in which the sum of cubes of the digits is the same for each term. We will now describe a sequence in which the sum of the cubes of the digits gives different cubes for different terms.

Theorem: If k is a positive integer then the number $(10^{n+2} - 4)^3$ is a member of the Smarandache Fermat Additive Cubic Sequence when n is can be expressed in the form $[4* \{(10^{3k} - 1)/27\} - 1]$. The sum of the cubes of the digits will then equal $(6*10^k)^3$.

Proof:
Consider the following patterned perfect cube sequence

996^3, 9996^3, 99996^3, . . .

where the sequence of cubes is

988047936, 998800479936, 999880004799936, . .

The general term of the original sequence is given by $(10^{n+2} - 4)$, which upon being cubed, becomes

$T_n = (10^{n+2} - 4)^3 = 10^{3(n+2)} - 12*10^{2(n+2)} + 48*10^{(n+2)} - 64$.

The n^{th} term of the sequence of cubes is given by

$(10^n-1)*10^{2n+6} + 88*10^{2n+4} + 47*10^{n+2} + (10^n-1)*10^2 + 36$.

Which can be reduced to the sequence T_n.

The sum of the cubes of the digits of T_n is

$s(d) = 2n*9^3 + 2*8^3 + 4^3 + 7^3 + 3^3 + 6^3 = 1458n + 1674$.

If s(d) is a cube say r^3, then we have

$r^3 = 1458n + 1674$ or $r^3 = 27(54n + 62)$.

If r = 3s then we have $27s^3 = 27(54n + 62)$ or

$s^3 = (54n + 62) = 54(n+1) + 8$, as s is even.

Let s = 2u, then

$8u^3 = 54(n+1) + 8$ or $8(u^3 - 1) = 54(n+1)$ or $n = 4(u^3 - 1)/27 - 1$

and n would be an integer if 27 divides u^3-1.

We have $999 = 27*37 = 10^3 - 1$ and $(10^3 - 1)$ divides $(10^{3k} - 1)$.

Therefore, 27 divides $(10^k)^3 - 1$, giving $u = 10^k$ for all values of k.

Expressed another way, $n = [4* \{(10^{3k} - 1)/27\} - 1]$.

Now

$r^3 = 1458n + 1674 = 1458[4(10^{3k} - 1)/27 - 1] + 1674 =$
$216*10^{3k} = \{6*10^k\}^3$.

This completes the proof.

Exploring similar sequences for higher powers can extend this idea.

Open problem: Is the sequence of additive fourth powers infinite?

Open problem: Is the sequence where the terms as well as the fourth powers of the digits are fourth powers finite or infinite?

Section 26

The Smarandache nn^2 Sequence Contains No Perfect Squares

Definition: The Smarandache nn^2 sequence is the set of numbers formed by concatenating n and n^2.

The first few terms of this sequence are

11, 24, 39, 416, 525, 636, 749, 864, 981, 10100, 11121, 12144, .

and it is clear that the general term of the sequence is given by

$a(n) = n*10^r + n^2$, where $r = d(n^2)$, the number of digits of n^2.

It has been conjectured in that there are no perfect squares in this sequence. In this section, we prove a theorem verifying that conjecture.

Theorem: The necessary condition on n that gives a perfect square term of the Smarandache nn^2 sequence is

a) $n \equiv 8$ or 0 (mod 9).
b) In the case where n =9m, m is not a square free number.

Proof of (a):

Definition: A number d is said to be a valid digits sum if

$d \equiv 1 (\text{mod } 3)$, or $d \equiv 0 (\text{mod } 9)$.

Proposition: The digits sum of a perfect square is necessarily a valid digits sum.

Proof of proposition: Consider the squares of numbers 1 through 9, 1,4, 9, 16, 25, 36, 49, 64, 81, where the digits sums are 1, 4, 9, 7, 7, 9, 4, 1, 9, all of which are a valid digits sum. It can also proved using the properties of congruence that the digits sum of the product of two numbers is the product of the digits sums. Therefore, the proof of the proposition is complete.

The following sequence is constructed from the final digit sum of the elements of the Smarandache nn^2 sequence.

2, 6, 3, 2, 3, 3, 2, 9, 9, 2, 6, 3, 2, 3, 3, 2, 9, 9, 2, 6, 3, 2, 3, 3, 2, 9, 9, 2.

The following two properties are easy to verify.

i) The sequence of digits {6, 3, 2, 3, 3, 2, 9, 9, 2} repeats periodically.
ii) The only valid digit sum is 9 and it occurs only for $n \equiv 8$ or $0 \pmod 9$.

Proof of (b): Let $a(n) = k^2$ be a perfect square in the sequence. Then we have $a(n) = k^2 = n*10^r + n^2$. If $n = 9m$, where m is a square free number, then

$k^2 – n^2 = n*10^r$ or $(k+9m)(k-9m) = 9m*10^r = m*3^2 * 2^r * 5^r$.
Now, it is evident that a prime divisor of m divides either k+9m or k-9m, and in either case m must divide k.
If k = p *m, then the expression can be written as

$(p *m+9m)(p *m-9m) = 9m*10^r = m*3^2 * 2^r * 5^r$, or

$(p+9)(p-9) = 9*10^r = *3^2 * 2^r * 5^r$.

On similar lines 3 divides p giving that p can be expressed in the form p = 3 *q, giving us

$(3q +9)(3q-9) = 3^2 * 2^r * 5^r$ or $(q +3)(q-3) = 3 * 2^r * 5^r$.

If we also have that q = 3 *s, we have $(s+1)(s-1) = 2^r * 5^r$, or $s^2 = 10^r +1$.

The digit sum of the right member is 2, which is not a valid digit sum for a perfect square. Therefore the expression has no solution in integers. Also, for even r = 2t, the two consecutive numbers 10^{2t} and $10^{2t}+1$ cannot both be perfect squares.

This completes the proof of part (b) of the theorem.

Definition: The Reduced Smarandache nn^2 sequence is given by b(n) = a(n)/n. The first few terms of the sequence are

11, 12, 13, 104, 105, 106, 107, 108, 109, 1010, 1011, 1012, 1013, . . .

Conjecture: There are infinitely many primes in the reduced Smarandache sequence.

Section 27

Primes In the Smarandache nn^m Sequence

Definition: For m >0, the Smarandache nn^m sequence is formed by concatenating n with the mth power of m.

For m = 3, the first few numbers of the sequence are

11, 28, 327, 464, 5125, 6216, 7343, 8512, 9729, 101000, 111331, 121728, . . .

A natural question to ask is to determine how many primes are in the sequence for specific values of m. The following theorem answers the general question for any value of m.

Theorem: The only prime that occurs in the Smarandache nn^m sequence for any value of m is 11.

Proof: It is clear that the general term a(n) of the Smarandache nn^m generalized sequence is given by $a(n) = n*10^k + n^m$ where $k = d(n^m)$, the number of digits of n^m. It follows that $a(n) = n\{ 10^k + n^{m-1})$.

For n = 1, a(n) =11 (independent of m.) and is a prime. If n>1, obviously a(n) is divisible by n and therefore is composite.

Definition: The Reduced Smarandache nn^m sequence is the set of numbers b(n) = a(n)/n, where a(n) is the element of the Smarandache nn^m generalized sequence. In this case, we have $b(n) = 10^k + n^{m-1}$ where $k = d(n^m)$ is the number of digits of n^m.

For m = 3, we have

a(n): 11, 28 ,327, 464, 5125 , 6216, 7343, 8512 , 9729, 101000, 111331, 121728, . . .

b(n): 11, 14, 109, 116, 1025, 1036, 1049, 1064, 1081, 10100, 10121, . . .

Open problem: How many terms in this sequence are prime?

Open problem: How many terms in the general b(n) sequence are prime?

Section 28

Some Ideas On the Smarandache nkn Sequence

Definition: Let k > 0 be a fixed integer. The elements of the Smarandache nkn Generalized Sequence are formed by concatenating n and k*n. The nth term of the sequence is given by $a(n) = k*n + n*10^r$, where r = d(k*n), the number of digits of k*n.

Example:
For k = 2, the first few terms of the sequence are

12, 24, 36, 48, 510, 612, 714, 816, . . . 1224, 1326, 1428,. .

where each term is formed by concatenating n and 2n. The n^{th} term is given by $a(n) = 2*n + n*10^r$, where r = d(2n), the number of digits of 2n.

It has been conjectured that the number of perfect squares in the previous sequence (for k =2) is finite. In this chapter, we analyze the sequence for k = 8 and prove that there are infinitely many perfect squares in the sequence.

For k = 8, the first terms of the sequence are

18, 216, 324, 432, 540, 648, 756, 864, 972, 1080, 1188, 1296, 13104, 14112, …

The general term is given by $a(n) = 8*n + n*10^r$, where r = d(8*n), the number of digits of 8 *n.

Proposition: a(n) is a perfect square if $n = (10^k + 8)/9$.

Proof: Let $n = (10^k +8)/9$, which has exactly k digits and a(n) is given by

$a(n) = 8*\{ 10^k + 8 \}/9 + [\{ 10^k + 8 \}/9]*10^k$

$a(n) = \{(10^k + 8)/9\} * \{ 8 + 10^k\}$

$a(n) = 9*\{(10^k + 8)/9\} * \{ (8 + 10^k)/9\}$

$a(n) = 9*\{(10^k + 8)/9\}^2$

$a(n) = (3n)^2$ = a perfect square.

This completes the proof.

The sequence of numbers defined by the formula $(10^k +8)/9$ is

12, 112, 1112, 11112, ...

and the corresponding n8n sequence is

1296 , 112896, 11128896 , 1111288896, . . .

Which is the same as

36^2 , 336^2 , 3336^2 , 33336^2 . . .

Open problem: Other than those defined by this formula, how many perfect squares are in the sequence? An example is $324 = 18^2$, which does not match the pattern.

Section 29

Some Notions On Least Common Multiples

Definition: The Smarandache LCM sequence (SLS) is defined by

T_n = least common multiple of all integers 1 through n.

The first few numbers of this sequence are

1, 2, 6, 60, 60, 420, 840, 2520, 2520, . . .

It is well known that n! divides the product of any set of n consecutive numbers. We will use this idea in combination with the SLS to define another sequence.

Definition: The terms of the Smarandache LCM Ratio sequence of the rth kind (SLRS(r)) are given by

${}_rT_n$ =LCM (n , n+1, n+2, . . .n+r-1) /LCM (1, 2, 3, 4, . . . r)

Examples:
SLRS(1)

1, 2, 3, 4, 5, . . . , ${}_1T_n$ (=n).

SLRS(2)

1, 3, 6, 10, . . . ${}_2T_n$ = n(n+1)/2 (triangular numbers).

SLRS(3)

LCM (1, 2, 3)/ LCM (1, 2, 3), LCM (2, 3, 4)/ LCM (1, 2, 3),
LCM (3, 4, 5)/ LCM (1, 2, 3), LCM (4, 5, 6)/ LCM (1, 2, 3),
LCM (5, 6, 7)/ LCM (1, 2, 3)
or
1, 2, 10, 10, 35 . . .

SLRS(4)

1, 5, 5, 35, 70, 42, 210, . . .

Note: It appears that for r > 2, the sequences do not follow a pattern.

Open problem: Search for patterns in the SLRS(r) sequences and find reduction formulas for the elements ${}_rT_n$.

Definition: For $n \geq r$

${}^nL_r$ = LCM (n, n-1, n-2, . . . n-r+1) / LCM (1, 2, 3, . . .r)

where the numerator is the least common multiple of the last r numbers up to n and the denominator is the least common multiple of the first r number. By definition, we will have ${}^0L_0 = 1$.

Starting at the beginning, we have

${}^{0}L_0 = 1$ ${}^{1}L_0 = 1$, ${}^{1}L_1 = 1$, ${}^{2}L_0 = 1$, ${}^{2}L_1 = 2$, ${}^{2}L_2 = 2$.

Which can be used to form the triangle

1

1, 1

1, 2, 1

1, 3, 3, 1

1, 4, 6, 2, 1

1, 5, 10, 10, 5, 1

1, 6, 15, 10, 5, 1, 1

1, 7, 21, 35, 35, 7, 7, 1

1, 8, 28, 28, 70, 14, 14, 2, 1

1, 9, 36, 84, 42, 42, 42, 6, 3, 1

1, 10, 45, 60, 210, 42, 42, 6, 3, 1, 1

Definition: The triangle formed from the ${}^{n}L_r$ numbers is called the Smarandache AMAR LCM triangle.

Note: As r! divides the product of r consecutive integers so does the

LCM (1, 2, 3, … r) divide the LCM of any r consecutive numbers Therefore, the elements of the Smarandache AMAR LCM Triangle are all integers.

The following properties of the Smarandache AMAR LCM Triangle are easy to see.

1. The first column and the leading diagonal elements are all unity.
2. The k^{th} column are the elements of SLRS(k).
3. The first four rows are the same as that of the Pascal's Triangle.
4. The second column is the set of natural numbers.
5. The third column is the set of the triangular numbers.
6. If p is a prime then p divides all the terms of the p^{th} row except the first and the last which are unity. In other words Σ p^{th} row $\equiv 2 \pmod p$.

By careful observation, additional problems present themselves. For example, in the ninth row, 42 appears in three consecutive places.

Open problem: Do sequences of equal values of arbitrary length appear in the Smarandache AMAR LCM triangle?

Open problem: Find a formula for the sum of the rows.

Open problem: Search for congruence properties when n is composite.

The Smarandache function S(n) = k, is defined as the smallest integer such that n divides k!. The following function is defined in a similar way.

Definition: The Smarandache LCM function $S_L(n) = k$, is defined as the smallest integer k such that n divides LCM(1, 2, 3, . . . , k).

Let $n = p_1^{a1}\, p_2^{a2}\, p_3^{a3} \ldots p_r^{ar}$ be the prime factorization of n and let p_m^{am} be the largest divisor of n with only one prime factor, then it follows that $S_L(n) = p_m^{am}$.

If n =k! then S(n) = k and $S_L(n) > k$.

If n is a prime then we have $S_L(n) = S(n) = n$.

Clearly $S_L(n) \geq S(n)$ the equality holding for n a prime or n = 4, n=12.

Also $S_L(n) = n$ if n is a prime power. ($n = p^a$).

Open problem: Are there numbers n >12 for which $S_L(n) = S(n)$?

Open problem: Are there numbers n for which $S_L(n) = S(n) \leq n$?

Section 30

An Application of the Smarandache LCM Sequence and the Largest Number Divisible By All the Integers Not Exceeding Its rth Root

Definition: The numbers which are divisible by all numbers not exceeding their square root are 2, 4, 6, 8, 12 and 24. These numbers will be called the Smarandache Murty numbers of order 2. The largest number will be called the Smarandache Pati number of order 2.

The numbers which are divisible by all whole numbers not exceeding their cube root are

2,3, 4,5, 6,7, all even numbers from 8 to 26, 36, 48, 60, 72, …120, 180, 240, 300, 420.

We have six consecutive numbers from 2 to 7, ten consecutive even numbers from 8 to 26, eight consecutive multiples of 12 from 36 to 120, four consecutive multiples of 60 from 120 to 300 and finally 420, which is the Smarandache Pati number of order 3.

If we add these consecutive properties together, we have

6 + 10 + 8 + 4 + 1 = 29.

Note: As $7^3 = 343 < 420 < 512 = 8^3$ and $840 > 729 = 9^3$ and furthermore

$2520 > 1331 = 11^3$, it is evident that 420 is the largest number which is divisible by all the whole numbers not exceeding its cube root.

Let $m = [n^{1/3}]$, where [ ] stands for the greatest integer function. Then n will be a Smarandache Murthy number of order 3 if the LCM of the numbers from 1 to m divides n.

Our next point of consideration will be to find an upper limit for the Smarandache Pati number of order r.

Consider the case for r=3. Obviously, n is divisible by all the primes less than m. Let there be s primes less than m. L(m) is also divisible by all the primes $p_1, p_2, \ldots p_s$ and let $L(m) = p_1^{k1} * p_2^{k2} p_3^{k3} \ldots p_s^{ks}$. Also, by the choice of the numbers we have the inequality $p_i^{ki} \leq n^{1/3} < p_i^{ki+1}$.

Multiplying all the inequalities we get

$$p_1^{k1} * p_2^{k2} p_3^{k3} \ldots p_s^{ks} \leq \{n^{1/3}\}^s < p_1^{k1+1} * p_2^{k2+1} p_3^{k3+1} \ldots p_s^{ks+1}$$

$$\{n^{1/3}\}^s < \{p_1^{k1} p_2^{k2} p_3^{k3} \ldots p_s^{ks}\} * \{p_1 * p_2 * \ldots * p_s\} < L(m) * L(m) = \{L(m)\}^2.$$

Also, as L(m) divides n we have $p_1 * p_2 * \ldots * p_s < L(m) < n$.

Therefore, we have $\{n^{1/3}\}^s < n^2$ and it follows that $s < 6$. As all the primes p_1, to p_5 are less than $n^{1/3}$, $p_6 = 13 > n^{1/3}$ or $n < 13^3 = 2197$. On investigating all the numbers smaller than 2197 we find that there are 29 Smarandache Murthy numbers of order 3 and 420 is the Smarandache Pati number of order 3.

Along similar lines we can prove that the Smarandache Pati number of order r $< p_{2r}^{r}$. The largest number which is divisible by all the whole numbers not exceeding its r^{th} root is less than p_{2r}^{r}.

Open problem: For large values of r this upper limit may be too big an estimate. Readers are encouraged to try to reduce the upper bound.

Open problem: For r= 2, there are 6 Smarandache Murthy numbers of order 2. For r =3, there are 29 Smarandache Murthy numbers of order 3 . The question is to find a general result on the total number of Smarandache Murthy numbers of order r.

Open problem: Find an expression for the Smarandache Pati number of order r.

Section 31

The Number of Primes in the Smarandache Multiple Sequence

Definition: The Smarandache nkn Generalized sequence is formed by concatenating all of the numbers n, 2n, 3n, . . . , n*n. The first few terms are

1 ,24 , 369 , 481216 , 510152025 , 61218243036, 7142128354249 , 816243240485664,

where 510152025 is formed by concatentating 5, 2*5, 3*5, 4*5 and 5*5.

The question has been raised regarding how many primes are contained in this sequence, so our next step will be to show that there are no primes in the sequence.

Proposition: n divides a(n), where a(n) is the nth term of the Smarandache nkn sequence. Therefore, there are no primes in the sequence.

Proof: Consider the n^{th} term a(n). Let k*n have d_k digits and let

$s(n-k) = d_n + d_{n-1} + \ldots + d_{n-k+1}$,

then by the very definition of the sequence, we have

$$a(n) = n^2 + n.(n-1).10^{s(n)} + n.(n-2).10^{s(n-1)} + n(n-3).10^{s(n-2)} + \ldots n(n-k).10^{s(n-k)} + n.2.10^{s(2)} + n.1.10^{s(1)}.$$

For example, $a(4) = 481216 = 4^2 + 4.3.10^2 + 4.2.10^4 + 4.1.10^5$
$= 16 + 1200 + 80000 + 400000.$

Therefore, it is clear that n divides a(n), and the proof is complete.

Definition: The Smarandache Reduced Multiple sequence is formed by dividing the terms of the nkn sequence by n.

The first few terms of the sequence are

1, 12, 123, 120304, 102030405, 10203040506, 1020304050607, . . .

a(13) = 13263952657891104117130143156169
b(13) = 102030405060700800901001101 2013.

Open problem: How many terms of the Smarandache reduced multiple sequence are prime?

Section 32

More on the Smarandache Square and Higher Power Bases

Definition: The Smarandache Square Base is the expression of a number as the sum of distinct squares greater than one plus e, where e = 0, 1, 2, or 3. It is known that every positive number can be expressed in Smarandache square base form.

In a similar manner, the Smarandache base cube and higher power bases can be defined.

Definition: The Smarandache Square Part Residue Zero sequence is the set of numbers that can be expressed as the sum of two or more perfect squares that are greater than or equal to one.

The first few terms of this sequence are

5, 10, 13, 14, 17, 20, 21, 25, 26, 29, 30, 34, . . .

Open problem: How many of the numbers in the Smarandache square part residue zero sequence are perfect squares?

Note 1: All numbers of the form $(a^2+b^2)^2$ { the largest members of the Pythagorean triplets} are members of the sequence. Do all the squares in this sequence have this form?

Note 2: For note 1, it follows that infinitely many fourth powers are members of the sequence.

Open problem: How many fifth or higher powers are members of the sequence?

Open problem: How many elements in the sequence can be expressed as the sum of squares in two or more ways?

Some examples are

45 = 36 + 9 = 25 + 16 + 4 , 125 = 100 + 25 = 121 + 4.

Open problem: Can one find a sequence of consecutive integers of arbitrary length in the sequence?

Along similar lines the Smarandache Square part residue unity the Smarandache Square part residue two and the Smarandache Square part residue three sequences can also be defined.

The same ideas can be used to define the Smarandche Cube Part Residue Zero sequence. The first few terms in the sequence are

9, 28, 35, 36, 65, 72, 73, 91, 92, 100, . . .

Open problem: How many elements in the Smarandche cube part residue zero sequence are perfect cubes?

Some examples are 216, 3375, 9261.

$216 = 1^3 + 2^3 + 3^3 = 1 + 8 + 27$, $3375 = 15^3 = 1^3 + 2^3 + 3^3 + 4^3 + 6^3 + 11^3 + 12^3 = 1 + 8 + 27 + 64 + 216 + 1331 + 1728$, $21^3 = 9261 = 1^3 + 3^3 + 6^3 + 9^3 + 15^3 + 17^3 = 2^3 + 6^3 + 13^3 + 14^3 + 64^3$.

Open problem: How many elements in the sequence can be expressed as the sum of perfect cubes in more than one way?

Some examples are: 1729 (the Ramanujan number) $= 12^3 + 1 = 10^3 + 9^3$,

$21^3 = 9261 = 1^3 + 3^3 + 6^3 + 9^3 + 15^3 + 17^3 = 2^3 + 6^3 + 13^3 + 14^3 + 64^3$.

Open problem: Is it possible to find a sequence of consecutive integers or arbitrary length in the above sequence?

The square of every triangular number $\{n*(n+1)/2\}^2$, $n > 1$ is a member of the above sequence. Also, since there are infinitely many square triangular numbers, the above sequence contains infinitely many fourth powers.

Open problem: How many fifth or higher powers are in the sequence?

Section 33
Smarandache Fourth and Higher Patterned/ Additive Perfect Power Sequences

Consider the following patterned fourth power sequence

99^4, 999^4, 9999^4, 99999^4, . . .

where the fourth powers are

96059601, 996005996001, 9996000599960001, 99996000059999600001, . . .

Proposition 1: The number $(10^n – 1)^4$ is a member of Smarandache fourth power additive sequence if $n = 2^{4m+3}*3^{4k+2} - 1$, for all m and k.

Proof: It can be proved that the sum of the digits $d(T_n)$ of the n^{th} term T_n is given by

$d(T_n) = 9+6+5+9+6+1 + 18*(n-1) = 18*(n+1) .= 2*3^2*(n+1)$.

If $n+1 = 2^{4m+3}*3^{4k+2}$ then $d(T_n) = 2^{4m+4}*3^{4k+4} = \{2^{m+1}*3^{k+1}\}^4$ which is a perfect fourth power. Since there are no restrictions on the values of m and k, this generates infinitely many terms of the Smarandache fourth power additive sequence.

Note: More generally, n can be chosen as $r^4* 2^{4m+3}*3^{4k+2} – 1$. (r, m, k chosen arbitrarily.)

The Smarandache Patterned/Additive fifth power sequence.

99^5, 999^5 , 9999^5, 99999^5 , . . .

9509900499, 995009990004999, 99950009999000049999, . . .

In this sequence, the sum of the digits $d(T_n)$ of the n^{th} term T_n are given by $T_n = 54 + 27(n-1) = 27(n+1)$. And if $n = r^5* 3^{5m+2} - 1$, the conditions are satisfied.

Smarandache Patterned sixth power sequence.

99^6, 999^6 , 9999^6, 99999^6 , . . .

941480149401, 994014980014994001, 999400149980001499940001, . . .

It can be proved that there is no term in this sequence for which the sum of the digits is a perfect sixth power.

Open problem: Are there infinitely many terms in the Smarandache sixth power additive sequence?

Smarandache Patterned/ Additive seventh power sequence.

$99^7, 999^7, 9999^7, 99999^7, \ldots$
93206534790699, 993020965034979006999, 9993002099650034997900069999, . . .

$d(T_n) = 72 + 36*(n-1) = 36(n+1)$. For $n= r^7*9^{7k+3} - 1$ we satisfy the conditions for which $d(T_n) = \{r*9^{(k+1)}\}^7$.

Smarandache Patterned/ Additive eighth power sequence.

$99^8, 999^8, 9999^8, 99999^8, \ldots$
9227446944279201, 992027944069944027992001,
99920027994400699944002799920001

$d(T_n) = 72 + 36(n-1) = 36(n+1)$, For $n= r^8*9^{8k+5} - 1$ we satisfy the conditions for which $d(T_n) = \{r*9^{(k+1)}\}^8$.

Additional problem to consider:
1) Determine the values of m for which the sequence $(10^n - 1)^m$ gives infinitely many terms of the Smarandache m^{th} power Additive sequence.
(For m = 6 there is no such term).
Conjecture: For every m there are infinitely many terms in the Smarandache m^{th} power additive sequence.

Section 34

The Smarandache Multiplicative Cubic Sequence and More Ideas on Digit Sums

Definition: The Smarandache multiplicative cube sequence is defined as a sequence of perfect cubes in which the product of the digits is also a cube.

The first few terms of the sequence are:

1, 8, 24389, 226981,9393931,11239424, 17373979, 36264691, 66923416,

94818816, **. . .**

Which are

$1, 2^3, 29^3, 61^3, 211^3, 224^3, 259^3, 331^3, 406^3, 456^3, \ldots$
respectively.

It is an open problem as to whether this sequence is finite or infinite. To prepare for an attack on this problem, two additional problems will be examined first.

Problem 1: Are there infinitely many cubes where no digit is zero?

Problem 2: Is there any pattern of numbers that generate perfect cubes with no zero digit?

It seems that the answer to both these two questions is no.

Some additional sequences based on the digit sum will now be given.

Smarandache sequence of numbers where the digits sum to a perfect square.

The first few terms are

1, 4 , 9 , 10, 13, 18, 22, 27, 31, 36, 40, 45, 54 79, 81, 90, 97, 100,103, 108, 112, 117, 121, 126, 130, 135, 144, 153, 162, 169, 171, 178, 180 ,187, 196,202, 207, 211, 216, 220, 225, . . .

Smarandache sequence of the smallest number T_n whose digits sum to n^2,

$d(T_n) = n^2$, T_n is the smallest such number.

The first few terms of the sequence are:

1, 4 , 9 , 79, 799, 9999, 499999, 19999999, 999999999, 199999999999, . . .

The Smarandache sequence of the smallest squares T_n whose digits sum to n^2

$d(T_n) = n^2$, where $d(T_n)$ = the sum of the digits of the n^{th} term, T_n is the smallest such number.

The first few terms of this sequence are:

1, 4 , 9 , 169 , 4489 ,69696,. . .

Open problem: Find a formula for the general term of this sequence.

The Smarandache sequence of numbers with digits whose first digit sum is a perfect cube.

The first few terms of the sequence are:

1,8,10, 17, 26, 35, 44,53,62,71,80,100, 107, 116, 125, 134,. . . 800, 999, 1000, 1007, 1016, . . .1899, 1989, 1998, 2006, 2015, . . . 2799, 2979, 2997, 3005, . . .19999999, . . .

The Smarandache sequence of the smallest numbers whose digits sum to n^3.

The first few terms in this sequence are:

1 , 8 , 999 , 19999999 ,…

The Smarandache sequence of the smallest cubes whose digits sum to n^3.

The first few terms of the sequence are:

1, 8 , 19683, 999400119992, 999998500000749999875, . . .
$1, 2^3 , 27^3 , 9998^3, 99995^3, 999999999999^3$,. . .
d(999400119992) = 64 , d(999998500000749999875) = 125

$d\{(999999999999)^3\}$ = d(999999999997000000000002999999999999) = 216

On a similar line, the general questions of the Smarandache sequence of the smallest number whose digits sum to n^m , and the Smarandache sequence of the smallest m^{th}

power whose digits sum to n^m can be posed.

Open problem: For what values of $m > 3$, is the Smarandache sequence of the smallest m^{th} power whose digits sum to an m^{th} power, finite?

Section 35
Smarandache Prime Generator Sequence

It has been proven that for every prime p, there is a prime of the form kp+1. This is evident from the fact that all the divisors of the number $(10^p-1)/9$ are of the form k*p+1, which itself is of the form k*p+1.

Starting from 2, form a sequence of primes in which

$T_1 = 2$, $T_{n+1} = k^* T_n + 1$, where k is the smallest number yielding a prime.

The first few terms of this sequence are

2, 3, 7, 29, 59, 709, 2837,. . .

The smallest prime not included in this sequence is 5.

Then, starting with $T_1 = 5$, $T_{n+1} = k^* T_n + 1$, where k is again the smallest number yielding a prime, we get the sequence,

5, 11, 23, 47, 283, 1699, . . .

The smallest prime not already in a sequence is 13, so starting with 13 and then repeating the process using additional primes, we get the following sequences

13, 53, 107, 643, 7717,. . .

17, 103, 619, 2477, . . .

19, 191, 383, 769, 7691, . . .

31, 311, 1867, . . .

The Smarandache Prime Generator sequence will be constructed using the first terms of these sequences.

2, 5, 13, 17, 19, 31, . . .

Conjecture: The Smarandache prime generator sequence is finite.

Is there any prime that is a member of more than one of the sequences that generate the Smarandache prime generator sequence? Let p be a common member of two sequences with the first term as p_1 and p_2 respectively. Then, for some k_1 and k_2 we get

$P = 2k_1{}^*p_1 + 1 = 2k_2{}^* p_2 +1$, or $k_1{}^* p_1 = k_2{}^* p_2$, or $k_1 = r^* p_2$ and $k_2 = r^*p_1$.

Conjecture: All of the sequences that generate the Smarandache prime generator sequence are distinct.

Additional open problems that I thought of regarding the Smarandache additive prime sequence will close this chapter.

Open problem: Are there arbitrarily long sequences of consecutive primes having the same first sum of digits?

Open problem: Are there arbitrarily long sequences of consecutive primes having the same final sum of digits?

Open problem: Are there arbitrarily long sequences of consecutive terms of the Smarandache additive prime sequence with the same first sum of digits?

Open problem: Are there arbitrarily long sequences of consecutive terms of the Smarandache additive prime sequence with the same final sum of digits?

Section 36

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Chapter 3
Miscellaneous Topics

In the previous chapters, an underlying similarity or theme was used to group the material together. However, not all of the Smarandache notions can be so easily categorized. Therefore, this chapter serves as a repository for those topics that are considered different enough that placing them in one of the previous chapters is considered inappropriate.

Section 1

Exploring Some New Ideas On Smarandache Type Sets, Functions And Sequences

(1)Smarandache Patterned Perfect Square Sequences. Consider the following sequence of numbers

13, 133, 1333, 13333, . . .-------- (1)

which is formed by the squares of the numbers

169, 1*7*6*8*9, 1*77*6*88*9, 1*777*6*888*9, . . . ------- (2)

Sequence (1) is called the root sequence of (2) and it is clear that there is a pattern to the numbers of both sequences. The root sequence is a one followed by a sequence of n 3's and the elements of the product sequence are a one, followed by (n-1) sevens, a 6, (n-1) eights and ending in a nine.

There are a finite number of such patterned perfect square sequences and here is a list of the root sequences.

(I) 13, 133, 1333, 13333, . . .
(2) 16, 166, 1666, 16666, . . .
(3) 19, 199, 1999, 19999, . . .
(4) 23, 233, 2333, 23333, . . .
(5) 26, 266, 2666, 26666, . . .
(6) 29, 299, 2999, 29999, . . .

Along similar lines, we have root sequences where the first terms are

(7) 33 (8) 36 (9) 39 (10) 43 (11) 46 (12) 49 (13) 53 (14) 66 (15) 73
(16) 79 (17) 93 (18) 96 (19) 99.

Open Problems:
(1) Are there any patterned perfect cube sequences?

(2) Are there any patterned perfect power sequences for a power greater than 3?

Smarandache Breakup Square Sequences

The sequence

4, 9, 284, 61209, . . .

is defined by

$4 = 2^2$
$49 = 7^2$
$49284 = 222^2$
$4928461209 = 70203^2$

T_n is the smallest sequence of digits such that the concatenation

$T_1T_2 \ldots T_{n-1}\, T_n$

is a perfect square. The following limit

$$\lim_{n\to\infty} \frac{(T_1T_2 \ldots T_{n-1}\, T_n)^{1/2}}{10^k}$$

where k is the number of digits in the numerator, is close to either 2.22 . . . or 7.0203 . . .

Smarandache Breakup Cube Sequences

Along similar lines, the Smarandache Breakup Cube Sequence, where cubes are used instead of squares. By using larger exponents, we can define Smarandache Breakup Perfect Power Sequences.

Smarandache Breakup Incremented Perfect Power Sequences

1, 6, 6375, …

$1 = 1^1$, $16 = 4^2$, $166375 = 55^3$, etc.

T_n is the smallest number whose digits concatenated with the previous numbers in the sequence

$T_1T_2T_3 \ldots T_n$ yields a perfect nth power.

Smarandache Breakup Prime Sequence

The Smarandache Breakup Prime Sequence is formed by finding the smallest number T_n such that if it is concatenated on the right to the concatenation of all previous terms, a prime number is formed. In other words,

2, 3, 3, . . .

2, 23, 233 etc. are primes.

$T_1T_2 \ldots T_{n-1}T_n$ is a prime

Smarandache Symmetric Perfect Square Sequence

This sequence is the set of perfect squares that are also palindromic. The first few terms are

1, 4, 9, 121, 484, 14641, . . .

Smarandache Symmetric Perfect Cube Sequence

This sequence is the set of perfect cubes that are also palindromic. The first few terms are

1, 8, 343, 1331, . . .

Smarandache Symmetric Perfect Power Sequence

This is the general sequence, for $n \geq 2$, the set of n powers that are also palindromes.

Smarandache Divisible by n Sequence

The terms of this sequence (T_k) are the smallest numbers such that k divides $T_1T_2 \ldots T_k$. The first few terms of this sequence are

1, 2, 0, 4, 0, 2, . . .

Which is a consequence of the computations

1 | 1, 2 | 12, 3 | 120, 4 | 1204, 5 | 12040, 6 | 120402, . . .

Smarandache Sequence of Numbers Where the Sum of the Digits Is Prime

2, 3, 5, 7, 11, 12, 14, 16, 20,21,23,25,29, . . .

Smarandache Sequence of Primes Where the Sum of the Digits Is Prime

2, 3, 5, 7, 11, 23, 29, 41, 43, 47, 61, 67, 83, 89, . . .

Smarandache Sequence of Primes p where 2p + 1 is Also Prime

2, 3, 5, 11, 23, 29, 41, 53, . . .

Smarandache Sequence of Primes p where 2p - 1 is Also Prime

3, 7, 19, 31, . . .

Smarandache Sequence of Primes p where $p^2 + 2$ is Also Prime

3, 17, . . .

Smarandache Sequence of the Smallest Primes Which Differs by 2n From the Previous Prime

5, 17, 29, 97, . . .

$S_1 = 5 = 3 + 2$, $S_2 = 17 = 13 + 4$, $S_3 = 29 = 23 + 6$, $S_4 = 97 = 89 + 8$.

Smarandache Sequence of Smallest Prime For Which p + 2r is Prime

Element r is the smallest prime p, such that p + 2r is prime. The first few terms are

3, 13, 23, 89, . . .

Since 3 + 2*1 = 5, 13 + 2*2 = 17, 23 + 2*3 = = 29, 89 + 2*4 = 97.

Smarandache Sequence s_n of the Smallest Number Whose Sum of Digits is n

The first few elements are

1, 2, 3, 4, 5, 6, 7, 8, 9, 19, 29, 39, 49, 59, 69, 79, 89, 99, 199, 299, 399, 499, 599, . . .

This is a sequence of numbers satisfying the property

$$N + 1 = \prod_{r=1}^{k} (a_r + 1)$$

where a_r is the rth digit of the number.

Proof:
Let $N = a_r a_{r-1} \ldots a_1$ such that

$$N + 1 = \prod_{r=1}^{k} (a_r + 1) \qquad (3)$$

The largest k-digit number is $N = 10^k - 1$, where all the digits are 9. It can be verified that this is a solution. Are there other solutions?

Let the m^{th} digit be changed from 9 to a_m ($a_m < 9$). Then the right member of (3) becomes $10^{(k-1)} (a_m + 1)$. This amounts to the reduction in value by $10^{(k-1)}(9-a_m)$. The value of the k-digit number N goes down by $10^{(m-1)}(9-a_m)$. For the new number to be a solution these two values have to be equal which occurs only at m = k. This gives 8 more solutions. In all there are 9 solutions given by $a.10^k - 1$, for a = 1 to 9.
For k = 3 the solutions are
199, 299, 399, 499, 599, 699, 799, 899, 999, …

Question: Are there infinitely many primes in this sequence?

Smarandache Sequence of Numbers Such That the Sum of the Digits Divides n

1, 3, 6, 9, 10, 12, 18, 20, 21, 24, 27, 30, 36, 40, 42, 45, 48, 50, 54, 60, 63, 72, 80, 81, 84, 90, 100, 102, 108, 110, 112, 114, 120, 126, 132, 133, 135, 140, 144, 150, . . .
Smarandache Sequence Of Numbers Such That Each Digit Divides n

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 20, 22, 24, 30, 33, 36, 40, 44, 50, 55, 60, 66, . . .

Smarandache Power Stack Sequence For n (SPSS(n))

SPSS(2)

The n^{th} term is obtained by concatentating the digits of the powers of 2 starting from 2^0 to 2^n and moving left to right. The first few digits of this sequence are

1, 12, 124, 1248, 12416, 1241632, . . .

SPSS(3)

The n^{th} term is obtained by concatentating the digits of the powers of 3 starting from 3^0 to 3^n and moving left to right. The first few digits of this sequence are

1, 13, 139, 13927, . . .

Question: If n is an odd number not divisible by 5, how many terms in the SPSS(3) sequence are prime? It is clear that $n \mid s_n$ if and only if $n \equiv 0 (\mathrm{mod}\ 5)$.

Smarandache Self Power Stack Sequence (SPSS)

The kth term in the sequence is formed by concatentating the numbers

$1^1, 2^2, 3^3, \ldots, k^k$

starting from the left. The first few terms are

1, 14, 1427, 1427256, 14272563125, 142725631257776, . . .

Smarandache Perfect Square Count Partition Sequence (SPSCPS(n))

The kth term (starting at zero) of this sequence is defined as the number of perfect squares m that satisfy the inequality

$$nk + 1 \leq m \leq nk + n.$$

For n = 12, the first few terms in the sequence SPSCPS(12) are

3, 1, 2, . . .

since the number of perfect squares less than 12 is 3 and the number of perfect squares between 13 and 24 is 1.

Smarandache Perfect Power Count Partition Sequence (SPPCPS(n,k))

The rth term (starting at zero) of SPPCPS(n,k) is the number of kth powers m that satisfy the inequality

$$nr + 1 \leq m \leq nr + n.$$

For example, the first term of SPPCPS(100,3) is 4, as $1^3, 2^3, 3^3, 4^3$ are all less than 100.

Question: Does Σ (Tr/(nr)) converge as $n \rightarrow \infty$?

Smarandache Bertrand Prime Sequence

According to Bertrand 's postulate, there exists a prime between n and 2n. Starting from 2, form a sequence by taking the largest prime less than double the previous prime in the sequence . The first few elements of the sequence are

2, 3, 5, 7, 13, 23, 43, 83, 163, . . .

Smarandache Semi-perfect Number Sequence

A semi-perfect number is one that can be expressed as the sum of a subset of its distinct divisors. For example,

12 = 2 + 4 + 6 = 1 + 2 + 3 + 6
20 = 1 + 4 + 5 + 10
30 = 2 + 3 + 10 + 15 = 5 + 10 + 15 = 1 + 3 + 5 + 6 + 15.

It is clear that every perfect number is also semi-perfect.

Theorem: There are infinitely many semi-perfect numbers.

Proof: We shall prove that $N = 2^n p$, where p is prime less than $2^{n+1} - 1$ is a semi-perfect number.

The divisors of N are

$1, 2, 2^2, \ldots, 2^n$

$p, 2p, 2^2p, \ldots, 2^n p$.

Summing the second row, we have

$$\sum_{r=0}^{n-1} 2^r p = p(1 + 2 + 2^2 + 2^3 + \ldots 2^{n-1}) = p(2^n - 1) = M.$$

The difference between N and M is p. It is known that every number is expressible as the sum of powers of two and we have selected p so that it is less than the largest power in the above list. Therefore, we can express p as the sum of powers of two

$$p = \sum_{r=0}^{n} a_r * 2^r, \quad \text{where } a_r = 0 \text{ or } a_r = 1.$$

Since none of these factors were used in the previous sum, we can express N as the sum of a subset of its' divisors.

Remark: There are many additional examples of semi-perfect numbers. Readers are encouraged to search for additional families of semi-perfect numbers.

Smarandache Co-prime But No Prime Sequence

This sequence starts with four and each subsequent term is the smallest number that is relatively prime to the previous term and is not prime. More formally,

The nth term T_n is defined as follows

$T_n = \{ x \mid (T_{n-1} , x) =1$, x is not a prime and $(T_{n-1} , y) \neq 1$ for $T_{n-1} < y < x \}$.

The first few terms of the sequence are

4, 9, 10, 21, 22, 25, 26, 27, 28, 33, 34, 35 , 36 ,49, 50, 51, 52 , . . .

Open problem: Does the Smarandache Co-prime but no Prime Sequence contain arbitrarily long sequences of integers of the form k, k+1, k+2,
k+3, . . .,k+n?

Definition: We define a prime p_r to be a week prime if

$p_r < (p_{r-1} + p_{r+1}) /2$.

It is a balanced prime if

$p_r = (p_{r-1} + p_{r+1}) /2$.

It is a strong prime if

$p_r > (p_{r-1} + p_{r+1}) /2$.

For example:

$3 < (2+5)/2$ is weak prime .
$5= (3 + 7)/2$ is a balanced prime .
$71 > (67 + 73)/2$ is a strong prime .

Smarandache Weak Prime Sequence

3, 7, 13, 19, 23, 29, 31, 37, . . .

Smarandache Strong Prime Sequence

11, 17, 41, . . .

Smarandache Balanced Prime Sequence

5, . . .

It is clear that for a balanced prime > 5, $p_r = p_{r-1} + 6k$.

Section 2

Fabricating Perfect Squares With a Given Valid Digit Sum

Introduction:

While studying the Smarandache additive square sequence [48-1] (sequence of squares in which the digits sum is also a square), a question popped into my mind. Given a number d can one get a perfect square whose digit sum is d? In this chapter some results pertaining to this question have been established.

Definition: Given any integer $a_n a_{n-1} \ldots a_0$ the digit sum is

$a_n + a_{n-1} + \ldots a_0$.

If this sum has more than one digit, repeatedly take the sum of the digits until the result is a one-digit number.

Definition: A number d is called a valid digits sum if

$d \equiv 1 \pmod 3$, or $d \equiv 0 \pmod 9$ ------------ (1)

Proposition I: The digit sum of a perfect square is a valid digit sum.

Proof: Consider the squares of numbers 1 through 9

1, 4, 9, 16, 25, 36, 49, 64, 81

the digit sums are

1, 4, 9, 7, 7, 9, 4, 1, 9

which definitely are of type (1) .

It can also be proved using the properties of congruence that the digits sum of the product of two numbers is the product of the digits sums. Since this can be repeated an arbitrary number of times, the proof is complete.

Theorem: If d is a valid digit sum then there exist infinitely many perfect squares whose digits sum is d.

Proof:
Consider the following four Smarandache Patterned perfect square sequences [48-2] along with their root sequences.

(I) 9, 99, 999, 9999 , . . .
81, 9801, 998001, 99980001, . . .

We have

$T_n = 10^n -1$, . $T_n^2 = 10^{2n} -2* 10^n + 1. = 10^{n+1} (10^{n-1} - 1) + 8* 10^n + 1.$

Hence, the sum of the digits of $T_n^2 = 9(n-1) + 8 + 1 = 9n.$

(II) 1, 19, 199, 1999 , . . .
1, 361, 39601, 3996001 , . . .

$T_n = 2 * 10^{n-1} - 1$, $T_n^2 = 4*10^{2(n-1)} - 4* 10^{n-1} + 1.$

$T_n^2 = 3*10^{2(n-1)} + 10^n * (10^{n-2} -1) + 6*10^{n-1} + 1.$

The sum of the digits of $T_n^2 = 3 + 9(n-2) + 6 + 1 = 9(n-1) + 1.$

(III) 2 , 29, 299, 2999, . . .
4, 841, 89401, 8994001, . . .

$T_n = 3 * 10^{n-1} - 1$, $T_n^2 = 9*10^{2(n-1)} - 6* 10^{n-1} + 1.$
$T_n^2 = 8*10^{2(n-1)} + 10^n * (10^{n-2} -1) + 4*10^{n-1} + 1.$

The sum of the digits of $T_n^2 = 8 + 9(n-2) + 4 + 1 = 9(n-1) + 4.$

(IV) 5, 59, 599 , 5999 . . .
25, 3481, 358801, 35988001, . . .

For n = 1 and n=2, the sum of the digits of T_n^2 are 7 and 16 respectively.

For $n \geq 3$ $T_n = 6* 10^{n-1} - 1$, and $T_n^2 = 36*10^{2(n-1)} -12* 10^{n-1} +1$

$T_n^2 = 35* 10^{2(n-1)} + 10^{n+1} *(10^{n-3} -1) + 88*10^{n-1} + 1.$

The sum of the digits of Tn2 = 3 + 5 + 9(n-3) + 8 + 8 + 1 = 9 (n-1) + 7.

This pattern also holds for n = 1 and 2 as well.

Since every number of the type 3*r+1 is congruent to 1, 4 or 7 (mod 9) the previous four cases cover all the valid digits sums.

Therefore, we have proved that there exists a perfect square with a given valid digits sum. By adding an even number of zeros we get infinitely many such numbers and the proof of the theorem is complete.

Example: Let d = 124, so our goal is to find a perfect square N with digits sum =124. We have d = 124 ≡ 7(mod 9). The required number is the member of sequence (IV), for which n = 14 , [124 = 9 (14 - 1) + 7.]

$N = (6 * 10^{13} - 1)^2 = 3599999999999880000000000001.$

Conjecture: For a given valid digits sum d there exists infinitely many nontrivial perfect squares whose digits sum is d. (If N is a solution then $N*10^{2n}$ is a nontrivial one.)

Generalizing this result, if we define a valid digit sum for a cube as a number congruent to 0, 1 ,or 8 (mod 9), then we can put forward the following conjectures.

Conjecture: For a given valid digit sum d there exists infinitely many nontrivial perfect cubes whose digits sum is d. (If N is a solution then $N*10^{3n}$ is a nontrivial one.)

Conjecture: For a given valid digit sum d there exists infinitely many nontrivial perfect mth powers whose digit sum is d. (If N is a solution then $N*10^{m*n}$ is a nontrivial one.)

Section 3

Fabricating Perfect Cubes With a Given Valid Digit Sum

Introduction:

In the previous section, given an arbitrary number d, the question to consider was whether there was a perfect square with a digit sum equal to d. In this section, we will consider the similar problem where square is replaced by cube.

Definition: A number d is called a valid digit sum for a cube is d ≡ 0, 1, or 8 (mod 9). In other words, the digit sum is 9k, 9k + 1 or 9k – 1.

Theorem: For a given valid digit sum for cube d, there exists infinitely many perfect cubes whose digit sum is d, when d is of the form 18k, 9k+1 or 9k - 1.

To prove this theorem, we start with the following proposition.

Proposition: For a perfect cube, the digit sum necessarily is a valid digits sum for a cube satisfying condition (1).

Proof of the proposition: Examining the cubes of the numbers 1 through 9, we get 1, 8, 27, 64, 125, 216, 343, 512, and 729. The corresponding digits sums are 1, 8, 9, 10, 8, 9, 10, 8, 18, all of which reduce down to 1, 8 or 9. This is consistent with the definition of the valid digits sum for a cube. Also, it can be proved using the properties of congruence that the digits sum of the product of two numbers is the product of the digits sums. The proof is complete.

Proof of the theorem:
Consider the following Smarandache patterned perfect cube sequences with the corresponding root sequences

(I)
991, 9991, 99991, 999991, . . .

973 242 271, 9 973 0 242 9 271, 99 973 00 242 99 271, 999 973 000 242 999 271

we have $T_n = 10^{n+2} - 9$, for the root sequence , and

$T_n^3 = 10^{3n+6} - 27*10^{2n+4} + 243*10^{n+2} - 729$.

The general term of the cube sequence $T_n^3 = t_n$ is given by

$t_n = 10^{2n+7} *(10^{n-1} - 1) + 973*10^{2n+4} + 242*10^{n+2} + 10^3 *(10^{n-1} - 1) + 271$.

Upon simplification we have

$t_n = 10^{3n+6} - 27*10^{2n+4} + 243*10^{n+2} - 729 = T_n^3$.

The sum of the digits of

$t_n = 9(n-1) + 9+ 7 + 3 + 2 + 4 +2 + 9(n-1) + 2 + 7 + 1$

equals $18(n+1) + 1 = 9 (2m) + 1$, $m > 1$.

With $25^3 = 15625$, the sum of the digits is 19, the case where m = 1 is also included.

(II)
995, 9995, 99995, 999995, . . .
985 074 875, 9 985 0 074 9 875, 99 985 00 074 99 875, 999 985 000 074 999 875.

We have for the root sequence

$T_n = 10^{n+2} - 5$, $T_n^3 = 10^{3n+6} -15*10^{2n+4} + 75*10^{n+2} - 125$.

For the cube sequence $t_n = T_n^3$

$t_n = 10^{2n+7}*(10^{n-1} – 1) + 985*10^{2n+4} + 74*10^{n+2} + 10^3*(10^{n-1} – 1) + 875$.

When this is simplified, we have

$t_n = 10^{3n+6} –15*10^{2n+4} + 75*10^{n+2} –125 = T_n^3$.

The sum of the digits of t_n equals 9(n-1) + 9+ 8 + 5 + 7 + 4 + 9(n-1) + 8 + 7 + 5

= 18(n-1) + 53 = 9(2m) - 1, m >2.

With 17^3 =4913, the sum of the digits is 17, and 95^3 =857375, sum of digits = 35, the case m = 1 and m = 2 are also included.

(III)
9, 99, 999, 9999, . . .
729, 970299, 997002999, 999700029999, . . .

Consider the root sequence $T_n = 10^n - 1$, $T_n^3 = 10^{3n} - 3*10^{2n} + 3*10^n -1$.

For the cube sequence $t_n = T_n^3$

$t_n = 10^{2n+1} * (10^{n-1} -1) + 7*10^{2n} + 2*10^n + (10^n - 1)$.

When this expression is simplified, we have

$t_n = 10^{3n} - 3*10^{2n} + 3*10^n -1 = T_n^3$.

The sum of the digits for this number equals

9(n-1) + 7 + 2 + 9n = 18n = 9 (2n) = 9(2m) .

With the above three sequences we have taken care of the digit sums 9k, 9k+1 and 9k-1, for k even.

We will now consider the sequences for k odd.

(IV)
97, 997, 9997, 99997,. . .
912673, 991026973, 999100269973, 999910002699973, . . .

Consider the root sequence

$T_n = 10^{n+1} – 3$, $T_n^3 = 10^{3n+3} –9*10^{2n+2} + 27*10^{n+1} –27$.

For the cube sequence $t_n = T_n^3$

$t_n = 10^{2n+4} (10^{n-1} –1) + 91*10^{2n+2} + 26* 10^{n+1} + 100*(10^{n-1} – 1) + 73$.

Which yields the following when simplified

$T_n = 10^{3n+3} –9*10^{2n+2} + 27*10^{n+1} –27 = T_n^3$.

The sum of the digits for

$t_n = 9(n-1) + 9 + 1 + 2 + 6 + 9(n-1) + 7 + 3 = 18(n-1) + 28 = 9(2n+1) + 1.$

With $7^3 = 343$, digits sum = 10 , which is the case where n = 0.

(V)
98, 998, 9998, 99998,. . .
941192, 994011992, 999400119992, 999940001199992, . . .

We have the root sequence

$T_n = 10^{n+1} - 2$, $T_n^3 = 10^{3n+3} - 6*10^{2n+2} + 12*10^{n+1} - 8$.

For the cube sequence $t_n = T_n^3$

$$t_n = 10^{2n+4}(10^{n-1} - 1) + 94*10^{2n+2} + 11*10^{n+1} + 100*(10^{n-1} - 1) + 92.$$

Which on simplification gives

$T_n =.10^{3n+3} - 6*10^{2n+2} + 12*10^{n+1} - 8 = T_n^3$.

The sum of the digits for

$t_n = 9(n-1) + 9 + 4 + 1 + 1 + 9(n-1) + 9 + 2 = 18(n-1) + 26 = 9(2n+1) - 1.$

$8^3 = 512$, with digit sum 8, is the case where n = 0.

Therefore, the theorem is proven for d = 18k, 9k+1 and 9k-1.

Example: Given d = 118, find a perfect cube N such that the digit sum is equal to d.
We have d =118 = 9*13 + 1. Hence N is a member of sequence (IV) and
$N = (10^{14} - 3)^3$.

Note: Readers are encouraged to look for other exhaustive sets of sequences.

Open Problem: Find a sequence of cubes, the sum of whose digits is an odd multiple of 9.

Consider the following table

N	N^3	Sum of digits of N^3 = d	d/ 9
3	27	9	1
33	35937	27	3
333	36926037	36	4
3333	37025927037	45	5
33333	37035925937037	63	7
333333	37036925926037037	72	8
3333333	37037025925927037037	81	9
33333333	37037035925925937037037	99	11
333333333	37037036925925926037037037	108	12

Based on the above table, we make the following conjectures:

Conjecture I: (a) If $N = (10^{3k} - 1)/3$ then the sum of the digits of N^3 is 9(4k).

Conjecture I: (b) If $N = (10^{3k-1} - 1)/3$ then the sum of the digits of N^3 is 9(4k –1).

Conjecture I: (b) If $N = (10^{3k+1} - 1)/3$ then the sum of the digits of N^3 is 9(4k +1).

Conjecture II: For a given valid digit sum d there exist infinitely many nontrivial perfect cubes whose digit sum is d. (If N is a solution then $N*10^{3n}$ is a nontrivial one.)

Note: If conjecture I is true, it will take care of d = 9(2k+1) , an odd multiple of 9.

Together with the theorem I, it would lead to the truth of conjecture II.

Section 4

Smarandache Perfect Powers With Given Valid Digit Sum

In [2] the Smarandache additive square sequence is defined as the sequence of squares in which the digit sum is also a square. The valid digit sum for a square was defined in a previous section as a number d such that d ≡ 1(mod 3) or d ≡ 0(mod 9).

In this section, we define a Smarandache sequence of perfect squares with a given digit first sum as the sequence of perfect squares whose digits sum is the same.

Examples:
If digit sum = 1, there is the sequence

1, 100, 10000, . . . , 10^{2n}, . . .

For digit sum = 4

1, 4, 121, 10201, . . .

For digit sum = 7

16, 25, 1024, 2401, . . .

For digit sum = 9

9, 36, 81, 144, 225, 324, 441, 900, . . .

For digit sum 10

64, 361, . . .

For digit sum 13

49, 256, 625, 841, . . .

Using the first terms of the above sequences, we can define the Smarandache sequence of smallest perfect squares with valid digit first sums

1, 4, 16, 9, 64, 49, 169, 576, 289, . . .

where the digit sums are

1, 4, 7, 9, 10, 13, 16, 18, 19, . . .

Open problem: There are three consecutive terms in increasing order 49, 169, 576. Is it possible to have an arbitrary number of terms in increasing or decreasing order?

We define the Smarandache sequence of perfect squares with a given digits final sum

{1} 1, 64, 100, 289, 361, 676, 784, 1225, 1369, . . .

where the root sequence is

1, 8, 10, 17, 19, 26, 28, 35, 37, . . . ------------ (a).

Additional sequences are

{4} 4 , 49, 121 , 256, 400, 625 , . . . ------------ (b)
(7) 16, 25, 169, 196, 484, 529, . . . ------------ (c)

{9}   9, 36 , 81 , 144 , 225 , . . .   ------------ (d).

Note that we have $T_n = 9*n^2$ in sequence (d).

Open Problem: To find an expression for the nth term for the sequences (a) , (b) , and (c).

The above idea can be generalized by defining

(A) Smarandache sequence of perfect cubes with a given digit first sum as listed below where the number in the braces {} is the digit sum.

{1} 1, 1000, 1000000, . . . 103n . . .
{8} 8, 125, 512, 1331, . . .
{9} 27, 216, 27000, . . .
{10} 64, 343, 64000, . . .
{17} 2744, 4913, 12167, . . .
{18} 729, 1728, 3375, 5832, 9261, 13824, 91125, . . .

We have similar sequences for 19, 26, 27, 28 and so forth. Note that for all the numbers d, $d \equiv 0$, 1 or 8 (mod 9).

(B) Smarandache sequence of perfect cubes with a given digit final sum.

{1} 1, 64, 343, 1000, 4096, 6859, . . .
{8} 8, 125, 512, 1331, 2744, 4913, 12167 , . . .
{9} 27, 216, 729, . . . , $T_n = 27*n^3$.

Open Problem: To find the general term for the sequences where the sums are 1 and 8.

(C) Smarandache sequence of smallest perfect cubes with valid digit first sums is defined in the following way:

1, 8, 27, 64, 2744, 729, 2197, . . .

with the digit sums

1, 8, 9, 10, 17, 18, 19, . . .

Open problem: We have five consecutive terms in increasing order: 1, 8, 27, 64, 2744. Can we have an arbitrary number of terms in increasing or decreasing order?

Generalization: The Smarandache Sequence of perfect m^{th} powers with a given digit first sum, the Smarandache Sequence of perfect m^{th} powers with a given digit final sum and the Smarandache sequence of smallest perfect m^{th} powers with valid digit first sums can all be defined on similar lines.

Section 5

Numbers That Are a Multiple of the Product of Their Digits And Related Ideas

Smarandache Proud Pairs of Numbers

Definition: We say that a pair of integers (m,n) is a Smarandache proud pair if
$n = m = Pd(n)$, where $Pd(n)$ is the product of the digits of n.

Examples:
For the single digit numbers

$n = 1 * n$

$36 = 2 * (3 * 6)$, $15 = 3 *(1 * 5)$, $24 = 3 * (2 * 4)$, $175 = 5 * (1 * 7 * 5)$

So, (1,1), (2,2), (3,3), . . . , (9,9), (3,15), (3, 24), (5,175) are Smarandache proud pairs.

Conjecture: For every m having no zero digit, there exists a number n such that (m,n) is a Smaradache proud pair.

Conjecture: For every m, there exists infinitely many n such that (m,n) is a Smarandache proud pair.

Numbers For Which the m^{th} Power of the Sum of the Digits Equals the Sum of the Digits of the m^{th} Power

Numbers for Which the Square of the Sum of the Digits Equals the Sum of the Digits of the Square

If d(n) is the sum of the digits of n, then these numbers satisfy the formula

$d(n^2) = [d(n)]^2$.

Examples:

$11^2 = 121$, $(1+1)^2 = 1 + 2 + 1$
$12^2 = 144$, $21^2 = 441$, $(2+1)^2 = 4 + 4 + 1$
$22^2 = 484$, $(2+2)^2 = 4 + 8 + 4$.
$13^2 = 169$, $31^2 =961$, $(1+3)^2 = 1+6+9$
$111^2 =12321$, $(1+1+1)^2 = 1 +2 + 3 + 2 + 1$
$212^2 = 44944$, $(2+1+2)^2 = 4 + 4 + 9 + 4 + 4$.

There are infinitely many such numbers is evident from the fact that $\{10^k +1)$ and $2\{10^k +1)$ satisfy the conditions for all values of k. If these are to be considered a bit trivial, then here are additional, nontrivial patterns.

n = 212, 2102 , 21002 , 210002, . . .
n^2 = 44944, 4418404, 441084004, . . .
n = 122 , 1022 , 10022, 100022, . . .
n^2 = 14884, 1044484, 100440484, . . .

Numbers For Which the Cube of the Sum of the Digits Equals the Sum Of the Digits of the Cube

In this case, we are looking for solutions to $d(n^3) = [d(n)]^3$

The family 101, 1001, 10001, can be considered trivial examples. Nontrivial examples are

$11^3 = 1331$, $(1+1)^3 = 1+3+3+1$
$111^3 = 1367631$, $(1+1+1)^3 = 1 + 3 + 6 + 7 + 6 + 3 + 1$.

Numbers of the form 1011, 10011, 100011, . . . and 1101, 11001, 110001, . . also satisfy the expression.

The two additional examples are:

$(1010010001)^3 = 1030331606363361603330030001$
d(1030331606363361603330030001) = 64

$(11010010001)^3 = 1334636937969963961633330030001$
d(1334636937969963961633330030001) =125

which suggests the following conjecture.

Conjecture: For every positive integer m there exists a number n such that
$m^3 = d(n^3) = \{d(n)\}^3$.

Numbers For Which the Fourth Power of the Sum of the Digits Equals the Sum of the Digits of the Fourth Power

In this case, we are searching for solutions to the expression $d(n^4) = [d(n)]^4$.

Examples:
$11^4 = 14641$, $(1+1)^4 = 1 + 4 + 6 + 4 + 1$
101, 1001, 10001, . . .

Open problem 1: Are there numbers n, such that $m^4 = d(n^4) = [d(n)]^4$, where $m > 2$?
Open problem 2: Are there numbers n, such that $d(n^k) = [d(n)]^k$, where $k > 4$?

Generalization: Additional Smarandache digital sequences can be defined by studying relations between a function of the digits and another function of the number itself.

m^{th} powers where permutations of the digits are also m^{th} powers

For m = 2

$144 = 12^2$, $441 = 21^2$, $169 = 13^2$, $196 = 14^2$, $961 = 31^2$,
$1296 = 36^2$, $2916 = 54^2$, $9216 = 96^2$, $9261 = 21^3$, $1089 = 33^2$, $9801 = 99^2$.
$1024 =32^2 = 2^{10}$, $2401 = 49^2 = 7^4$, $4761= 69^2$, $1764= 42^2$, $1936 = 44^2$, $1369 = 37^2$
$178^2 = 31684$, $191^2 = 36481$, $196^2 =38416 =14^4$, $209^2 = 43681$.

For m = 3

$125= 5^3$, $512 = 8^3$, $331^3 = 36264691$, $406^3 = 66923416$.

For m = 4

$256 = 4^4$, $625 = 5^4$.

Open problem: Are there numbers m and n such that the digits of m^k are a permutation of the digits of n^k for all $k > 1$?

Section 6

The Largest and Smallest m^{th} Power Whose Digit Sum/Product Is Its' m^{th} Root

Introduction: While studying the Smarandache additive square sequence [7] (sequence of squares in which the digits also sum to a square), a problem occurred to me. Are there perfect squares whose digit sum is the same as the square root? If so, then there must be a smallest and a largest such number. A similar question can be asked for higher powers. In this section that question is examined.

We will refer to numbers that are perfect powers of the sum of their digits Smarandache Anurag numbers. That such numbers exist can be seen from $9^2 = 81$, and the sum of the digits of 81 = 9. For purposes of this analysis, we will consider one to be a trivial solution and ignore it. We will show that 81 is the only nontrivial perfect square with this property and we will call it the Smarandache Shikha number for two. By default, 81 is the largest perfect square whose digits sum to the square root. It is also the smallest such number, so it will also be the Smarandache Anirudh number for 2.

We will now prove that 81 is the only Smarandache Anurag number. As there is no perfect square number with this property smaller than $1296 = 36^2$, and 36 being the largest possible sum of a 4 digit number it is evident that 81 is the only such perfect square.

Moving on to the cubes, we have

$8^3 = 512 = (5+1+2)^3$, $17^3 = 4913 = (4+9+1+3)^3$, $18^3 = 5832 = (5+8+3+2)^3$, $26^3 = 17576 = (1+7+5+7+6)^3$ and $27^3 = 19683 = (1+9+6+8+3)^3$.

As $45^3 = 91125$ (a five digit number) and 45 is the largest possible sum of a 5-digit number, it is evident that 19683 is the largest cube with this property and is the Smarandache Shikha number for 3. Let the smallest such number (8) be called the Smarandache Anirudh number for 3.

Considering the fourth powers we have

$7^4 = 2401 = (2+4+1)^4$, $22^4 = 234256$, $25^4 = 390625$, $28^4 = 614656$, and $36^4 = 1679616$.

There is no number between 36^4 and 72^4 with $72^4 = 26873856$ (an eight digit number). Since 72 is the largest possible sum of an 8- digit number, $36^4 = 1679616$ is the Smarandache Shikha number for 4 and 7 is the Smarandache Anirudh number for 4.

Considering the fifth powers we have

$28^5 = 17210368$, $35^5 = 52521875$, $36^5 = 60466176$, and $46^5 = 205962976$.

It is to be noted that only numbers with final sum of digits 1, 8 or 9 qualify to be Smarandache Anurag numbers for five.

Conjecture: For every m > 2, there exists at least two Smarandache Anurag numbers. In other words, the Smarandache Shikha and Smarandache Anirudh numbers are distinct.

Conjecture: The total number of Smarandache Anurag numbers for the m^{th} powers are more that that for the $(m+1)^{th}$ power.

Definition: A number (where no digit is zero) divisible by the product of its' digits is called a Smarandache Meenakshi Number. The sequence obtained by applying this property is called the Smarandache Meenakshi Sequence.

The first few terms are

1, 2, . . . ,9, 12, 15, 24, 36, . . .

Let Pd(n) denote the product of the digits of N. With this notation, we have

Pd(36) = 18, Pd(144) = 9
$2916 = 54^2$. Pd(2916) = 2*9*1*6 = 108, 2916/108 = 27
$248832 = 12^5$, Pd(248832) = 3072 = 248832/3072 = 81
$429981696 = 144^6$, Pd(429981696) = 1679616 , 429981696/1679616 = 256.

Proposition: There are infinitely many terms in the Smarandache Meenakshi Sequence.

Proof:
Let $N = (10^n - 1)/9$. It is clear that Pd(N) = 1*1 . . . *1 = 1 and is an element of the Smarandache Meenakshi Sequence. These numbers will be considered trivial solutions.

Definition: The following will be considered semi-trivial elements of the Smarandache Meenakshi Sequence.

12, 112, 1112, 11112, . . .
15, 115, 1115, 11115, . . .

Additional sequences are

1113, 1111113, 1111111113, . . .

where $T_n = 10 * (10^{3n} - 1)/9 + 3$,

1111117, 1111111111117, 1111111111111111117, . . .

where $T_n = 10 * (10^{6n} - 1)/9 + 7$.

A proof that 7 divides each element of the last sequence is given in [7]. All other terms of the Smarandache Meenakshi Sequence will be considered non-trivial.

Conjecture: There are infinitely many non-trivial terms in the Smarandache Meenakshi Sequence.

Open problem: Is there any m^{th} power whose m^{th} root equals the product of its' digits? In other words, are there solutions to the equation $[Pd(N)]^m = N$?

Note: If N is a solution to $[Pd(N)]^m = N$ then it is evident that N takes the canonical form , $N = 2^a * 3^b * 5^c * 7^d$, where a, b , c , d are non negative integers.

Section 7

A Conjecture on d(N), the Divisor Function Itself As A Divisor with Required Justification

Introduction: The number of divisors of a natural number varies quite irregularly as N increases. It is known that $d(N) < 2N^{1/2}$, but there is no definite relationship as d(N) depends on the indices of the canonical form rather than the value of N. In this section, the condition where d(N) divides N is considered.

Definition: N is a Smarandache beautiful number if d(N) divides N. If d(N) divides N, then I = N/d(N) is the Index of Beauty of N.

Conjecture: For every number I, there is a number N such that I is the index of beauty of N.

The conjecture will be established for several cases.

Theorem: If p is prime, then there is a number N such that p = N/d(N).

Proof:
For I = 2, we have N = 8, d(N) = 4, N/d(N) = 2 = I.
For I = 3, we have N = 9, d(N) = 3, N/d(N) = 3 = I.
For $I = p \geq 5$ a prime, choose $N = 12p = 2^2 3p$. Then, d(N) = 12, N/d(N) = p = I.

(N = 8p could also have been used when p > 2.)

Theorem: If I is the index of beauty of M and if

$I = n_1 * n_2 * \ldots n_r$

is the Smarandache factor partition (a breakup of I as the product of its divisors), then

$J = p_1^{(n1-1)} * p_2^{(n2-1)} * p_r^{(nr-1)}$ is the index of beauty of M*J when (M,J) = 1.

Proof: Let N = M*J (1), then d(N) = d(M) * d(J) as (M,J) = 1.
$d(N) = d(M) * n_1 * n_2 * \ldots n_r$
d(N) = d(M) * I
d(N) = M (2), as I is the index of beauty for M. From (1) and (2) we get N/d(N) = J.

Therefore, J is the index of beauty of M*J.

Corollary: If I is the index of beauty of M then p^{I-1} is the index of beauty for $M*p^{I-1}$ if (M,p) = 1.

The following results provide the motivation for this conjecture.

Definition: For convenience we use the symbol p[r] for $p_1 * p_2 * p_3 * p_4$, the product of four primes.

When examining the contents of the following table, this list of points should be kept in mind.

a) All p's and q's are primes.
b) If $N = M * p^r$ then $(M,p) = 1$.
c) If $N = M * p[r]$, then $(M, p[r]) = 1$.

S.N.	I	N	d(N)
1	p^2	$24p^2$, $18p^2$, $9p^2$	24, 18, 9
2	p^3	$36p^3$	36
3	p^4	$40p^4$, $60p^4$	40, 60
4	p^5	$72p^5$	72
5	p^6	$84p^6$	84
6	p^7	$96p^7$, $80p^7$	96 , 80
7	p^8	$108p^8$	108
8	p^9	$180p^9$	180
9	p^{10}	$132p^{10}$	132
10	p^{11}	$240p^{11}$	240
11	p[1]	8p, 12p	8, 12
12	p_1p_2 = p[2]	36 p[2]	36
13	p[3]	96 p[3]	96
14	p[4]	$2^5 3^2$ p[4]	$2^5 3^2$
15	p[5]	2^6 7 p[5]	2^6 7
16	p[7]	2^9 5 p[7]	2^9 5
17	p[8]	2^{11} 3 p[8]	2^{11} 3
18	p[11]	2^{15}p[11], 2^{12}.13 p[11] , 2^{13} 7 p[11]	2^{15} , 2^{12}.13 , 2^{13} .7
19	p[12]	2^{14} 3*5 p[12]	2^{14} 3*5
20	p[13]	2^{14} 5* 3^2 p[13]	2^{14} 5* 3^2
21	p[16]	2^{19} 5 p[16]	2^{19} 5
22	p[20]	2^{21} 11 p[20]	2^{21} 11
23	p[23]	$2^{24}5^2$ 3 p[23]	$2^{24}5^2$ 3
24	p[p'-2]	$2^{p'-1}$*p'*p[p'-2]	$2^{p'-1}$*p'

25	$p[p^2-2]$	$2^{p2\ -1}\ p^2\ 3\ p[p^2-2]$	$2^{p2\ -1}\ p^2\ 3$
26	$p[p^3-3]$	$2^{p3-1} * p^3\ p[p^3-3]$	$2^{p3-1} * p^3$
27	$p[p^4-2]$	$2^{p4-1}\ p^4\ 5\ p[p^4-2]$	$2^{p4-1}\ p^4\ 5$
28	p[2p-3]	2^{2p-1} *p* p[2p-3]	2^{2p-1} *p

In the rows of the preceding table, I is the index of beauty for the corresponding structure of N.

Though the conjecture can be established/ verified for a number of canonical forms, the proof for the general case will most likely give mathematicians many more sleepless nights. Along similar lines, the following conjectures are put forward.

Conjecture: For every positive integer k there exists a number N such that N / S (N) = k, where S(N) is the Smarandache function.

Examples:
For k =2 , N = 6 , 8; for k = 3, N = 12; for k = 4, N = 20; for k =5, N = 50.

Conjecture: For every positive integer k there exists a number N such that N / φ(N) = k. (Euler's function.)

Conjecture: For every positive integer k, there exists a number N such that σ (N) /N = k. (For k = 2 we have N, a perfect number).

Section 8

Smarandache Fitorial and Supplementary Fitorial Functions

The Smarandache Fitorial Function, denoted by FI (N) is defined as the product of all the ϕ (N) numbers relatively prime to and less than N.

Examples:

FI(6) = 1*5 = 5 , FI(7) = 6! = 720, FI(12) = 1*5*7*11 =385.

The Smarandache Supplementary Fitorial Function, denoted by SFI (N) is defined as the product of all the remaining N - ϕ (N) numbers less than or equal to N which are not relatively prime to N.

Examples:

SFI (6) = 2*3*4*6 = 144, SFI (7) = 7, SFI(11) = 11,
SFI(12) = 2*3*4*6*8*9*10*12 = 1244160 .

Theorem: FI(N) and SFI(N) satisfy the following properties:

1. FI(N) * SFI(N) = N!.
2. SFI(p) = p , and FI(p) = (p-1)! iff p is a prime.
3. FI(N) < $(N/2)^{\varphi(N)}$

Proof:

1. This formula follows from the definition of the functions.
2. If p is prime, then every number less than p is relatively prime to p. Therefore, FI(p) = (p-1)! And SFI(p) = p. If p is not prime, then there is at least one number k less than p that is not relatively prime to p. Therefore, FI(p) < (p-1)! and SFI(p) > p.
3. The sum of the $\phi(N)$ numbers relatively prime to N is given by $(N\phi(N))/2$. Therefore, the arithmetic mean (A. M.) of the numbers in the sum is N/2. Their geometric mean is given by

$$G.\ M. = \{FI(N)\}^{1/\phi(N)}$$

Using the relationship between these means when the numbers are not all equal (A. M. > G. M.), we have

$$\{FI(N)\}^{1/\phi(N)} < N/2$$

and taking both sides to the $\phi(N)$ power

$$\{FI(N)\} < (N/2)^{\phi(N)} \quad \text{--------------} \ (1)$$

4. $N! * (N/2)^{-\phi(N)} < SFI(N) <$

$$(N/2)^{N-\phi(N)} * \{1 + 1/(N - \phi(N)\}^{N-\phi(N)} \quad \text{-------}(A)$$

Proof:

We have the sum of the numbers not relatively prime to N, $S_2 = \sum N - N\phi(N)/2$

$= N(N+1)/2 - N\phi(N)/2$.

Hence the Arithmetic Mean of the numbers relatively prime to N is

$S_2/(N - \phi(N)) = (N/2)*(1 + 1/(N - \phi(N)))$ ----------- (2).

Their Geometric Mean is given by

$G_2 = SFI(N)^{(N - \phi(N)}$ -------------- (3).

From (2) and (3) we have $S_2 > G_2$ and finally

$SFI(N) < (N/2)^{N - \phi(N)} * \{ 1 + 1/ (N - \phi(N)\}^{N - \phi(N)}$ ------ (4)

Also by definition we have FI(N)*SFI(N) = N!, hence SFI(N) = N! / FI(N) and from (1) we get

$N! * (N/2)^{-\phi(N)} < SFI(N)$ ---------------- (5).

Combining (4) and (5) we get (A).

The following result is a direct consequence of formula (A).

5. $SFI(N) < (N/2)^{N - \phi(N)} * e$ (where e is base of natural logarithm 2.71828. . .)

Proposition 1:

Applying properties (3) and (5) we get the formulas

$N! = FI(N)*SFI(N) < (N/2)^N *e$ or

$N! < (N/2)^N *e$ ------ (B).

Proposition 2: For large values of n, $SFI(2^n)/FI(2^n) \approx (\pi/2)^{1/2}$ ------- (C)

Justification:

If $N = 2^n$ then $FI(N) = 1*3*5*7* . . .(2^n -3)*(2^n -1)$. And

$SFI(N) = 2*4*6*. . .* (2^n -2)*(2^n)$

Then $\{ SFI(N)/FI(N)\}^2 = \{2*2*4*4*6*6*8*8. . . * (2^n -2)* (2^n -2)*(2^n)* (2^n) \}/ \{ 1*3*3*5*5*7*7. . .* . (2^n -3)*(2^n -3) *(2^n -1)*(2^n -1)\}$.

From the well-known result by John Wallis on the value of π as an infinite product we have

$$\pi/2 = \{2*2*4*4*6*6*8*. . .\}/ \{1*3*3*5*5*7*7. . .\} \quad \text{----------- (D)}$$

and formula (C) is a direct consequence.

Note: In formula (D), for an approximation close to π/2 the number of terms taken in the product in the numerator should be exactly the same as that used in the denominator.

The following open problems and conjectures are proposed.

Open Problem-I: For what values of N is, FI(N) < SFI(N)?

Open Problem-II: For what values of N is, FI(N) > SFI(N)?

Open Problem-III: If d(n) is the number of divisors, for what values of N is d(FI(N)) > d(SFI(N))?

Open Problem-IV: For what values of N is d(FI(N)) < d(SFI(N))?

Open Problem-V: For what values of N is σ(FI(N)) < σ(SFI(N))?

Open Problem-VI: For what values of N is $\sigma(FI(N)) > \sigma(SFI(N))$?

Define the sum of the ϕ (N) relatively prime numbers as A(N) and that of the remaining N - ϕ (N), numbers as B(N).

Then we have

$A(N) = N\phi(N)/2$

$B(N) = N(N+1)/2 - N\phi(N)/2$.

For example, if N= 12, A(12) = 24 , and B(N) = 54 and

for N = 15 , A(15) = 60 , and B(N) = 60.

Note: A(N) = B(N) , iff $\phi(N) = (N+1)/2$, provided that N is odd.

Readers are encouraged to explore this further.

Open Problem: For what values of N, is it true that A(N)) > B(N))?

The case where $N = 3^n$ is considered below.

For $N= 3^n$ we get, $A(N) = 3^n*\{3^n*(1- 1/3)\}/2 = 3^{2n-1}$

$B(N) = 3^n*\{3^n +1\}/2 - 3^{2n-1} = \{3^{2n} + 3^n - 2*3^{2n-1}\}/2 = \{3^{2n-1} +3^n\}/2$

For n=1 $3^n = 3^{2n-1}$, for n>1 , 2n-1 > n and $3^{2n-1} > 3^n$, hence

$B(N) < \{3^{2n-1} + 3^{2n-1}\}/2 = 3^{2n-1} = A(N)$.

Open Problem: For what values of N is it true that A(N)) < B(N))?

The case where $N=2^n$ is considered below.

For $N = 2^n$ we have, $A(N) = 2^n \{ 2^n *(1- ½)\}/2 = 2^{2n-2}$

and $B(N) = 2^n\{ 2^n +1 \}/2 - 2^{2n-2} = 2^{2n-2} + 2^{n-1} = A(N) + 2^{n-1}$.

Therefore, we have B(N) > A(N).

Section 9

Some More Conjectures On Primes and Divisors

There are an innumerable number of conjectures and unsolved problems in number theory based on prime numbers, which have been giving mathematicians sleepless nights all over the world for centuries. Here are a few more to add to their troubles:

(1) Every even number can be expressed as the difference of two primes.

(2) Every even number can be expressed as the difference of two consecutive primes. i.e. for every m there exists an n such that $2m = p_{n+1} – p_n$, where p_n is the nth prime.

(3) Every number can be expressed as N / d(N) , for some N, where d(N) is the number of divisors of N.

If d(N) divides N, we define N / d(N) = I as the index of beauty for N.

Conjecture: For every natural number M there exists a number N such that M is the index of beauty for N. In other words M = N/d(N) .

The conjecture is true for primes, which is easily proven.

We have 2 = 12 / d(12) = 12/ 6 , 2 is the index of beauty for 12.

3 = 9 /d(9) = 9 / 3 , 3 is the index of beauty for 9.

For a prime p >3 we have N = 12p, d(N) = 12 and N /d(N) = p .

(N= 8p can also be used).

The conjecture is true for a large number of families of numbers. However the proof of the general case is still unsolved.

(4) If p is a prime, then there are infinitely many primes of the form

(A) $2^n p + 1$.
(B) $2 * a^n p + 1$.

(5) It is a well-known fact that one can have arbitrarily large numbers of consecutive composite numbers. For example for any value of r:

(r+1)! +2, (r+1)! +3, (r+1)! +4, . . .(r+1)! + r-1 , (r+1)! + r

is a list of r consecutive composite numbers.

But this is not necessarily the smallest set of such numbers. Let us consider the smallest set of r consecutive composite numbers for the first few values of r.

r	**Smallest set of composite numbers**	**R/first composite number**
1	1	1/1
2	8,9	2/8
3	14, 15, 16	3/14
4	24, 25, 26, 27	4/24
5	24, 25, 26, 27, 28	5/24
6	90, 91, 92, 93, 94, 95	6/90
7	90, 91, 92, 93, 94, 95, 96	7/90
8	114, 115, … 121	8/114

Similarly for 9, 10, 11, 12, 13 the first of the composite numbers is 114.

Conjecture: The sum of the ratios in the third column is finite and > e.

(6) Given a number N, carry out the following steps to get a number N_1.

$N - p_{r1} = N_1$, where $p_{r1} < N < p_{r1} + 1$, p_{r1} is the r1th prime.

Repeat the previous step to get N_2.

$N_1 - p_{r2} = N_2, \quad p_{r2} < N_1 < p_{r2+1}$.

Repeat these steps until the result is $N_k = 0$ or 1.

The conjecture is

(a) For all N, $k < \log2 \log2 N$.

(b) There exists a constant C such that $k < C$.

Open Problem: If (b) is true, find the value of C.

Section 10

Smarandache Reciprocal Function and An Elementary Inequality

Definition: The Smarandache Reciprocal Function $S_c(n)$ is defined in the following way:

$S_c(n) = x$, where $x + 1$ does not divide $n!$ and for every $y < x$, $y \mid n!$.

Theorem: If $S_c(n) = x$ and $n \neq 3$, then $x+1$ is the smallest prime greater than n.

Proof: It is obvious that n! is divisible by all numbers 1, 2, 3, …, n. To prove the theorem, it is necessary to show that n! is also divisible by n+1, n+2, . . . , n+m, where n+m+1 is the smallest prime greater than n.

Let r be any of the composite numbers from n+1 through n+m. Since r is not prime, it must be possible to factor it into two factors, each of which is ≥ 2. Let $r = p*q$ be that factorization. If one of the factors (say q), is ≥ n, then $r = p*q \geq 2n$. But, according to Bertrand's postulate, there must be a prime between n and 2n, which yields a contradiction of the assumption that all of the numbers n+1 through n+m are composite. Therefore, each of the factors must be less than n.

There are two possibilities:

Case 1: $p \neq q$. In this case, each factor is less than n so $p*q = r$ divides n!.

Case 2: $p = q$ where they are prime. This means that $r = p^2$ and there are three subcases.

Subcase 1: $p = 2$. Then $r = 4$ and n must be 3. Since 4 does not divide 3!, we have the special case eliminated in the statement of the theorem.

Subcase 2: $p = 3$. Then $r = 9$ and n must be 7 or 8 and 9 divides both of these numbers.

Subcase 3: $p \geq 5$. Then $r = p^2 > 4p \Rightarrow 4p < r < 2n \Rightarrow 2p < n$. Therefore, both p and 2p are less than n, so p^2 divides n!.

Remark: n! is divisible by all composite numbers between n and 2n.

Note: It is well known that $S(n) \leq n$ for S(n) the Smarandache function.

From the previous theorem, it is possible to deduce the following inequality.

If p_r is the rth prime and $p_r \leq n < p_{r+1}$, then $S(n) \leq p_r$, which is a slight improvement on $S(n) \leq n$.

Writing out the sequence

$S(p_r) = p_r$, $S(p_r+1) < p_r$, $S(p_r+2) < p_r$, . . . , $S(p_{r+1} - 1) < p_r$, S $S(p_{r+1}) = p_{r+1}$.

Creating the sum for all number $p_r \leq n < p_{r+1}$, one gets

$$\sum_{t=0}^{p_{r+1} - p_r - 1} S(p_r + t) \leq (p_{r+1} - p_r)p_r$$

Summing up for all the numbers up to the sth prime, where $p_0 = 1$, we get

$$\sum_{k=1}^{p_s} S(k) \leq \sum_{r=0}^{s} (p_{r+1} - p_r)\, p_r \quad (1)$$

Smarandache (Inferior) Prime Part Sequence

For any positive real number n, one can define $p_p(n)$ as the largest prime number less than or equal to n. In [9] Prof. Krassimir T. Atanassov proves that the value of the nth partial sum of the sequence

$$X_n = \sum_{k=1}^{n} p_p(k)$$

is given by

$$X_n = \sum_{k=2}^{\pi(n)} (p_k - p_{k-1})*p_{k-1} + (n - p_{\pi(n)} + 1)*p_{\pi(n)} \quad (2)$$

From (1) and (2), we have

$$\sum_{k=1}^{n} S(k) \le X_n.$$

Section 11

Smarandache Maximum Reciprocal Representation Function

Definition: The Smarandache Maximum Reciprocal Representation Function (SMRR) is defined as follows:

$f_{SMRR}(n) = t$ if

$$\sum_{r=1}^{t} 1/r \quad \le n \le \sum_{r=1}^{t+1} 1/r$$

Definition: The Smarandache Maximum Reciprocal Representation Sequence (SMRRS) is defined as $T_n = f_{SMRR}(n)$

$f_{SMRR}(1) = 1$
$f_{SMRR}(2) = 3$, $(1 + 1/2 + 1/3 < 2 < 1 + 1/2 + 1/3 + ¼)$
$f_{SMRR}(3) = 10$ $\quad \sum_{r=1}^{10} 1/r \le 3 \le \sum_{r=1}^{11} 1/r$

The sequence of numbers is 1, 3, 10 , …

Note: The harmonic series $\sum 1/n$ satisfies the inequality

$\log(n+1) < \sum 1/n < \log n + 1 \quad (1).$

This inequality can be derived in the following way:

$e^x > 1+x$, $x > 0$, $\quad (1 + 1/n)^{(1+1/n)} > 1$, $n > 0$,

which gives

$1/(r+1) < \log(1 + 1/r) < 1/r$.

Summing up for r = 1 to n+1 and applying some algebraic rearrangement yields (1).

Applying (1), we get the following result for the SMRR function.

If SMRR(n) = m, then $[\log(m)] \approx n-1$, where $[x]$ is the integer value of x.

Conjectures:

1) Every positive integer can be expressed as the sum of the reciprocal of a finite number of distinct natural numbers in infinitely many ways.
2) Every natural number can be expressed as the sum of the reciprocals of a set of natural numbers in arithmetic progression.
3) Let $\sum 1/r \le n \le 1/(r+1)$ where $\sum 1/r$ is the sum of the reciprocals of the first r natural numbers. Let

$S_1 = \sum 1/r$
$S_2 = S_1 + 1/(r + k_1)$ such that $S_2 + 1/(r + k_1 + 1) > n \ge S_2$.
$S_3 = S_2 + 1/(r+k_2)$ such that $S_3 + 1/(r+k_2 + 1) > n \ge S_3$.
And so on.

Then, there exists a finite m such that

$S_{m+1} + 1/(r + k_m) = n$.

Remarks:

a) There are infinitely many disjoint sets of natural numbers the sum of whose reciprocals is unity.
b) Among the sets mentioned in (a), there are sets which can be organized in an order such that the largest element of any set is smaller than the smallest element of the next set.

Section 12

Smarandache Determinant Sequences

Definition: The Smarandache Cyclic Determinant Natural Sequence is defined as the determinants of the following sequence of matrices

$$\begin{vmatrix} 1 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$ and so on.

The determinants of the first four matrices are

1, -3, -18, and 160.

These initial values suggest the following general formula

$T_n = (-1)^{[n/2]} \{(n+1)/2\} * n^{n-1}$ where [x] is the integer part of x.

This formula will be verified for the case where n = 5, which will demonstrate how the general case is handled.

$$T_5 = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{vmatrix}$$

By carrying out the following elementary row operations

a) R_1 = sum of all the rows.
b) Taking 15 from the first row.
c) Replacing C_k, the kth column by $C_k - C_1$, we have

$$=15 \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 3 & -1 \\ 3 & 1 & 2 & -2 & -1 \\ 4 & 1 & -3 & -2 & -1 \\ 5 & -4 & -3 & -2 & -1 \end{vmatrix} \qquad = 15 \begin{vmatrix} 1 & 2 & 3 & -1 \\ 1 & 2 & -2 & -1 \\ 1 & -3 & -2 & -1 \\ -4 & -3 & -2 & -1 \end{vmatrix}$$

$R_1 - R_2$, $R_3 - R_2$, $R_4 - R_2$,

$$15 \begin{vmatrix} 0 & 0 & 5 & 0 \\ 1 & 2 & -2 & -1 \\ 0 & -5 & 0 & 0 \\ -5 & -5 & 0 & 0 \end{vmatrix}$$ = 1875, the value suggested by the formula.

Although the proof of the general case is clumsy, it is based on similar lines.

Generalization:
This sequence can be further generalized by using an arithmetic progression with a the first term and common difference d. We will define the Smarandache Cyclic Arithmetic determinant sequence in the following way:

$$\begin{vmatrix} a \end{vmatrix} \quad \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix} \quad \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a \\ a+2d & a & a+d \end{vmatrix} \quad \text{and so on.}$$

Conjecture:

$T_n = (-1)^{[n/2]} * S_n * d^{n-1} * n^{n-2} = (-1)^{[n/2]} * \{a + (n-1)d\}*(1/2)*(nd)^{n-1}$

where S_n is the sum of the first n terms of the arithmetic progression.

Open problem: Develop a formula for the sum of the first n terms of this sequence.

Definition: The Smarandache bisymmetric determinant natural sequence is the determinants of the following sequence of matrices.

$$\begin{vmatrix} 1 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

where the matrices are symmetric across both main diagonals.

The determinants of these matrices are

1, -2, -12, 40, . . .

The values of these first few terms suggests that the general formula is

$T_n = (-1)^{[n/2]} (n(n+1))*2^{n-3}$

We will verify that this formula also holds for n =5 and the general case can be dealt with using a similar sequence of operations.

$$T_5 = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 4 \\ 3 & 4 & 5 & 4 & 3 \\ 4 & 5 & 4 & 3 & 2 \\ 5 & 4 & 3 & 2 & 1 \end{vmatrix}$$

Carrying out the following sequence of row operations

a) R_1 = sum of all the rows.
b) Take 15 from the first row to get

$$15\begin{vmatrix} 1 & 2 & 3 & 2 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & -2 \\ -1 & -2 & -3 & -4 \end{vmatrix}$$

$R_1 = R_1 + R_4$ gives

$$15\begin{vmatrix} 0 & 0 & 0 & -2 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & -2 \\ -1 & -2 & -3 & -4 \end{vmatrix} = 120,$$ which is the value predicted from the suggested general formula.

The proof of the general case is based on similar operations.

Generalization: This sequence of determinants can be also be generalized using the elements of an arithmetic progression.

$$\begin{vmatrix} a \end{vmatrix} \quad \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix} \quad \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a+d \\ a+2d & a+d & a \end{vmatrix}$$

Conjecture: The general term of this sequence of determinants is given by

$T_n = (-1)^{[n/2]} * (a + (n+1)d)*2^{n-3}d^{n-1}$.

Section 13

Expansion of x^n in Smarandache Terms of Permutations

Definition: Given the following expansion of x^n

$x^n = b_{(n,1)}x + b_{(n,2)}x(x-1) + \ldots + b_{(n,n)}{}^{x}P_n$

we define $b_{(n,r)}x(x-1)(x-2)\ldots(x-r+1)(x-r)$ as the rth Smarandache term in the expansion.

In this section, we will examine some of the properties of the coefficients and encounter some fascinating coincidences.

We will start by examining the values of some terms for specific values of x.

For x = 1, $b_{(n,1)} = 1$.

For x = 2, $b_{(n,2)} = (2^n - 2)/2$.

For x = 3, $b_{(n,3)} = [3^n - 3 - 6(2^n - 2)/2]/6$

$$= (1/3!)*(1*3^n - 3*2^n + 3*1^n - 1*0^n).$$

For x = 4, $b_{(n,4)} = (1/4!)*[1*4^n - 4*3^n + 6*2^n - 4*1^n + 1*0^n]$.

These initial values suggest the following theorem.

Theorem:

$$b_{(n,r)} = (1/r!) \sum_{k=1}^{r} (-1)^{r-k} * {}^rC_k * k^n = a_{(n,r)}.$$

First proof:
The first step is to prove the proposition

$$b_{(n+1,r)} = b_{(n,r-1)} + r*b_{(n,r)}.$$

Starting with

$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) +. . .+ b_{(n,n)}\, {}^xP_n$

replacing x with r, we have

$r^n = b_{(n,1)}\, r + b_{(n,2)}\, r(r-1) + b_{(n,3)}\, r(r-1)(r-2) +. . .+ b_{(n,n)}\, {}^rP_n$.

Multiplying both sides by r

$r^{n+1} = b_{(n,1)}\, r*r + b_{(n,2)}\, r*r(r-1) + b_{(n,3)}\, r*r(r-1)(r-2) +. . .+ b_{(n,r)}\, r* {}^rP_r$ +

terms equal to zero.

Using slightly different notation, the expression is equivalent to

$r^{n+1} = b_{(n,1)}\, r* {}^rP_1 + b_{(n,2)}\, r * {}^rP_2 + b_{(n,3)}\, r * {}^rP_3 +. . .+ b_{(n,r)}\, r* {}^rP_r$.

Using the identity $r*{}^rP_{k+1} + k* {}^rP_k$, the expression can be rewritten as

$r^{n+1} = b_{(n,1)}\, \{{}^rP_2 + 1 * {}^rP_1\} + b_{(n,2)}\, \{{}^rP_3 + 2 * {}^rP_2\} +. . .+$

$\quad b_{(n,r)}\, \{ {}^rP_r + r *{}^rP_{r-1} \}$

$r^{n+1} = b_{(n,1)}\, {}^rP_1 + \{ b_{(n,1)} + 2 * b_{(n,2)}\}\, {}^rP_2 + \{ b_{(n,2)} + 3 * b_{(n,3)}\}\, {}^rP_3 + . . . +$

$\{ b_{(n,r-1)} + r * b_{(n,r)}\}\, {}^rP_r$

$r^{n+1} = b_{(n+1,1)}\, {}^rP_1 + b_{(n+1,2)} * {}^rP_2 + b_{(n+1,3)} * {}^rP_3 +. . .+ b_{(n+1,r)} * {}^rP_r$.

The coefficients of ${}^rP_t$ $(t < r)$ are independent of r, so they can be equated separately, giving

$$b_{(n+1,r)} = b_{(n,r-1)} + r * b_{(n,r)} .$$

We will proceed by induction. Let

$$b_{(n,r)} = (1/r!) \sum_{k=0}^{r} (-1)^{r-k} * {}^rC_k * k^n$$

$$b_{(n,r-1)} = (1/(r-1)!) \sum_{k=0}^{r-1} (-1)^{r-1-k} * {}^{r-1}C_k * k^n$$

be true as the inductive hypothesis. Then, the sum $b_{(n,r-1)} + r*b_{(n,r)}$ equals

$$(1/(r-1)!) \sum_{k=0}^{r-1} (-1)^{r-1-k} * {}^{r-1}C_k * k^n + r * (1/r!) \sum_{k=0}^{r} (-1)^{r-k} * {}^rC_k * k^n$$

$$= \quad ((-1)^{r-1}/r!) \left[\sum_{k=0}^{r-1} (-1)^{-k}\, r \{ {}^{r-1}C_k - {}^rC_k\} k^n \right] + r^{n+1}/r!$$

$$= \quad ((-1)^{r-1}/r!) \left[\sum_{k=0}^{r-1} (-1)^{-k} \{ -k * {}^rC_k\} k^n \right] + r^{n+1}/r!$$

$$= (1/r!) \sum_{k=0}^{r-1} (-1)^{r-k}\ {}^rC_k\, k^{n+1} .$$

This gives us

$$b_{(n+1,r)} = (1/r!) \sum^{r-1} (-1)^{r-k}\ {}^rC_k\, k^{n+1}$$

$k=0$

$b_{(n+1,r)}$ also takes the same form. Hence by induction, the proof is complete.

Second proof: This proof is based on a combinatorial approach.

If n objects, where no two are alike, are to be distributed in x boxes, no two alike, and each box can contain an arbitrary number of objects, the number of ways this can be done is x^n, since there are n alternatives for disposals of the first object, n alternatives for the disposal of the second, and so on.

Alternately, let us use a different approach. Consider the number of distributions in which exactly n objects are to be placed in a given set of r boxes (the rest are empty). Let the number of distributions be represented by f(n,r).

We derive a formula for f(n,r) by using the inclusion/exclusion principle. The method is illustrated by the computation of f(n,5). Consider the total number of arrangements, 5^n of n objects in 5 boxes. Say that such an arrangement has property 'a'. In case the first box is empty, property 'b' in case the second box is empty, and similar property 'c', 'd', and 'e' for the other three boxes respectively. To find the number of distributions with no box empty, we simply count the number of distributions having none of the properties 'a', 'b', 'c', . . .etc.

We can apply the formula

$$N - {}^rC_1.N(a) + {}^rC_2.N(a,b) - {}^rC_3.N(a,b,c) + \ldots$$

Here, $N = 5^n$ is the total number of distributions. By N(a), we mean the number of distributions with the first box empty, so $N(a) = 4^n$. Similarly, N(a,b) is the number of distributions where the first two boxes are empty. However, this is the same as the number of distributions into 3 boxes and $N(a,b) = 3^n$. Thus, we can write

$$N = 5^n,\ N(a) = 4^n,\ N(a,b) = 3^n \text{ etc. } N(a,b,c,d,e) = 0.$$

Applying the previous formula, we get

$$f(n,5) = 5^n - {}^5C_1.4^n + {}^5C_2.3^n - {}^5C_3.2^n + {}^5C_4.1^n - {}^5C_5.0^n.$$

By generalizing this and replacing 5 with r, we have

$$f(n,r) = r^n - {}^rC_1 * (r-1)^n + {}^rC_2 * (r-2)^n - {}^rC_3 * (r-3)^n + \ldots$$

$$f(n,r) = \sum_{k=0}^{r} (-1)^k\, {}^rC_k\, (r-k)^n$$

$f(n,r) = r! * a_{(n,r)}$, from theorem (3.1) of ref. [12].

Now, these r boxes out of the given x boxes can be chosen in ${}^xC_r$ ways. Hence, the total number of ways in which n distinct objects can be distributed into x distinct boxes, occupying exactly r of them (with x-r boxes empty), defined as d(n,r/x), is given by

$$d(n,r/x) = r! * a_{(n,r)}\ {}^xC_r$$

$$d(n,r/x) = a_{<n,r)} * {}^xP_r.$$

Summing up all the cases for r=0 to r=x, the total number of ways in which n distinct objects can be distributed in x distinct boxes is given by

$$\sum_{r=0}^{x} d(n,r/x)) = \sum_{r=0}^{x} {}^xP_r\ a_{(n,r)}$$

Equating the two results obtained by the two different approaches, we get

$$x^n = \sum_{r=0}^{n} {}^xP_r\ a_{(n,r)}.$$

Remarks: If n distinct objects are to be distributed in x distinct boxes with no box empty, then it is necessary for n < x. For example, 5 objects cannot be places in 7 boxes where no box is empty. Therefore, we get the following result

$$f(n,r) = 0, \text{ for } n < k.$$

$$f(n,r) = \sum_{k=0}^{r} (-1)^k\ {}^rC_k\ (r-k)^n = 0 \text{ if } n \geq r.$$

Further generalization:

1) The expansion of x^n can also be expressed in the form

$$x^n = g_{(n/k,1)}\ x + g_{(n/k,2)}\ x(x-k) + g_{(n/k,3)}\ x(x-k)(x-2k) +\ldots+$$

$$g_{(n/k,n)}\ x(x-k)(x-2k)\ldots(x-(n-1)k)(x-nk+k)$$

$$g_{(n/k,r)} = b(n,r) = a(n,r)$$

One can also look for interesting patterns when k = 2, 3, . . .

We can also define $g_{(n/k,r)}\, x(x-k)(x-2k)(x-(n-1)k)(x-rk+k)$ as the rth Smarandache term of the kth kind.

2) Another generalization could be

$$x^{T(n)} = c_{(n/k,1)}\,(x-k) + c_{(n/k,2)}\,(x-k)(x^2-k) + c_{(n/k,3)}\,(x-k)(x^2-k)(x^3-k) +\ldots + c_{(n/k,n)}\,(x-k)\,(x^2-k)(x^3-k)\ldots(x^n - k)$$

where $T(n) = n(n+1)/2$ is the n^{th} triangular number.
For k=1, if we denote $c_{(n/k,r)} = c_{<n,r)}$, for simplicity we get

$$x^{T(n)} = c_{(n,1)}\,(x-1) + c_{(n,2)}\,(x-1)(x^2-1) + c_{(n,3)}\,(x-1)(x^2-1)(x^3-1) +\ldots + c_{(n,n)}\,(x-1)\,(x^2-1)(x^3-1)\ldots(x^n - 1)$$

We can define

$$c_{(n/k,r)} *(x-k)\,(x^2-k)(x^3-k)\ldots(x^r - k)$$

as the rth Smarandache factorial term of the kth kind in the expansion of $x^{n!}$. One can also search for interesting patterns in the coefficients $c_{(n/k,r)}$.

Section 14

Miscellaneous Results and Theorems on Smarandache Terms and Factor Partitions

In section 8, $b_{(n,r)}\, x(x-1)(x-2)\ldots(x-r+1)(x-r)$ was defined as the rth Smarandache term in the expansion of

$$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) +\ldots+ b_{(n,n)}\, {}^xP_n.$$

In this section, we present some more results depicting how closely the coefficients of the Smarandache term and SFPs are related.

In the previous section, the formula

$$x^n = \sum_{r=0}^{n} {}^xP_r\, a_{(n,r)}$$

was proven. This leads to the beautiful formula

$$\sum_{k=1}^{x} k^n = \sum_{k=1}^{x} \sum_{r=1}^{k} {}^kP_r\, a_{(n,r)}.$$

Writing this in matrix notation where x = 4 = n, we have

$$\begin{pmatrix} {}^{1}P_1 & 0 & 0 & 0 \\ {}^{2}P_1 & {}^{2}P_2 & 0 & 0 \\ {}^{3}P_1 & {}^{3}P_2 & {}^{3}P_3 & 0 \\ {}^{4}P_1 & {}^{4}P_2 & {}^{4}P_3 & {}^{4}P_4 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1^1 & 1^2 & 1^3 & 1^4 \\ 2^1 & 2^2 & 2^3 & 2^4 \\ 3^1 & 3^2 & 3^3 & 3^4 \\ 4^1 & 4^2 & 4^3 & 4^4 \end{pmatrix}$$

In general,

$P * A' = Q$ where $P = \left({}^{i}P_j \right)_{n \times n}$.

$A = \left(a_{(I,j)} \right)_{nxn}$ and $Q = \left(i^j \right)_{nxn}$

(A' is the transpose of A).

Consider the expansion of x^n again

$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) + \ldots + b_{(n,n)}\, {}^{x}P_n$.

For x = 3, we have

$x^3 = b_{(3,1)}\, x + b_{(3,2)}\, x(x-1) + b_{(3,3)}\, x(x-1)(x-2)$.

Comparing the coefficients of the powers of x on both sides, we have

$b_{(3,1)} - b_{(3,2)} + 2\, b_{(3,3)} = 0$

$b_{(3,2)} - 3\, b_{(3,3)} = 0$

$b_{(3,3)} = 1$.

Expressing this in matrix form,

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ & & \end{pmatrix} * \begin{pmatrix} b_{(3,1)} \\ b_{(3,2)} \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \end{pmatrix}$$

0 0 1 $b_{(3,3)}$ 1

$C_3 * A_3 = B_3$

$A_3 = C_3^{-1} * B_3$

$$C_3^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(C_3^{-1})' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}.$$

Similarly, it has been observed that

$$(C_4^{-1})' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix}.$$

This observation leads to a theorem.

Theorem:
In the expansion of x^n as

$x^n = b_{(n,1)}\, x + b_{(n,2)}\, x(x-1) + b_{(n,3)}\, x(x-1)(x-2) +. . .+ b_{(n,n)}\, {}^{x}P_n$

if C_n is the coefficient matrix of equations obtained by equating the coefficient of powers of x on both sides, then

$$(C_n^{1})' = \underset{n \times n}{\left(a_{(I,j)} \right)} = \text{star matrix of order n.}$$

Proof: It is clear that C_{pq}, the element of the pth row and qth column of C_n is the coefficient of x^p in ${}^xP_q$. Also, C_{pq} is independent of n. The coefficient of x^p on the RHS is the coefficient of

$$x^p = \sum_{q=1}^{n} b_{(n,q)} C_{pq} .$$

The coefficient of $x^p = 1$ if p = n and is zero if $p \neq n$.

In matrix notation

$$\text{Coefficient of } x^p = \left(\sum_{q=1}^{n} b_{(n,q)} C_{pq} \right)$$

$$= \left(\sum_{q=1}^{n} b_{(n,q)} C'_{qp} \right)$$

$= i_{np}$ where $i_{np} = 1$, if $n = p$ and $i_{np} = 0$, if $n \neq p$.

$= I_n$ (identity matrix of order n.)

$$\left(b_{(n,q)} \right) \left(C_{p,q} \right)' = I_n$$

$$\left(a_{(n,q)} \right) \left(C_{p,q} \right)' = I_n \quad \text{as } b_{(n,q)} = a_{(n,q)}.$$

$$A_n * C_n' = I_n$$

$$A_n = I_n \left[C_n' \right]^{-1}$$

$$A_n = I_n \left(C_n' \right)^{-1}$$

This completes the proof of the theorem.

Theorem: If $C_{k,n}$ is the coefficient of x^k in the expansion of ${}^xP_n$, then

$$\sum_{k=1}^{n} F(1\#k)\, C_{k,n} = 1$$

Proof: The following property of the Smarandache Star Triangle (lower triangle of the matrix array $a_{(I,j)}$) can be established using the result of section 8 of chapter 1.

$$F'(1\#n) = \sum_{m=1}^{n} a_{(n,m)} = B_n .$$

In matrix notation for n = 4, it can be expressed in the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} .$$

In general,

$$\underset{1 \times n}{\begin{bmatrix} 1 \end{bmatrix}} \quad * \quad \underset{\substack{(c_n^{-1})' \\ n \times n}}{\begin{bmatrix} a_{(I,j)} \end{bmatrix}'} \quad = \quad \underset{1 \times n}{\begin{bmatrix} B_i \end{bmatrix}}$$

$$\underset{1 \times n}{\begin{bmatrix} B_i \end{bmatrix}} \quad * \quad \underset{\substack{(c_n) \\ n \times n}}{\begin{bmatrix} \quad \end{bmatrix}} \quad = \quad \underset{1 \times n}{\begin{bmatrix} 1 \end{bmatrix}} .$$

In $C_{n,n}$, $C_{p,q}$ the pth row and qth column is the coefficient of x^p in ${}^xP_q$. Hence we have

n n

$$\sum_{k=1} F(1\#k)\, C_{k,n} = 1 = \sum_{k=1} B_k\, C_{k,n}.$$

Theorem:

$$\sum_{k=1}^{n} F(1\#(k+1))\, C_{k,n} = n + 1 = \sum_{k=1}^{n} B_{k+1}\, C_{k,}$$

Proof: It has already been established that

$$B_{n+1} = \sum_{m=1}^{n} (m+1)\ a_{(n,m)}.$$

In matrix notation

$$\underset{1 \times n}{\left[\ j+1\ \right]} \ * \ \underset{\substack{n \times n \\ (c_n^{-1})}}{\left[a_{(I,j)} \right]}{}' = \underset{1 \times n}{\left[\ B_{j+1}\ \right]}\ .$$

$$\underset{1 \times n}{\left[\ j+1\ \right]} \ = \ \underset{1 \times n}{\left[B_{j+1} \right]} \ = \underset{\substack{n \times n \\ (c_n)}}{\left[\qquad \right]}.$$

$$\sum_{k=1}^{n} B_{k+1}\, C_{k,n} = n + 1.$$

There exists ample scope for deriving many more results.

Section 15

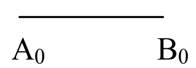
Smarandache-Murthy’s Figures of Periodic Symmetry of Rotation Specific to an Angle

Preliminary: Start with a given line segment. If we rotate the line segment about one end, starting with an angle x , and continue to do so with angles which are multiples of the initial angle, each time alternating the end about which the rotation is executed , we get fascinating figures that are unique to the value of x. In this section, we analyze these figures.

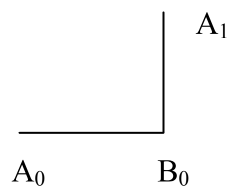
The figures in this section are constructed by applying the following sequence of steps.

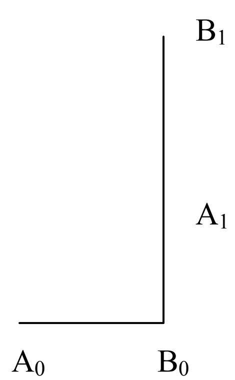
1) Begin with a line segment A_0B_0, and select an angle x = 360/n, where n is an integer.
2) From the end B_0, attach a new segment B_0A_1 of the same length by rotating x degrees in the clockwise direction. The points B_i and A_j alternate.
3) Repeat step 2 by rotating additional segments through the angles 2x, 3x, 4x, . . .

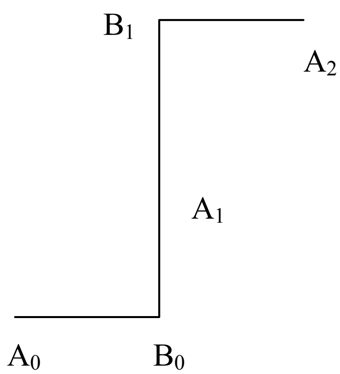
For example, start with the segment

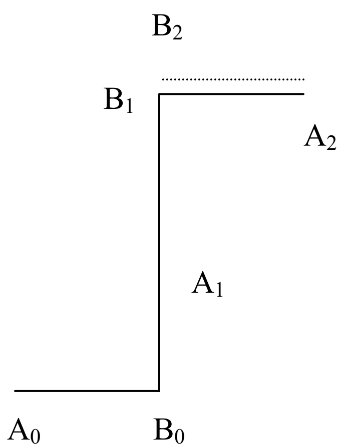


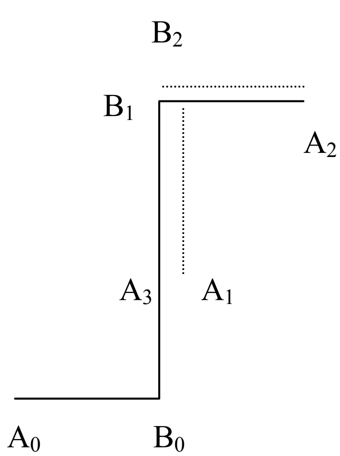
and use x = 360/4 = 90^0.

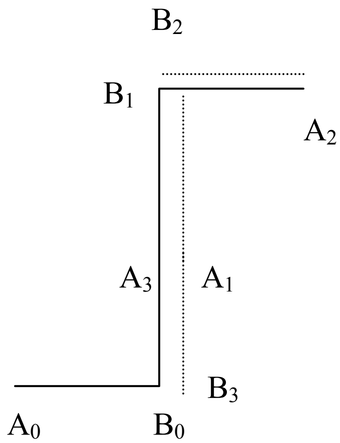


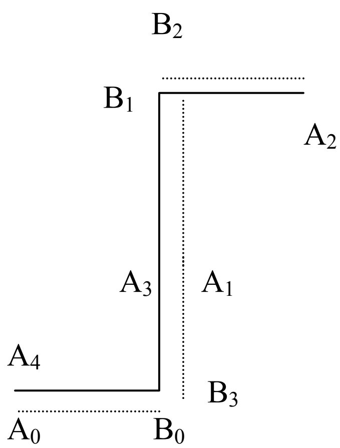



B1
A2
A1
A0
B0


B2
B1
A2
A1
A0
B0


B2
B1
A2
A3
A1
A0
B0

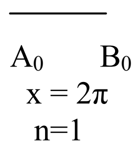


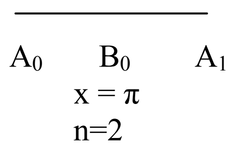


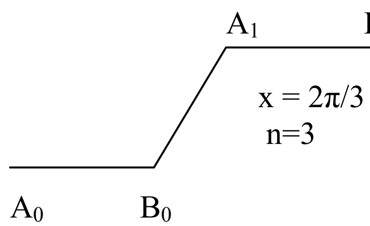
The dashed segments represent where the result of the rotation is a segment that occupies the same location as a previous segment.

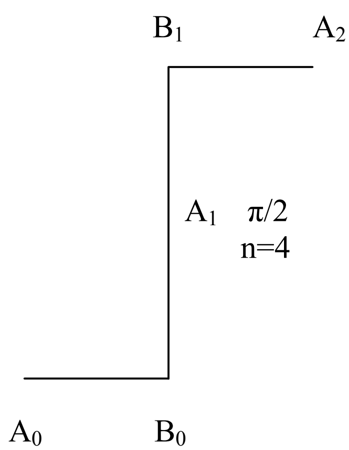
The final figure represents the Smarandache-Murthy's figure of rotation of periodic symmetry for a right angle. Different figures are constructed when different initial angles are used. However, each of the figures that are constructed conforms to the following rules.

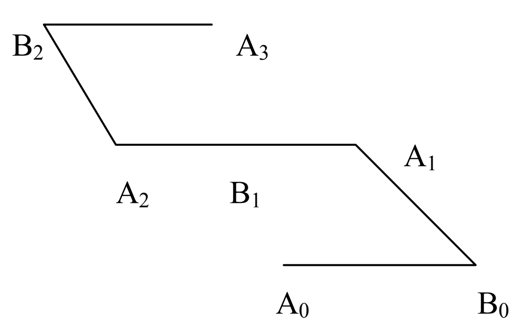
a) Each figure is periodic in nature.
b) The number of segments (each of length A_0B_0) in the figure is n, where $n=720/x$.
c) They exhibit different symmetries for even and odd values of n.
d) For $n > 9$, we get closed segments as part of the figure, with complete triangles for $n=10$.


A0 B0
x = 2π
n=1

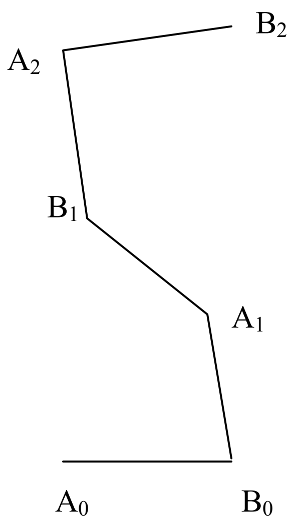

A0 B0 A1
x = π
n=2


A1 B1
x = 2π/3
n=3
A0 B0

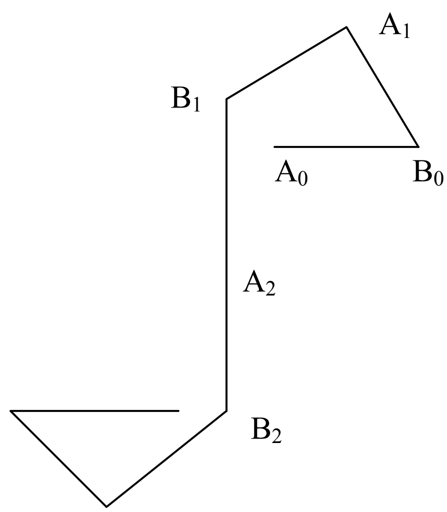

B1 A2
A1 π/2
n=4
A0 B0


B2 A3
A1
A2 B1
A0 B0

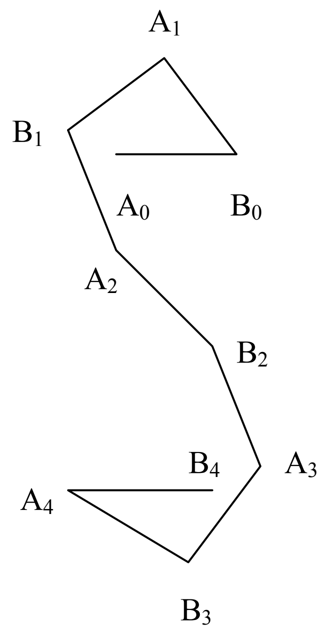
$\pi/3$ $n = 6$



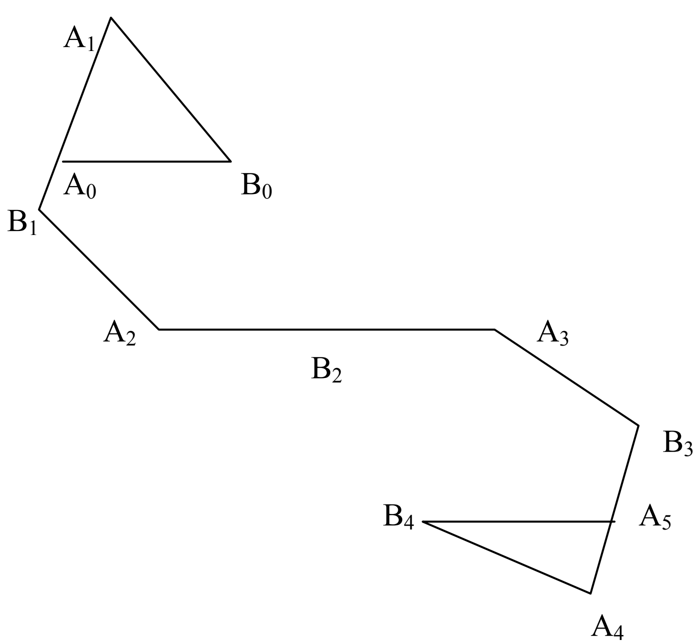
2π/5 n = 5



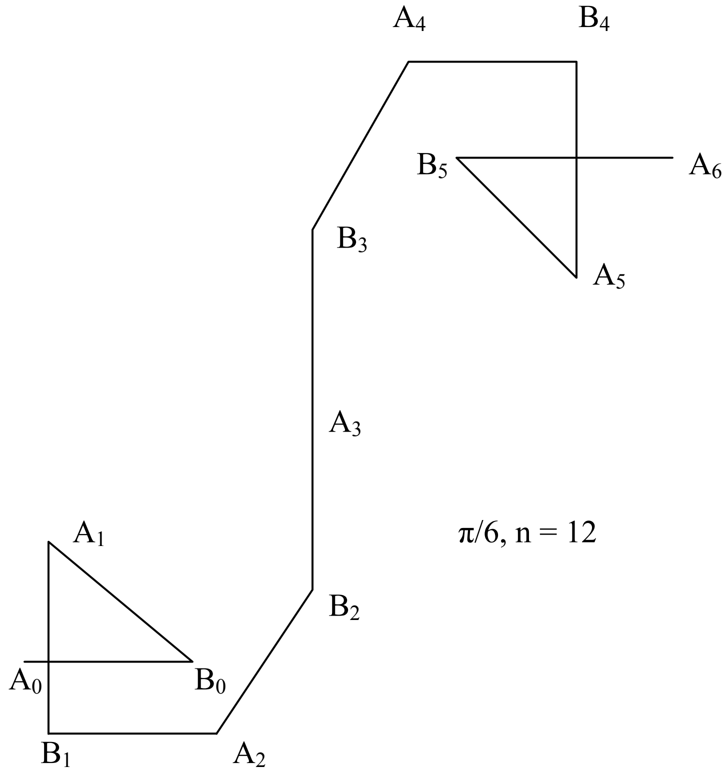
π/4 n = 8



$2\pi/9 \quad n = 9$



$2\pi/10$, $n = 10$



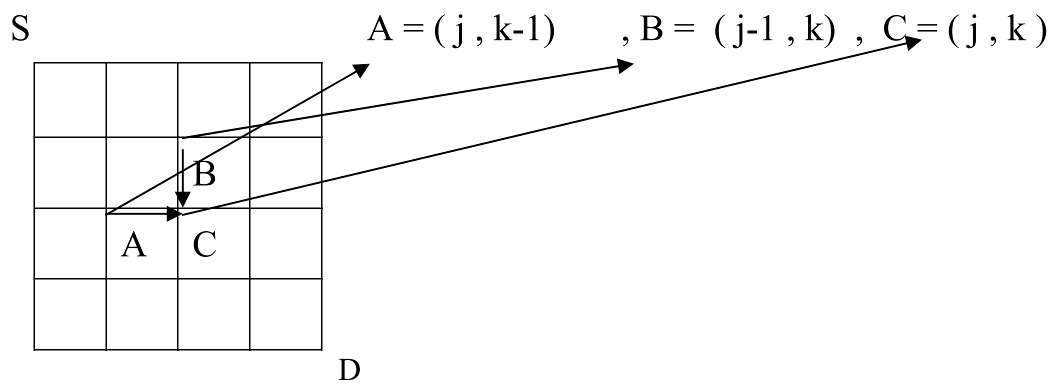
$\pi/6$, n = 12

Section 16

Smarandache Route Sequences

Consider a rectangular city with a mesh of tracks which are of equal length and which are either horizontal or vertical and meeting at nodes. If one row contains m tracks and one column contains n tracks then there are (m+1)(n+1) nodes. To begin, let the city be of a square shape i.e. m = n.

Consider the possible number of routes R where a person at one end of the city can take from a source S (starting point) to reach the diagonally opposite end D the destination.



(m rows and m columns).

Refer to the previous figure to see the derivation of the following values.

For m = 1, the number of routes R = 1.
For m = 2, the number of routes R = 2.
For m = 3, the number of routes R = 12.

We see that for the shortest routes, one has to travel 2m units of track length. There are routes of 2m+2 units in length up to the longest that are 4m+4 units long.

Definition: The Smarandache Route Sequence (SRS) is defined as the number of all possible routes for a square m x m city. The routes will be of length 2m up through 4m+4.

Open problem: Derive a general formula for SRS.

Our next step here will be to derive a reduction formula, which is a general formula for the number of shortest routes.

Reduction formula for the number of shortest routes:
Let $R_{j,k}$ = number of shortest routes from S to node (j,k). Node (j,k) can be reached only from node (j-1,k) or from node (j,k-1), as only shortest routes are to be considered. It is clear that there is only one way of reaching node (j,k) from node (j-1,k). Similarly, there is only one way of reaching node (j,k) from node (j,k-1). Hence, the number of shortest routes to (j,k) is given by

$$R_{j,k} = 1*R_{j-1,k} + 1*R_{j,k-1} = R_{j-1,k} + R_{j,k-1}.$$

This gives the reduction formula for $R_{j,k}$. Applying this reduction formula to fill the chart we observe that the total number of shortest routes to the destination (the other end of the diagonal) is ${}^{2n}C_n$. This can be established by induction.

We can further categorize the routes by the number of turning points (TPs) it is subjected to. The following chart contains the number of turning points for a city with nine nodes.

No. of TPs	1	2	3	4
No. of routes	2	2	2	5

For further investigation:

1) Explore for patterns in the total number of routes, number of turning points and develop formulas for square as well as rectangular grids.
2) To study how many routes pass through a given number/set of nodes. How many of them pass through all the nodes?

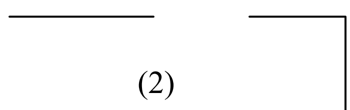
Section 17

Smarandache Geometrical Partitions and Sequences

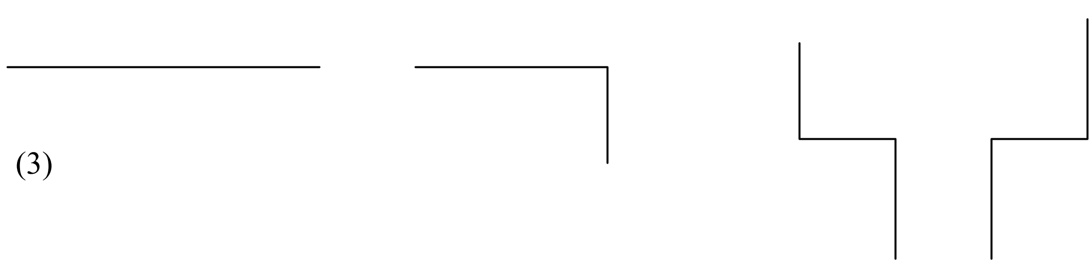
1) Smarandache Traceable Geometrical Partition

Consider a chain having identical links (sticks) which can be bent at the hinges to give it different shapes. For example, consider the following sets of one through four links.

(1)



(2)



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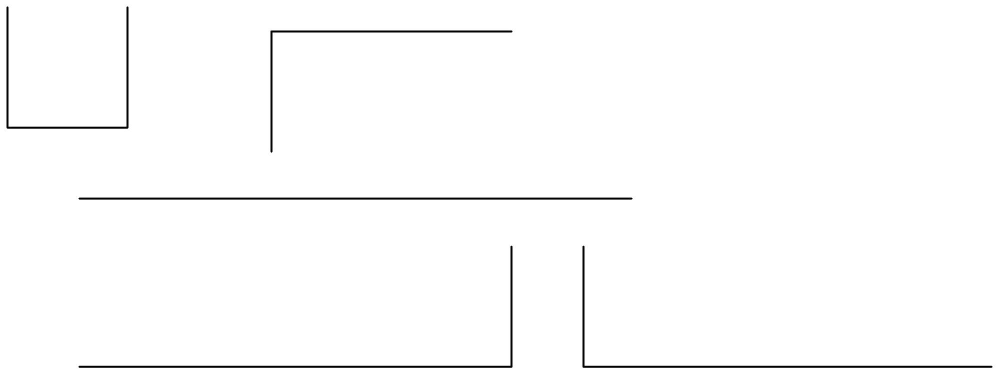

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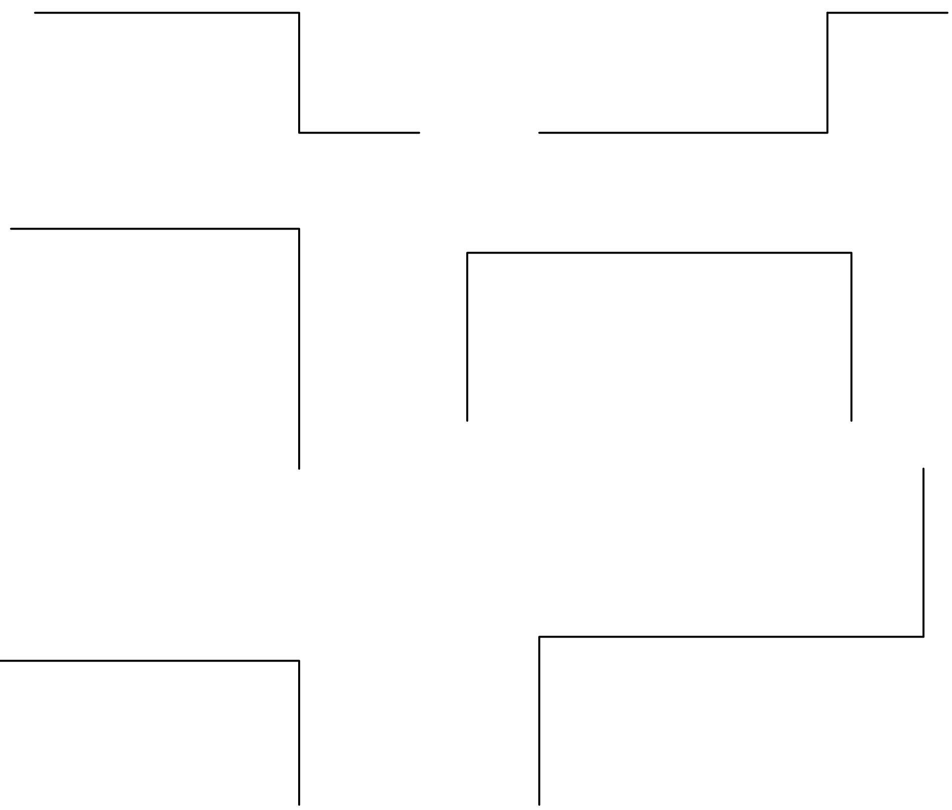
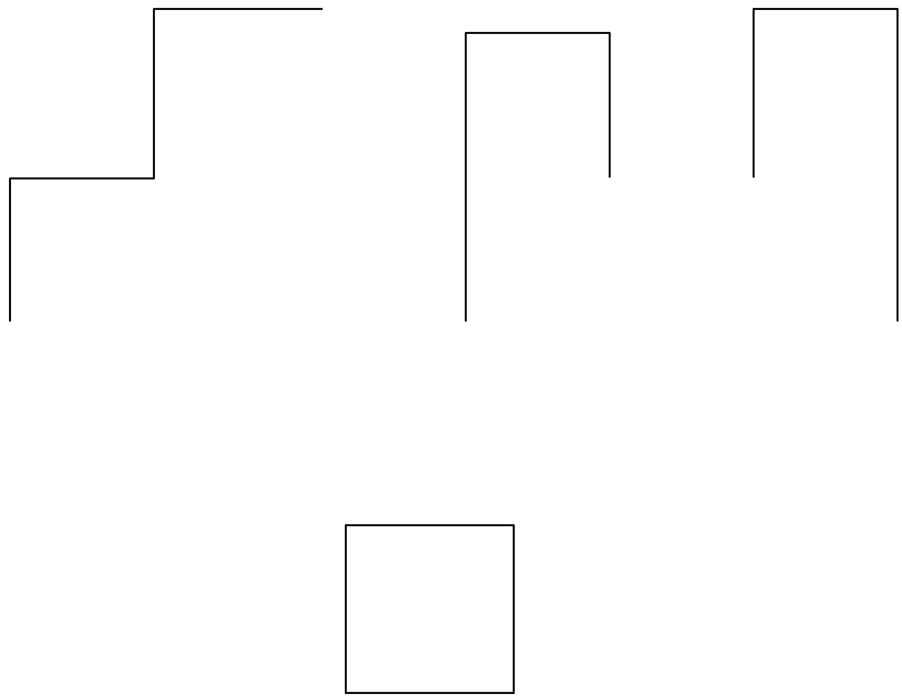




Note that the shapes of the figures satisfy the following rules:

1) The links are either horizontal or vertical.
2) No figure can be obtained from any other by rotation only. It must also be lifted from the horizontal plane.
3) As the links are connected, there are only two ends and one can travel from one end to the other traversing all the links. There are at the most two ends (there can be zero ends in case of a closed figure) to each figure. These are the nodes which are connected to only one link.

Definition: For n the number of connected sticks, we define the Smarandache Traceable Geometric Partition $S_{gp}(n)$ to be the number of different figures that can be constructed using those hinged, connected sticks. The sequence of numbers is called the Smarandache Traceable Geometric Partition Sequence (STGPS).

The first few numbers of STGPS are

1, 2, 6, 15, . . .

Open problem:

To derive a reduction formula for STGPS.

Definition: A bend is defined as a point where there is an angle of 90^0 between the connected sticks.

The following is a table of the number of sticks that have specific numbers of bends in them.

	Number of bends				
	0	1	2	3	4
No. of sticks					
1	1	0	0	0	0
2	1	1	0	0	0
3	1	2	3	0	0
4	1	3	7	3	1

Readers are encouraged to extend this table and look for patterns in the number of bends.

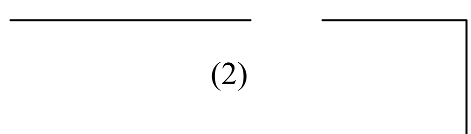
2) Smarandache Comprehensive Geometric Partition

Start with a set of identical sticks and connect them as before. However, in this case, we will relax the previous rules to allow:

1) Sticks can have more than one end.
2) It may not be possible to travel from one end of the figure to the other and traverse each stick only one.

With this relaxation of the rules defining the figures, we have the following sets of figures.

(1)



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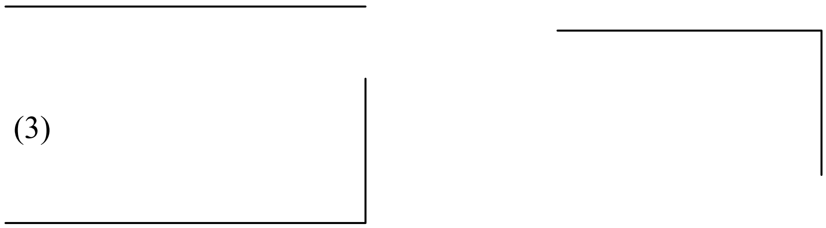

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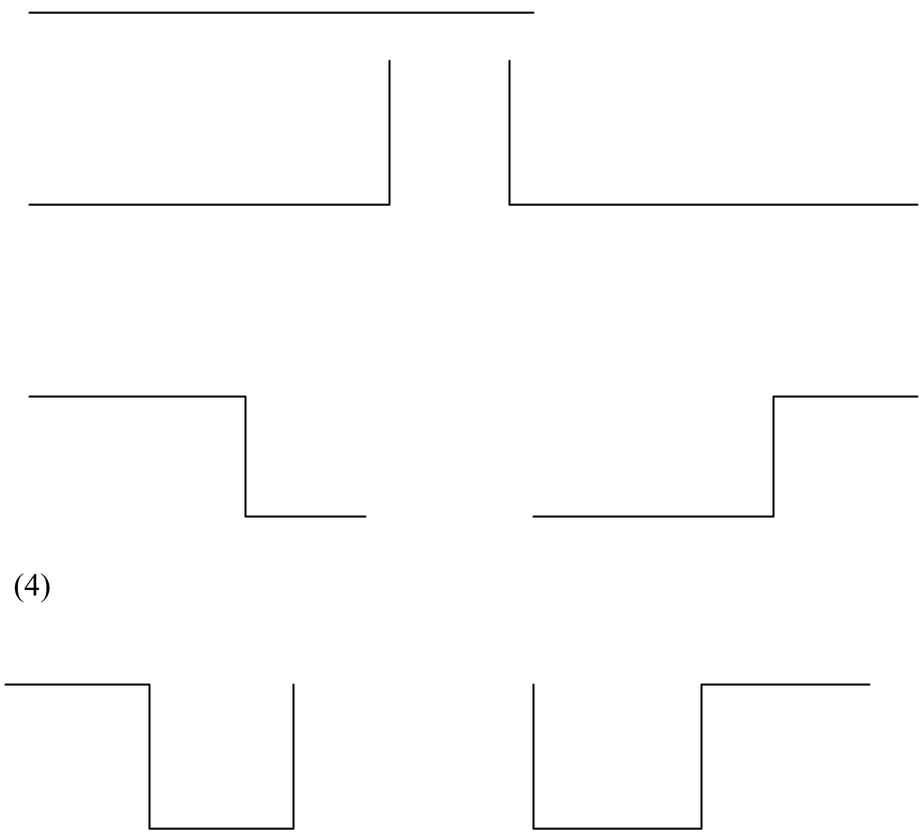

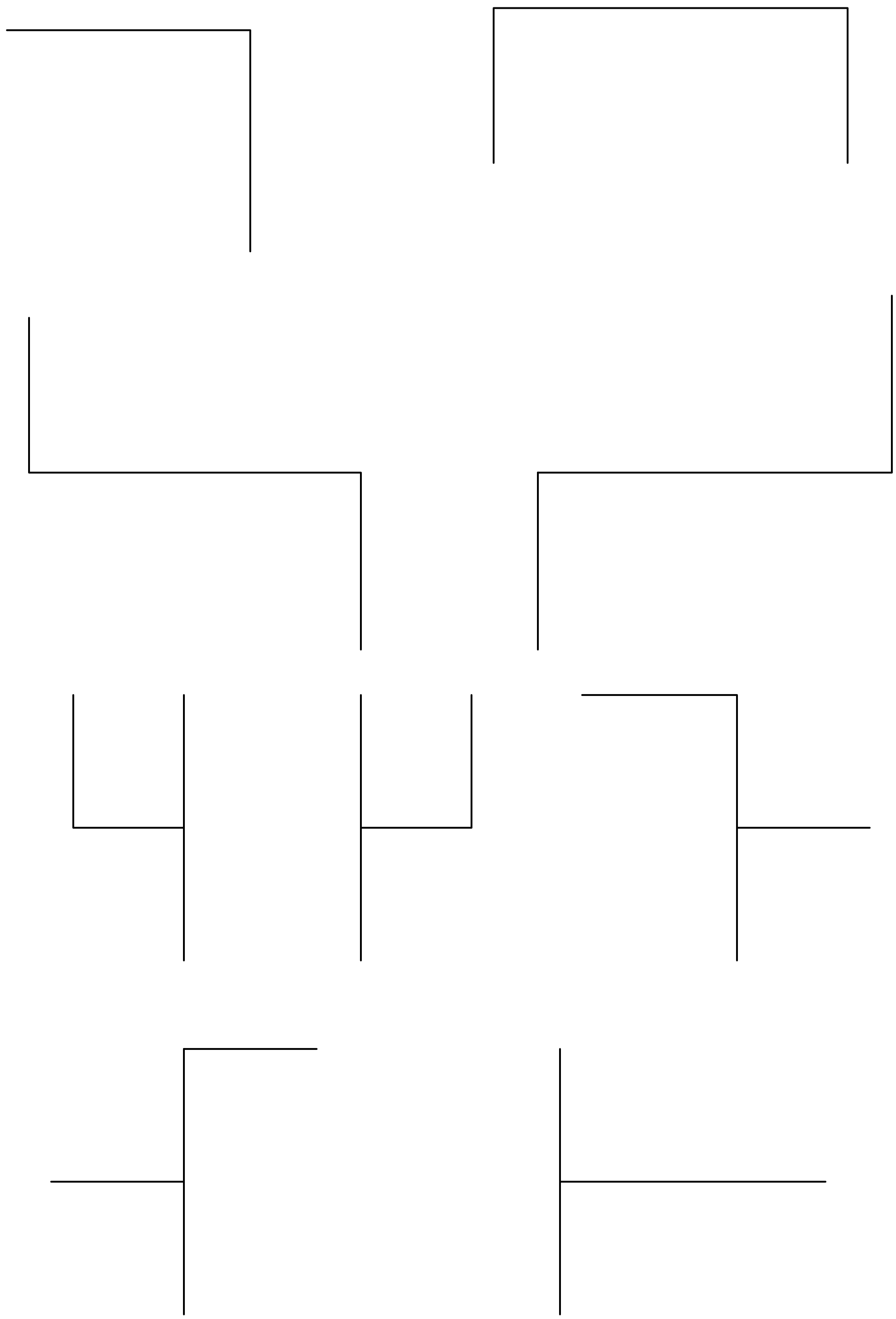

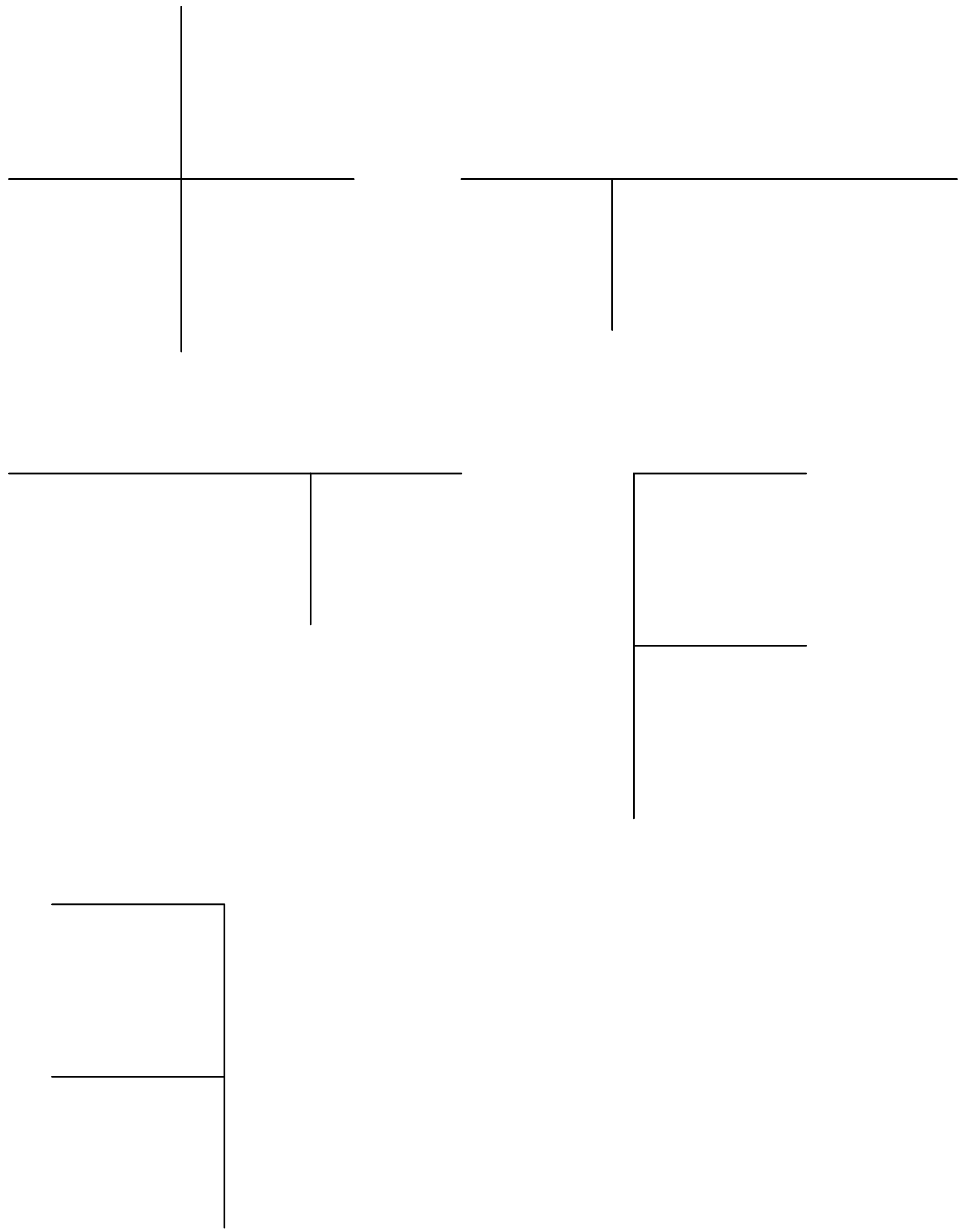

Definition: The sequence of figures that can be created in this way is called the Smarandache Comprehensive Geometric Partition Function (SCGP) and the sequence of numbers formed by the number of such figures for n sticks is known as the SCGPS.

The first few terms of the SCGPS are 1, 2, 7, 25, . . .

The following table summarizes the number of figures having a specific number of ends.

	Number of sticks			
	1	2	3	4
No. of ends				
0	0	0	0	1
1	0	0	0	0
2	1	2	6	14
3	0	0	1	9
4	0	0	0	1

The table can be extended by increasing the number of sticks and readers are encouraged to search for patterns in the table.

Open problem: Derive a reduction formula for SCGPS.

Further consideration: This idea can be extended by allowing the bends to be angles other than 90^0.

Section 18

Smarandache Lucky Methods in Algebra, Trigonometry and Calculus

Definition: A number is said to be a Smarandache Lucky Number if an incorrect calculation leads to a correct result. For example, in the fraction 64 / 16 if the 6's are incorrectly canceled (simplified) the result (4) is still correct. More generally a Smarandache Lucky Method is said to be any incorrect method that leads to a correct result. In [8], the following question is asked:

(1) Are there infinitely many Smarandache Lucky Numbers?

We claim that the answer is yes.

Also in this section, we give some fascinating Smarandache Lucky methods in algebra, trigonometry and calculus.

The following are some examples of Smarandache Lucky Numbers.

(1) 64 /16 = 4 / 1 = 4. (Canceling the 6 from numerator and denominator).

(2) 95 / 19 = 5/ 1 = 5. (Canceling the 9 from numerator and denominator).

(3) 136 / 34 = 16/ 4 = 4. (Canceling the 3 from numerator and denominator).

The following Smarandache Lucky Numbers can be used to generate many additional lucky numbers, although the family could be considered trivial.

4064 /1016, 40064 / 10016, ... In general

4 00000. . .64 / 100000. . . 16 in which both the numerator and denominator

contain the same number (n) of zeroes and the sixes are cancelled.

A Smarandache Lucky Method In Trigonometry

Some students who have just been introduced to the concept of function misinterpret f(x) as the product of f and x. In other words, they consider sin(x) to be the product of sin and x. This gives rise to a funny, lucky method applicable to the following identity.

To prove

$$\sin^2(x) - \sin^2(y) = \sin(x + y)\sin(x - y)$$

LHS = $\sin^2(x) - \sin^2(y)$

$= \{ \sin(x) + \sin(y) \} * \{ \sin(x) - \sin(y) \}$. (A)

Factoring the "common" sin from all "factors"

$= \{ \sin(x + y) \} * \{\sin(x - y) \}$

= RHS.

The correct method from point (A) onwards should have been

$\{2 \sin((x + y)/2) * \cos((x - y)/2)\} * \{2 \cos((x + y)/2) * \sin((x - y)/2)\}$.

$= \{ 2 \sin((x + y)/2) * \cos((x + y)/2)\} * \{2 \cos((x - y)/2) * \sin((x - y)/2)\}$.

$= \{ \sin(x + y) \} * \{\sin(x - y) \}$

= RHS.

A Smarandache Lucky Method In Algebra

In vector algebra the dot product of two vectors $(a_1 i + a_2 j + a_3 k)$ and $(b_1 i + b_2 j + b_3 k)$ is given by

$(a_1 i + a_2 j + a_3 k) \bullet (b_1 i + b_2 j + b_3 k) = a_1 b_1 + a_2 b_2 + a_3 b_3$

If this same idea was applied to ordinary algebra

$(a + b)(c + d) = ac + bd.$ ---------- (B)

This wrong lucky method is applicable in proving the following algebraic identity.

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

RHS $= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= (a + b + c)\{(a^2 - bc) + (b^2 - ac) + (c^2 - ab)\}$

applying the wrong lucky method (B), one gets

$= a.(a^2 - bc) + b(b^2 - ac) + c(c^2 - ab)$

$= a^3 - abc + b^3 - abc + c^3 - abc$

$= a^3 + b^3 + c^3 - 3abc =$ LHS.

A Smarandache Lucky Method In Calculus

The fun involved in the following lucky method in calculus is twofold, and it goes like this. A student is asked to differentiate the product of two functions. Instead of applying the formula for the differentiation of the product of two functions, he applies the method of integration of the product of two functions (integration by parts) and gets the correct answer. The height of coincidence is if another student is given the same product of two functions and asked to integrate does the reverse of it i.e. he ends up applying the formula for differentiation of the product of two functions and yet gets the correct answer. I would take the liberty to call such a lucky method to be a Smarandache superlucky method.

Consider the product of two functions x and sin(x)

$f(x) = x$ and $g(x) = \sin(x)$.

$d\{f(x)g(x)\}/dx = f(x)\int g(x)dx - \int[\{d(f(x))/dx\}\int g(x)dx]dx$

$d\{(x)\sin(x)\}/dx = (x)\int \sin(x)dx - \int[\{d(x)/dx\}\int \sin(x)dx]dx$

$= -(x)(\cos(x)) + \sin(x)$
$= -x\cos(x) + \sin(x).$

The Smarandache lucky method of integration is as follows:

$\int \{(f(x))\, g(x)\}dx. = f(x)d\{g(x)\}/dx \quad + g(x)\, d\{f(x)\}/dx.$

If we use the same functions used in the previous example, when we apply this lucky method we get

$\int \{(x)\sin(x)\}dx = (x)\{\cos(x)\} + \{\sin(x)\}(1)$

or $\int \{(x)\sin(x)\}dx = x\cos(x) + \sin(x).$

By applying the correct methods, it can be verified that these are the correct answers.

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Index

Florentin Smarandache is an incredible source of ideas, only some of which are mathematical in nature. Amarnath Murthy has published a large number of papers in the broad area of "Smarandache Notions", which are math problems whose origin can be traced to Smarandache. This book is an edited version of many of those papers, most of which appeared in "Smarandache Notions Journal", and more information about SNJ is available at http://www.gallup.unm.edu/~smarandache/ .

The topics covered are very broad, although there are two main themes under which most of the material can be classified. A Smarandache Partition Function is an operation where a set or number is split into pieces and together they make up the original object. For example, a Smarandache Repeatable Reciprocal partition of unity is a set of natural numbers where the sum of the reciprocals is one. The first chapter of the book deals with various types of partitions and their properties and partitions also appear in some of the later sections.

The second main theme is a set of sequences defined using various properties. For example, the Smarandache n2n sequence is formed by concatenating a natural number and its' double in that order. Once a sequence is defined, then some properties of the sequence are examined. A common exploration is to ask how many primes are in the sequence or a slight modification of the sequence.

The final chapter is a collection of problems that did not seem to be a precise fit in either of the previous two categories. For example, for any number d, is it possible to find a perfect square that has digit sum d? While many results are proven, a large number of problems are left open, leaving a great deal of room for further exploration.

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