Nom Associative $\mathbf{SU(3)}_L \otimes U(1)_N$ Gauge Model and Predictions

N. Mebarki[(1)] and M.Boussahel[(1),(2)]
[(1)]Laboratoire de Physique Mathematique et Subatomique,
Mentouri University, Constantine, Algeria
[(2)]Physics Department, Faculty of Sciences,
Mohamed Boudiaf University, M'sila, Algeria

Abstract

A classical gauge model based on the Lie group $SU(3)_L \otimes U(1)_N$ with exotic quarks is reformulated within the formalism of non-associative geometry associated to an L-cycle. The N charges of the fermionic particles and the related parameters constraints are uniquely determinedalgebraic consequences. Moreover, the number of scalar particles are dictated by the non-associativity of the geometry. As a byproduct of this formalism, the scalar, charged and neutral gauge bosons masses as well as the mixing angles are derived. Furthermore, various expressions of the vector and axial couplings of the quarks and leptons with the neutral gauge bosons and lower bounds of the very heavy gauge bosons are also obtained.

PACS: 24.85.+p, 13.85.Hd, 13.87.-a

1 Introduction

One of the greatest achievement of the noncommutative geometry (NCG) is the geometrization of the standard model [1, 2, 3]. NCG provides a framework where the Higgs boson may be treated at the same level as the $W^{\pm}$and Z^0 bosons. In this approach, one introduces additional discrete dimensions to the ordinary space-time. If the gauge bosons are associated to the continuous space-time, the Higgs boson results from gauging the discrete directions. In the NCG Connes' approach, a universal formula for an action associated with a noncommutative geometry, defined by a spectal triple is proposed. It is based on the spectrum of the Dirac operator and is a geometric invariant. The new symmetry principle is the automorphism of an algebra A which combines both diffeomorphisms and internal symmetries. Applied to the geometry defined by the spectrum of the standard model gives an action that unifies gravity with the standard model at a very high energy scale. However, Connes' prescription is compatible only with linear representations of the matrix group which imposes very stringent constraints on gauge models. Indeed, it was shown that [4, 5] the only models which can be constructed in this approach are the standard model, the Pati-Salam [6] and the Pati-Mohapatra models [7, 8]. Now, if one takes into account the reality condition of the K-cycle [9], the last two models are ruled out leaving at the end the standard model as the unique model compatible with Connes' prescription. It is worth to mention that they are other formulations of NCG where there is no a such restriction [10, 11, 12, 13, 14]. Recently, Wulkenhaar has proposed a modification to the noncommutative geometry where the differential geometry is formulated in terms of a graded differential Lie algebras instead of unital associative algebras as it is the case in Connes' approach [15, 16]. Its application to a list of physical models has been successful. Among this list figure out the standard model [17], the flipped $SU(5)\otimes U(1)$ [18] , $SO(10)$ models [19] and left-right gauge model [20] etc.... The interesting feature of Wulkenhaar's approach or nonassociative geometry (NAG) is the use of a graded Lie algebra and the obtention of mass relations between fermions and bosons masses as in Connes' formalism. On the other hand, the number of fermion families in nature and the pattern of fermion masses and mixing angles are two of the most intriguing puzzles in modern particle physics. Over the last decade, the $3-3-1$ extension of the standard model (SM) for the strong and electroweak interactions, based on the local gauge group $SU_c(3) \otimes SU_L(3)L \otimes U_X(1)$ have been studied extensively [21, 22, 23, 24, 25, 26, 27, 28]. It provides an interesting attempt to answer the question on family replication. In fact, this extension has among its best features that several models can be constructed so that anomaly cancellation is achieved by an interplay between the families [21, 22, 23, 24, 25, 26, 27, 28]. In some of them [21, 22, 23, 24], the anomaly cancellation implies quarks with exotic electric charges $-4/3$ and $5/3$. Moreover, some models based on the $3-3-1$ local gauge structure are suitable to describe some neutrino properties, because they include in a natural way most of the ingredients needed to explain the masses and mixing. [28, 29, 30, 31, 32, 33]. The goal of this paper is to enlarge the list of NCG physical models [17, 18, 19, 20] by reformulating the $3-3-1$ model within the Wulkenhaar's approach of a graded Lie algebra and derive as an output of this prescription, the various mass spectrum relations, mixing angles and $V-A$ couplings in the weak neutral currents for quarks and leptons. The paper is organized as follows: In section2, we present the general formalism of Wulkenhaar's non associative geometry and the L-cycle. In section 3, we omit the strong interaction sector and construct (within this approach) the bosonic and fermionic actions of the $SU_L(3)L \otimes U_X(1)$ model for electroweak interactions and derive the fuzzy mass, mixing angles and couplings relations as an output. Finally, in section 4 we draw our conclusions.

2 General Formalism

The fundamental object in NAG is the $L-$cycle $(g, \mathcal{H}, D, \pi, \Gamma)$, constituted of:

1)A Hilbert space $\mathcal{H}$:

$$\mathcal{H} = L^2(X,S) \otimes \mathbb{C}^F \tag{1}$$

where $L^2(X,S)$ is a the square integrable bispinors Hilbert space.

2)A unitary skew adjoint Lie algebra g of bounded operators on the Hilbert space $\mathcal{H}$:

$$g = C^\infty(X) \otimes \boldsymbol{a} \tag{2}$$

where $\boldsymbol{a}$ and $C^\infty(X)$ are a matrix Lie algebra and an algebra of smooth functions on a compact Euclidean space-time manifold X respectively.

3)A total self adjoint non degenerate Dirac operator D with a compact resolvant such that :

$$D = \mathcal{D} \otimes \mathbf{1}_F + \gamma^5 \otimes \mathcal{M}$$

where $\mathcal{D}$ and γ^5 are the Dirac operator of the spin connection and the chirality

operator on $L^2(X,S)$ and $\mathcal{M}$ the Dirac operator associated to the discrete algebra $\boldsymbol{a}$.

4)A representation π of g on $\mathcal{H}$ which is given by:

$$\pi = \mathbf{I} \otimes \widehat{\pi} \tag{3}$$

where $\widehat{\pi}$ is the representation of $\boldsymbol{a}$ on $\mathbb{C}^F$ (F is the number of families).

5)A graded operator Γ, acting on $\mathcal{H}$ which has the form:

$$\Gamma = \gamma^5 \otimes \widehat{\Gamma} \tag{4}$$

with the following properties:

$$\Gamma^2 = id_{\mathcal{H}} \tag{5}$$

$$[\Gamma, \pi(g)] = 0 \tag{6}$$

and

$$\{\Gamma, D\} = 0 \tag{7}$$

Here $\widehat{\Gamma}$ is the graded operator acting on $\mathbb{C}^F$ and $id_{\mathcal{H}}$ stands for identity over the Hilbert space $\mathcal{H}$. Now, the prescription towards the construction of a classical

gauge theory associated to the L-cycle requires the following[17, 18, 19, 20]:

i) A space $\mathbf{\Omega}^1\mathbf{a}$ generated by elements ω^1 of the type:

$$\omega^1 = \sum_{\alpha, z \geq 0} \left[a_\alpha^z, ... \left[a_\alpha^1, da_\alpha^0\right] ...\right], \qquad a_\alpha^z \in \boldsymbol{a} \tag{8}$$

where d is the universal exterior derivative. The representation $\widehat{\pi}$ acts on the space $\mathbf{\Omega}^1\mathbf{a}$ as:

$$\begin{aligned}\widehat{\pi} &: \mathbf{\Omega}^1\mathbf{a} \to M_F(\mathbb{C}) \\ \tau^1 &= \widehat{\pi}\left(\omega^1\right) = \sum_{\alpha,z\geq 0}\left[\widehat{\pi}\left(a_\alpha^z\right),...\left[\widehat{\pi}\left(a_\alpha^1\right),\left[-i\mathcal{M},\widehat{\pi}\left(a_\alpha^0\right)\right]\right]...\right] \end{aligned} \tag{9}$$

We define also another mapping such that:

$$\begin{aligned}\widehat{\sigma} &: \mathbf{\Omega}^1\mathbf{a} \to M_F\mathbb{C} \\ \widehat{\sigma}\left(\omega^1\right) &= \sum_{\alpha,z\geq 0}\left[\widehat{\pi}\left(a_\alpha^z\right),...\left[\widehat{\pi}\left(a_\alpha^1\right),\left[\mathcal{M}^2,\widehat{\pi}\left(a_\alpha^0\right)\right]\right]...\right] \end{aligned} \tag{10}$$

ii) A space $\mathbf{\Omega}^n a$ spanned by elements ω^n of the form:

$$\omega^n = \sum_{\alpha,z\geq 0}\left[\omega_{n,\alpha}^1,\left[\omega_{n-1,\alpha}^1.....,\left[\omega_{2,\alpha}^1,\omega_{1,\alpha}^1\right]\right]...\right] \tag{11}$$

where $\omega_{i,\alpha}^1 \in \mathbf{\Omega}^1\mathbf{a}$. We can also extend $\widehat{\pi}$ and $\widehat{\sigma}$ to $\mathbf{\Omega}^n\boldsymbol{a}$ as follows:

$$\widehat{\pi}\left(\left[\omega^1,\omega^k\right]\right) := \widehat{\pi}\left(\omega^1\right)\widehat{\pi}\left(\omega^k\right) - (-1)^k\,\widehat{\pi}\left(\omega^k\right)\widehat{\pi}\left(\omega^1\right), \tag{12}$$

and

$$\widehat{\sigma}\left(\left[\omega^1,\omega^k\right]\right) := \widehat{\sigma}\left(\omega^1\right)\widehat{\pi}\left(\omega^k\right) - \widehat{\pi}\left(\omega^k\right)\widehat{\sigma}\left(\omega^1\right) - \widehat{\pi}\left(\omega^1\right)\widehat{\sigma}\left(\omega^k\right) - (-1)^k\,\widehat{\sigma}\left(\omega^k\right)\widehat{\pi}\left(\omega^1\right) \tag{13}$$

iii) Define for $k \geq 2$, $\widehat{\pi}\left(\Im^k\boldsymbol{a}\right)$ such that:

$$\widehat{\pi}\left(\Im^k\boldsymbol{a}\right) := \{\widehat{\sigma}\left(\omega^{k-1}\right), \omega^{k-1} \in \mathbf{\Omega}^{k-1}\mathbf{a} \cap \mathbf{ker}\,\widehat{\pi}\}. \tag{14}$$

iv) Define spaces $\mathbf{r}^0\ \boldsymbol{a} \subset M_F(\mathbb{C})$ and $\mathbf{r}^1\ \boldsymbol{a} \subset M_F(\mathbb{C})$ such that :

$$\begin{gathered}\mathbf{r}^0\boldsymbol{a} = -\left(\mathbf{r}^0\boldsymbol{a}\right)^* = \widehat{\Gamma}\left(\mathbf{r}^0\boldsymbol{a}\right)\widehat{\Gamma}, \\ \mathbf{r}^1\boldsymbol{a} = -\left(\mathbf{r}^1\boldsymbol{a}\right)^* = -\widehat{\Gamma}\left(\mathbf{r}^1\boldsymbol{a}\right)\widehat{\Gamma} \\ \left[\mathbf{r}^0\boldsymbol{a},\widehat{\pi}\left(\boldsymbol{a}\right)\right] \subset \widehat{\boldsymbol{\pi}}\left(\boldsymbol{a}\right), \\ \left[\mathbf{r}^0\boldsymbol{a},\widehat{\pi}\left(\Omega^1\boldsymbol{a}\right)\right] \subset \widehat{\boldsymbol{\pi}}\left(\mathbf{\Omega}^1\boldsymbol{a}\right) \\ \left\{\mathbf{r}^0\boldsymbol{a},\widehat{\pi}\left(\boldsymbol{a}\right)\right\} \subset \left\{\widehat{\boldsymbol{\pi}}\left(\boldsymbol{a}\right),\widehat{\boldsymbol{\pi}}\left(\boldsymbol{a}\right)\right\} + \widehat{\boldsymbol{\pi}}\left(\mathbf{\Omega}^2\boldsymbol{a}\right), \\ \left\{\mathbf{r}^0\boldsymbol{a},\widehat{\pi}\left(\Omega^1\boldsymbol{a}\right)\right\} \subset \left\{\widehat{\boldsymbol{\pi}}\left(\boldsymbol{a}\right),\widehat{\boldsymbol{\pi}}\left(\mathbf{\Omega}^1\boldsymbol{a}\right)\right\} + \widehat{\boldsymbol{\pi}}\left(\mathbf{\Omega}^3\boldsymbol{a}\right) \\ \left[\mathbf{r}^1\boldsymbol{a},\widehat{\pi}\left(\boldsymbol{a}\right)\right] \subset \widehat{\boldsymbol{\pi}}\left(\mathbf{\Omega}^1\boldsymbol{a}\right) \\ \left\{\mathbf{r}^1\boldsymbol{a},\widehat{\pi}\left(\Omega^1\boldsymbol{a}\right)\right\} \subset \left\{\widehat{\boldsymbol{\pi}}\left(\boldsymbol{a}\right),\widehat{\boldsymbol{\pi}}\left(\boldsymbol{a}\right)\right\} + \widehat{\boldsymbol{\pi}}\left(\mathbf{\Omega}^2\boldsymbol{a}\right)\end{gathered} \tag{15}$$

These additional structures are needed for the construction of a connection form

ρ and a curvature θ. The latter are not in general elements of $\mathbf{\Omega}^1\mathbf{a}$ and $\mathbf{\Omega}^2\mathbf{a}$. This is one of the main differences between Connes's and Wulkenhaar's approachs to NCG.

v) Define spaces $\mathbf{j}^0\boldsymbol{a}$, $\mathbf{j}^1\boldsymbol{a}$ and $\mathbf{j}^2\ \boldsymbol{a} \subset M_F(\mathbb{C})$ as:

$$\begin{aligned}
\mathbf{j}^0\boldsymbol{a} &= \mathbf{c}^0\boldsymbol{a} \\
\mathbf{j}^1\boldsymbol{a} &= \mathbf{c}^1\boldsymbol{a} \\
\mathbf{j}^2\boldsymbol{a} &= \mathbf{c}^2\ \boldsymbol{a} + \widehat{\pi}\left(\Im^2\boldsymbol{a}\right) + \{\widehat{\boldsymbol{\pi}}\left(\boldsymbol{a}\right), \widehat{\boldsymbol{\pi}}\left(\boldsymbol{a}\right)\}
\end{aligned} \tag{16}$$

and such that:

$$\begin{gathered}
\mathbf{c}^0\boldsymbol{a} = -\left(\mathbf{c}^0\boldsymbol{a}\right)^* = \widehat{\Gamma}\left(\mathbf{c}^0\boldsymbol{a}\right)\widehat{\Gamma} \\
\mathbf{c}^1\boldsymbol{a} = -\left(\mathbf{c}^1\boldsymbol{a}\right)^* = -\widehat{\Gamma}\left(\mathbf{c}^1\boldsymbol{a}\right)\widehat{\Gamma} \\
\mathbf{c}^2\boldsymbol{a} = -\left(\mathbf{c}^2\boldsymbol{a}\right)^* = -\widehat{\Gamma}\left(\mathbf{c}^2\boldsymbol{a}\right)\widehat{\Gamma} \\
\mathbf{c}^0\boldsymbol{a}.\widehat{\pi}\left(\boldsymbol{a}\right) = \mathbf{c}^1\boldsymbol{a}.\widehat{\pi}\ \left(\boldsymbol{a}\right) = 0 \\
\mathbf{c}^0\boldsymbol{a}.\widehat{\pi}\left(\Omega^1\boldsymbol{a}\right) = \mathbf{c}\ ^1\boldsymbol{a}.\widehat{\boldsymbol{\pi}}\ \left(\boldsymbol{\Omega}^1\boldsymbol{a}\right) = 0 \\
\left[\mathbf{c}^2\boldsymbol{a}, \widehat{\pi}\left(\boldsymbol{a}\right)\right] = 0 \\
\left[\mathbf{c}^2\boldsymbol{a}, \widehat{\pi}\left(\Omega^1\boldsymbol{a}\right)\right] = 0
\end{gathered} \tag{17}$$

vi) The connection form ρ has the following structure:

$$\rho = \sum_\alpha \left(c^1_\alpha \otimes m^0_\alpha + c^0_\alpha\gamma^5 \otimes m^1_\alpha\right) \tag{18}$$

where $c^1_\alpha \in \Lambda^1$, $c^0_\alpha \in \Lambda^0$, $m^0_\alpha \in \mathbf{r}^0\boldsymbol{a}$, $m^1_\alpha \in \mathbf{r}^1\boldsymbol{a}$ and Λ^k is the differential space of $k-$forms represented by the gamma matrices.

vii) The curvature θ is computed from the connection ρ as:

$$\theta = d\rho + \rho^2 - i\{\gamma 5 \otimes \mathcal{M}, \rho\} + \widehat{\sigma}_g(\rho)\gamma^5 + \mathbf{J}^2 g, \tag{19}$$

where

$$\mathbf{J}^2 g = \left(\Lambda^2 \otimes \mathbf{J}^0\boldsymbol{a}\right) \oplus \left(\Lambda^1\gamma^5 \otimes \mathbf{J}^1\boldsymbol{a}\right) \oplus \left(\Lambda^0 \otimes \mathbf{J}^2\boldsymbol{a}\right) \tag{20}$$

and $\widehat{\sigma}_g$ is the extension of $id \otimes \widehat{\sigma}$ to elements of the form as in eq.(19).

viii) Select the representative $e\left(\theta\right)$ ortogonal to $\mathbf{J}^2 g$ (elimination of the junks forms $j \in \mathbf{J}^2 g$) such that:

$$e(\theta) = d\rho + \rho^2 - i\{\gamma^5 \otimes \mathcal{M}, \rho\} + \widehat{\sigma}_g(\rho)\gamma^5 + j \tag{21}$$

and

$$\int_X dx\ Tr\left(e(\theta)j'\right) = 0 \qquad\qquad \forall j' \in \mathcal{J}^2 g \tag{22}$$

where the trace Tr inludes that in $M_F(\mathbb{C})$ and over gamma matrices.

ix)Calculate the bosonic and fermionic actions S_B and S_F respectively such that:

$$S_B = \int_X dx \frac{1}{g_0^2 F} Tr\,(e(\theta))^2 \tag{23}$$

and

$$S_F = \int_X dx \psi^* (D + i\rho)\, \psi \tag{24}$$

Here g_0 is a coupling constant and $\psi \in \mathcal{H}$ (matter fields).

x) Finally, perform a Wick rotation.

3 NAG Construction of the model

3.1 Motivation

The $3-3-1$ model, based on the gauge group $SU(3)_c \otimes SU(3)_L \otimes U(1)_N$ is particularly interesting and possibly the simplest way to enlarge the gauge group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$. The price we must pay is the introduction of exotic quarks with electric charges $5/3$ and $-4/3$. The main motivations to study this kind of model are: The natural prediction of three generations based on anomaly cancellation. Thus, the family number must be three. This result comes from the fact that the model is anomaly-free only if we have equal number of triplets and antitriplets, counting the $SU(3)_c$ colors, and requiring the sum of all fermion charges to vanish. The triangle anomaly cancellation occurs for the three, or multiply of three, together and not generation by generation like in the standard model (SM). It is worth to mention also that the incorporation of the third family of quarks differently from the other two, leads to potentially large flavor changing neutral currents. The extra neutral vector boson Z' conserves flavor in the leptonic but not in the quark sector. Moreover, it is the simplest model that includes bileptons. The lepton number is violated explicitly by the charged scalar and heavy vector bosons exchange including a vector field with a double electric charge. The model has also several sources of CP violation

implemented spontaneously or explicitly. Denoting by:

$$L^1 = \begin{pmatrix} \upsilon_e \\ e \\ e^c \end{pmatrix}; \qquad Q^1 = \sqrt{2} \begin{pmatrix} u \\ d \\ J_1 \end{pmatrix} \tag{25}$$

The triplet (resp. singlet) representations for the left (L) (resp. right (R)) handed fields (leptons $L^1$and quarks Q^1) are:

$$L_L^1 \sim \ (1,3,0); \qquad Q_L^1 \sim (3,3,+\frac{2}{3}) \tag{26}$$

and

$$\begin{aligned} L^1_{1R} &\sim (1,1,0);\;\; L^1_{2R}\sim(1,1,-1);\;\; L^1_{3R}\sim(1,1,+1) \\ Q^1_{1R} &\sim (3,1,+\frac{2}{3});\;\; Q^1_{2R}\sim(3,1,-\frac{1}{3});\;\; Q^1_{3R}\sim(3,1,+\frac{5}{3}) \end{aligned} \tag{27}$$

The numbers 0, 2/3 in eq.(26) and 2/3,−1/3 and 5/3 in eq.(27) are $U(1)_N$ charges. The normalization factor $\sqrt{2}$ is introduced for practical reasons as it will be clear later. The electric charge operator Q_e is defined in terms of the N charges as:

$$\frac{Q_e}{e} = \frac{1}{2}\left(\lambda_3 - \sqrt{3}\lambda_8\right) + N.\mathbf{1}_{3X3} \tag{28}$$

where λ_3 and λ_8 are the usual Gell-Mann matrices.The other two leptons and quarks generations

$$L^2 \sim \begin{pmatrix} \upsilon_\mu \\ \mu \\ \mu^c \end{pmatrix}; \qquad L^3 = \begin{pmatrix} \upsilon_\tau \\ \tau \\ \tau^c \end{pmatrix} \tag{29}$$

$$Q^2 = \begin{pmatrix} s \\ c \\ J_2 \end{pmatrix}; \qquad Q^3 = \begin{pmatrix} b \\ t \\ J_3 \end{pmatrix} \tag{30}$$

belong to the representations:

$$L^2_L \sim (1,3,0); \qquad L^3_L \sim (1,3,0) \tag{31}$$

$$L^2_{1R} \sim (1,1,0);\;\; L^2_{2R}\sim(1,1,-1);\;\; L^2_{3R}\sim(1,1,+1) \tag{32}$$

$$L^3_{1R} \sim (1,1,0);\;\; L^3_{2R}\sim(1,1,-1);\;\; L^3_{3R}\sim(1,1,+1) \tag{33}$$

$$Q^2_L \sim (3,3^*,-\frac{1}{3}); \qquad Q^3_L \sim (3,3^*,-\frac{1}{3}) \tag{34}$$

$$Q^2_{1R} \sim (3,1,-\frac{1}{3});\;\; Q^2_{2R}\sim(3,1,+\frac{2}{3});\;\; Q^2_{3R}\sim(3,1,-\frac{4}{3}) \tag{35}$$

$$Q^3_{1R} \sim (3,1,-\frac{1}{3});\;\; Q^3_{2R}\sim(3,1,+\frac{2}{3});\;\; Q^3_{3R}\sim(3,1,-\frac{4}{3}) \tag{36}$$

Here u, d, s, c, b, t and J_1, J_2, J_3 stand for the wave function of the up, down, strange, charm, bottom, top and exotic quarks respectively.

3.2 NAG construction of the 3-3-1 model

The discrete L-cycle denoted by $(a, \mathcal{H}, \mathcal{M})$ consists of the Lie algebra

$$\mathbf{a} = su(3) \oplus su(3) \oplus u(1) \ni \{a'_3, a_3, a_1\} \tag{37}$$

For the three generations, the total internal Hilbert space is $\mathbb{C}^{72}$ labelled by the elements:

$$(\ Q^1_{iL}, \ Q^2_{iL}, \ Q^3_{iL}, \ Q^1_{iR}, \ Q^2_{iR}, \ Q^3_{iR}, \ L^1_{iL}, \ L^2_{iL}, \ L^3_{iL}, \ L^1_{iR}, \ L^2_{iR}, \ L^3_{iR}, \ )^T \tag{38}$$

where $i = \overline{1,3}$, Q^1_{iL}, Q^2_{iL}, Q^3_{iL}, Q^1_{iR}, Q^2_{iR}, $Q^3_{iR} \in \mathbb{C}^3 \otimes \mathbb{C}^3$ and L^1_{iL}, L^2_{iL}, L^3_{iL}, L^1_{iR}, L^2_{iR}, $L^3_{iR} \in \mathbb{C}^3$. In what follows, we omit the strong interactions sector. In

this case, the Lie algebra $\mathbf{a}$ acts on $\mathcal{H}$ via the representation:

$$\widehat{\pi}(a_1, a_3) = \begin{pmatrix} \widehat{\pi}_Q(a_1, a_3) & \mathbf{0} \\ \mathbf{0} & \widehat{\pi}_l(a_1, a_3) \end{pmatrix} \tag{39}$$

Using the fact that the elements of the $su(3)$ algebra have the general representation:

$$a_3 = \begin{pmatrix} f_3 + f_8 & f_1 - if_2 & f_4 - if_5 \\ f_1 + if_2 & -f_3 + f_8 & f_6 - if_7 \\ f_4 + if_5 & f_6 + if_7 & -2f_8 \end{pmatrix} \tag{40}$$

and since one of the family of quarks is incorporated differently from the other two, the representation $\widehat{\pi}_Q(a_1, a_3)$ of quarks has to have the following form:

$$\widehat{\pi}_Q(a_1, a_3) = if_0 diag \left[\begin{array}{c} \begin{pmatrix} \widehat{\boldsymbol{\alpha}} & \mathbf{0}_2 \\ 0 & -\frac{1}{3} \otimes \mathbf{I}_2 \end{pmatrix}, \begin{pmatrix} \widehat{\boldsymbol{\alpha}} & \mathbf{0}_2 \\ 0 & -\frac{1}{3} \otimes \mathbf{I}_2 \end{pmatrix}, \begin{pmatrix} \widehat{\boldsymbol{\alpha}} & \mathbf{0}_2 \\ 0 & -\frac{1}{3} \otimes \mathbf{I}_2 \end{pmatrix}, \\ \widehat{\boldsymbol{\beta}} \otimes \mathbf{I}_3, \widehat{\boldsymbol{\gamma}} \otimes \mathbf{I}_3, \begin{pmatrix} \widehat{\boldsymbol{\delta}} & \mathbf{0}_2 \\ 0 & -\frac{4}{3} \otimes \mathbf{I}_2 \end{pmatrix} \end{array} \right] \tag{41}$$

$$+i \begin{pmatrix} f_3 + f_8 & \mathbf{0}_2 & f_1 - if_2 & \mathbf{0}_2 & f_4 - if_5 & \mathbf{0}_2 & \\ 0 & (f_3 + f_8) \otimes \mathbf{I}_2 & 0 & (f_1 - if_2) \otimes \mathbf{I}_2 & 0 & (f_4 - if_5) \otimes \mathbf{I}_2 & \\ f_1 + if_2 & \mathbf{0}_2 & f_8 - f_3 & \mathbf{0}_2 & f_6 - if_7 & \mathbf{0}_2 & \\ 0 & (f_1 + if_2) \otimes \mathbf{I}_2 & 0 & (f_8 - f_3) \otimes \mathbf{I}_2 & 0 & (f_6 - if_7) \otimes \mathbf{I}_2 & \mathbf{0}_9 \\ f_4 + if_5 & \mathbf{0}_2 & f_6 + if_7 & \mathbf{0}_2 & -2f_8 & \mathbf{0}_2 & \\ 0 & (f_4 + if_5) \otimes \mathbf{I}_2 & 0 & (f_6 + if_7) \otimes \mathbf{I}_2 & 0 & -2f_8 \otimes \mathbf{I}_2 & \\ & & & \mathbf{0}_9 & & & \mathbf{0}_9 \end{pmatrix}$$

where the f_i 's are real numbers and $\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\gamma}}$ and $\widehat{\boldsymbol{\delta}}$ are hermitique operators for which the eigenvalues on the states Q^j_i are denoted by

$$\begin{aligned} \widehat{\boldsymbol{\alpha}} Q^j_{iL} &= \left(\frac{1}{2}\delta_{1j} + \delta_{2j} + \delta_{3j} \right) \alpha^j Q^j_{iL} \\ \widehat{\boldsymbol{\beta}} Q^j_{iR} &= \left(\frac{1}{2}\delta_{1j} + \delta_{2j} + \delta_{3j} \right) \delta_{1i} \beta^j Q^j_{iR} \\ \widehat{\boldsymbol{\gamma}} Q^j_{iR} &= \left(\frac{1}{2}\delta_{1j} + \delta_{2j} + \delta_{3j} \right) \delta_{2i} \gamma^j Q^j_{iR} \\ \widehat{\boldsymbol{\delta}} Q^j_{iR} &= \left(\frac{1}{2}\delta_{1j} + \delta_{2j} + \delta_{3j} \right) \delta_{3i} \delta^j Q^j_{iR} \end{aligned} \tag{42}$$

(more properties of these operators will be discussed later). It is very important to mention that in order to keep our construction as clear as possible, we have choosen the operators coefficients associated with $u(1)$ algebra $\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\gamma}}$ and $\widehat{\boldsymbol{\delta}}$ a priori with arbitrary eigenvalues when acting on the quarks. As in ref.[20], we will see when we discuss the electromagnetic interaction terms in the fermionic action that (as expected)the eigenvalues α^j, β^j, γ^j and δ^j correspond exactlty to the N charges of the commutative model [21]. Similarly, for the leptons, the representation $\widehat{\pi}_l(a_1, a_3)$ has to have the form:

$$\begin{aligned} \widehat{\pi}_l(a_1, a_3) &= if_0 diag\left(\widehat{\boldsymbol{\alpha}}' \otimes \mathbf{I}_3, \widehat{\boldsymbol{\alpha}}' \otimes \mathbf{I}_3, \widehat{\boldsymbol{\alpha}}' \otimes \mathbf{I}_3, \widehat{\boldsymbol{\beta}}' \otimes \mathbf{I}_3, \widehat{\boldsymbol{\gamma}}' \otimes \mathbf{I}_3, \widehat{\boldsymbol{\delta}}' \otimes \mathbf{I}_3\right) \\ &+ i \begin{pmatrix} (f_3' + f_8') \otimes \mathbf{I}_3 & (f_1' - if_2') \otimes \mathbf{I}_3 & (f_4' - if_5') \otimes \mathbf{I}_3 & \\ (f_1' + if_2') \otimes \mathbf{I}_3 & (f_8' - f_3') \otimes \mathbf{I}_3 & (f_6' - if_7') \otimes \mathbf{I}_3 & \mathbf{0}_9 \\ (f_4' + if_5') \otimes \mathbf{I}_3 & (f_6' + if_7') \otimes \mathbf{I}_3 & -2f_8' \otimes \mathbf{I}_3 & \\ & \mathbf{0}_9 & & \mathbf{I}_3 \end{pmatrix} \end{aligned} \quad (43)$$

where the f_i' 's are real numbers. Similarly for leptons, we assign operators coefficients $\widehat{\boldsymbol{\alpha}}', \widehat{\boldsymbol{\beta}}', \widehat{\boldsymbol{\gamma}}'$ and $\widehat{\boldsymbol{\delta}}'$ arbitrary fo which their action (eigenvalues) α'^j, β'^j, γ'^j and δ'^j are such that:

$$\begin{aligned} \widehat{\boldsymbol{\alpha}}' L^j_{iL} &= \alpha'^j L^j_{iL} \\ \widehat{\boldsymbol{\beta}}' L^j_{iR} &= \beta'^j L^j_{iR} \\ \widehat{\boldsymbol{\gamma}}' L^j_{iR} &= \gamma'^j L^j_{iR} \\ \widehat{\boldsymbol{\delta}}' L^j_{iR} &= \delta'^j L^j_{iR} \end{aligned} \quad (44)$$

will be determined later and correspond to the leptons N charges like in the original commutative model[21]. Regarding the mass matrix $\mathcal{M}$ of the L-cycle, it has the form:

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_Q & \mathbf{0}_{18} \\ \mathbf{0}_{18} & \mathcal{M}_L \end{pmatrix} \quad (45)$$

where

$$\mathcal{M}_Q = \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & M_{Q^1} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & M_{Q^2} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & M_{Q^3} \\ M^*_{Q^1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & M^*_{Q^2} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & M^*_{Q^3} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix} \quad (46)$$

and

$$\mathcal{M}_L = \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & M_{L^1} & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & M_{L^2} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & M_{L^3} \\ M^*_{L^1} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & M^*_{L^2} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & M^*_{L^3} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \end{pmatrix} \quad (47)$$

Here M_{Q^i} and $M_{L^i} \in M_3(\mathbb{C})$ are the mass matrices of the fermions (quarks and leptons) such that:

$$M_{Q^1}=\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix},\quad M_{Q^2}=\begin{pmatrix} m_d & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix},\quad M_{Q^3}=\begin{pmatrix} m_{J_1} & 0 & 0 \\ 0 & m_{J_2} & 0 \\ 0 & 0 & m_{J_3} \end{pmatrix} \tag{48}$$

and

$$M_{L^1}=\begin{pmatrix} m_{\nu_e} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & m_{\nu_\mu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & m_{\nu_\tau} \end{pmatrix},\quad M_{L^2}=M_{L^3}=\begin{pmatrix} m_e & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & m_\mu & 0 \\ \mathbf{0} & 0 & m_\tau \end{pmatrix} \tag{49}$$

It is worth to mention that in this formalism the graded operator $\widehat{\Gamma}$ is given by:

$$\widehat{\Gamma}=diag\left(-\mathbf{I}_3,-\mathbf{I}_3,-\mathbf{I}_3,\mathbf{I}_3,\mathbf{I}_3,\mathbf{I}_3,-\mathbf{I}_3,-\mathbf{I}_3,-\mathbf{I}_3,\mathbf{I}_3,\mathbf{I}_3,\mathbf{I}_3\right) \tag{50}$$

The space $\widehat{\pi}\left(\mathbf{\Omega}^1\mathbf{a}\right)$ is generated by elements of the type:

$$\tau^1=\widehat{\pi}\left(\omega^1\right)=\sum_{\alpha,z\geq 0}\left[\widehat{\pi}\left(a_\alpha^z\right)...,\left[\widehat{\pi}\left(a_\alpha^2\right),\left[\widehat{\pi}\left(a_\alpha^1\right),\left[-i\mathcal{M},\widehat{\pi}\left(a_\alpha^0\right)\right]\right]...\right] \tag{51}$$

where

$$a_\alpha^z=\left(a_1^z,a_3^z\right) \tag{52}$$

Direct simplifications lead to the following decomposition:

$$\tau^1=\begin{pmatrix} \tau_Q^1 & \mathbf{0} \\ \mathbf{0} & \tau_l^1 \end{pmatrix} \tag{53}$$

where

$$\tau_Q^1=i\begin{pmatrix} \mathbf{0} & \tau_{1Q}^1 \\ \tau_{2Q}^1 & \mathbf{0} \end{pmatrix} \tag{54}$$

and

$$\tau_l^1=i\begin{pmatrix} \mathbf{0} & \tau_{1l}^1 \\ \tau_{2l}^1 & \mathbf{0} \end{pmatrix} \tag{55}$$

with

$$\tau_{1Q}^1=\begin{pmatrix} M_{Q^1}k_{11} & M_{Q^2}k_{12} & M_{Q^3}k_{13} \\ -M_{Q^1}^*k_{12}^* & M_{Q^2}k_{22} & M_{Q^3}k_{23} \\ -M_{Q^1}k_{13}^* & -M_{Q^2}k_{23}^* & M_{Q^3}k_{33} \end{pmatrix},\quad \tau_{2Q}^1=\begin{pmatrix} M_{Q^1}^*k_{11}^* & -M_{Q^1}^*k_{12} & -M_{Q^1}^*k_{13} \\ M_{Q^2}^*k_{12}^* & M_{Q^2}^*k_{22}^* & -M_{Q^2}^*k_{23} \\ M_{Q^3}^*k_{13}^* & M_{Q^3}^*k_{23}^* & M_{Q^3}^*k_{33}^* \end{pmatrix} \tag{56}$$

and

$$\tau_{1l}^1=\begin{pmatrix} M_{L^1}k_{11}' & M_{L^2}k_{12}' & M_{L^3}k_{13}' \\ -M_{L^1}^*k_{12}'^* & M_{L^2}k_{22}' & M_{L^3}k_{23}' \\ -M_{L^1}k_{13}'^* & -M_{L^2}k_{23}'^* & M_{L^3}k_{33}' \end{pmatrix},\quad \tau_{2l}^1=\begin{pmatrix} M_{L^1}^*k_{11}'^* & -M_{L^1}^*k_{12}' & -M_{L^1}^*k_{13}' \\ M_{L^2}^*k_{12}'^* & M_{L^2}^*k_{22}'^* & -M_{L^2}^*k_{23}' \\ M_{L^3}^*k_{13}'^* & M_{L^3}^*k_{23}'^* & M_{L^3}^*k_{33}'^* \end{pmatrix} \tag{57}$$

The coefficients k_{11}, k_{12}, k_{13}, k_{22}, k_{23}, k_{33}, k'_{11}, k'_{12}, k'_{13}, k'_{22}, k'_{23} and k'_{33} are given by:

$$\begin{aligned}
k_{12} &= if_1 + f_2 \\
k_{13} &= if_4 + f_5 \\
k_{23} &= if_6 + f_7 \\
k_{11} &= i\left(f_3 + f_8 + \left(\widehat{\boldsymbol{\alpha}} - \widehat{\boldsymbol{\beta}}\right) f_0\right) \\
k_{22} &= i\left(-f_3 + f_8 + \left(\widehat{\boldsymbol{\alpha}} - \widehat{\boldsymbol{\gamma}}\right) f_0\right) \\
k_{33} &= -i\left(-2f_8 + \left(\widehat{\boldsymbol{\alpha}} - \widehat{\boldsymbol{\delta}}\right) f_0\right)
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
k'_{12} &= if'_1 + f'_2 \\
k'_{13} &= if'_4 + f'_5 \\
k'_{23} &= if'_6 + f'_7 \\
k'_{11} &= i\left(f'_3 + f'_8 + \left(\widehat{\boldsymbol{\alpha}}' - \widehat{\boldsymbol{\beta}}'\right) f'_0\right) \\
k'_{22} &= i\left(-f'_3 + f'_8 + \left(\widehat{\boldsymbol{\alpha}}' - \widehat{\boldsymbol{\gamma}}'\right) f'_0\right) \\
k'_{33} &= -i\left(-2f'_8 + \left(\widehat{\boldsymbol{\alpha}}' - \widehat{\boldsymbol{\delta}}'\right) f'_0\right)
\end{aligned} \tag{59}$$

In what follows, and in order to have a consistency in the model within NAG, we take the real parameters f_i 's and f'_i's equal . Moreover, we will also concentrate only on the quark sector and give the corresponding results for the lepton sector accordingly. Regarding the elements $\tau^2 \in \widehat{\pi}\left(\Omega^2\mathbf{a}\right)$, they are obtained by summing up elements of the type:

$$\tau^2 = \left\{\tau^1, \tau^1\right\} \tag{60}$$

Straightforward calculations give the following form for the quarks, :

$$\tau_Q^2 = \begin{bmatrix} \tau_{1Q}^2 & \mathbf{0}_9 \\ \mathbf{0}_9 & \tau_{2Q}^2 \end{bmatrix} \tag{61}$$

where

$$\tau_{1Q}^2 = \begin{pmatrix} \zeta_{11}^1 & \zeta_{12}^1 & \zeta_{13}^1 \\ \zeta_{21}^1 & \zeta_{22}^1 & \zeta_{23}^1 \\ \zeta_{31}^1 & \zeta_{32}^1 & \zeta_{33}^1 \end{pmatrix} \tag{62}$$

and

$$\tau_{2Q}^2 = \begin{pmatrix} \zeta_{11}^2 & \zeta_{12}^2 & \zeta_{13}^2 \\ \zeta_{21}^2 & \zeta_{22}^2 & \zeta_{23}^2 \\ \zeta_{31}^2 & \zeta_{32}^2 & \zeta_{33}^2 \end{pmatrix} \tag{63}$$

with

$$
\begin{aligned}
\zeta_{11}^{1} &= -2\left(k_{11}k_{11}^{*}M_{Q^{1}}M_{Q^{1}}^{*}+k_{12}^{*}k_{12}M_{Q^{2}}M_{Q^{2}}^{*}+k_{13}^{*}k_{13}M_{Q^{3}}M_{Q^{3}}^{*}\right) && (64)\\
\zeta_{22}^{1} &= -2\left(k_{12}k_{12}^{*}M_{Q^{1}}M_{Q^{1}}^{*}+k_{22}k_{22}^{*}M_{Q^{2}}M_{Q^{2}}^{*}+k_{23}k_{23}^{*}M_{Q^{3}}M_{Q^{3}}^{*}\right)\\
\zeta_{33}^{1} &= -2(k_{13}k_{13}^{*}M_{Q^{1}}M_{Q^{1}}^{*}+k_{23}k_{23}^{*}M_{Q^{2}}M_{Q^{2}}^{*}+k_{33}k_{33}^{*}M_{Q^{3}}M_{Q^{3}}^{*})\\
\zeta_{21}^{1} &= 2\left(k_{12}^{*}k_{11}^{*}M_{Q^{1}}M_{Q^{1}}^{*}-k_{12}^{*}k_{22}M_{Q^{2}}M_{Q^{2}}^{*}-k_{13}^{*}k_{23}M_{Q^{3}}M_{Q^{3}}^{*}\right)\\
\zeta_{12}^{1} &= 2\left(k_{12}k_{11}M_{Q^{1}}M_{Q^{1}}^{*}-k_{12}k_{22}^{*}M_{Q^{2}}M_{Q^{2}}^{*}-k_{13}k_{23}^{*}M_{Q^{3}}M_{Q^{3}}^{*}\right)\\
\zeta_{31}^{1} &= 2(k_{13}^{*}k_{11}^{*}M_{Q^{1}}M_{Q^{1}}^{*}+k_{12}^{*}k_{23}^{*}M_{Q^{2}}M_{Q^{2}}^{*}-k_{13}^{*}k_{33}M_{Q^{3}}M_{Q^{3}}^{*})\\
\zeta_{13}^{1} &= 2(k_{13}k_{11}M_{Q^{1}}M_{Q^{1}}^{*}+k_{12}k_{23}M_{Q^{2}}M_{Q^{2}}^{*}-k_{13}k_{33}^{*}M_{Q^{3}}M_{Q^{3}}^{*})\\
\zeta_{32}^{1} &= 2(-k_{13}^{*}k_{12}M_{Q^{1}}M_{Q^{1}}^{*}+k_{23}^{*}k_{22}^{*}M_{Q^{2}}M_{Q^{2}}^{*}-k_{23}^{*}k_{33}M_{Q^{3}}M_{Q^{3}}^{*}) && (65)\\
\zeta_{23}^{1} &= 2(-k_{13}k_{12}^{*}M_{Q^{1}}M_{Q^{1}}^{*}+k_{23}k_{22}M_{Q^{2}}M_{Q^{2}}^{*}-k_{23}k_{33}^{*}M_{Q^{3}}M_{Q^{3}}^{*})
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta_{11}^{2} &= -2\left(k_{12}^{*}k_{12}+k_{11}^{*}k_{11}+k_{13}^{*}k_{13}\right)M_{Q^{1}}M_{Q^{1}}^{*} && (66)\\
\zeta_{22}^{2} &= -2\left(k_{23}^{*}k_{23}+k_{22}^{*}k_{22}+k_{12}^{*}k_{12}\right)M_{Q^{2}}M_{Q^{2}}^{*}\\
\zeta_{33}^{2} &= -2\left(k_{33}^{*}k_{33}+k_{23}^{*}k_{23}+k_{13}^{*}k_{13}\right)M_{Q^{3}}M_{Q^{3}}^{*}\\
\zeta_{21}^{2} &= 2\left(k_{12}^{*}\left(k_{22}^{*}-k_{11}^{*}\right)-k_{23}k_{13}^{*}\right)M_{Q^{1}}M_{Q^{2}}^{*}\\
\zeta_{12}^{2} &= 2\left(k_{12}\left(k_{22}-k_{11}^{*}\right)-k_{23}^{*}k_{13}\right)M_{Q^{2}}M_{Q^{1}}^{*}\\
\zeta_{31}^{2} &= 2\left(k_{13}^{*}\left(k_{33}^{*}-k_{11}\right)+k_{12}^{*}k_{23}^{*}\right)M_{Q^{1}}M_{Q^{3}}^{*}\\
\zeta_{13}^{2} &= 2\left(k_{13}\left(k_{33}-k_{11}^{*}\right)+k_{12}k_{23}\right)M_{Q^{3}}M_{Q^{1}}^{*}\\
\zeta_{32}^{2} &= 2\left(k_{23}^{*}\left(k_{22}-k_{33}^{*}\right)-k_{12}k_{13}^{*}\right)M_{Q^{2}}M_{Q^{3}}^{*}\\
\zeta_{23}^{2} &= 2\left(k_{23}\left(k_{22}^{*}-k_{33}\right)-k_{12}^{*}k_{13}\right)M_{Q^{3}}M_{Q^{2}}^{*}
\end{aligned}
$$

Regarding $\widehat{\sigma}\left(\omega^{1}\right)$, by using the expression of eq.(11) and after straightforward calculations, one can show that it takes (for the quarks) the form:

$$
\widehat{\sigma}\left(\omega^{1}\right)\rightarrow\sigma_{Q}=\left(\begin{array}{cccc}
\sigma_{Q_{11}} & \sigma_{Q_{12}} & \sigma_{Q_{13}} & \\
\sigma_{Q_{21}} & \sigma_{Q_{22}} & \sigma_{Q_{23}} & \mathbf{0}_{9}\\
\sigma_{Q_{31}} & \sigma_{Q_{32}} & \sigma_{Q_{32}} & \\
 & \mathbf{0}_{9} & & \mathbf{0}_{9}
\end{array}\right) \qquad (67)
$$

where

$$\begin{aligned}
\sigma_{Q_{11}} &= 2\left[\left(f_4^2+f_5^2\right)M_{Q^1Q^3}+\left(f_1^2+f_2^2\right)M_{Q^1Q^2}\right] \\
\sigma_{Q_{21}} &= \left(if_6+f_7\right)\left(-if_4+f_5\right)\left(M_{Q^2Q^1}+2M_{Q^1Q^3}\right)+2if_3\left(-if_1+f_2\right)M_{Q^2Q^1} \\
\sigma_{Q_{31}} &= \left(-if_4+f_5\right)\left(-if_3-3f_8\right)M_{Q^1Q^3}-\left(-if_6+f_7\right)\left(-if_1+f_2\right)\left(M_{Q^1Q^3}+2M_{Q^3Q^2}\right) \\
\sigma_{Q_{12}} &= \left(-if_6+f_7\right)\left(if_4+f_5\right)\left(M_{Q^2Q^1}+2M_{Q^1Q^3}\right)-2if_3\left(if_1+f_2\right)M_{Q^2Q^1} \\
\sigma_{Q_{22}} &= 2\left[\left(f_6^2+f_7^2\right)M_{Q^2Q^3}+\left(f_1^2+f_2^2\right)M_{Q^1Q^2}\right] \\
\sigma_{Q_{32}} &= \left(if_6+f_7\right)\left(3f_8-if_3\right)M_{Q^2Q^3}-\left(if_4+f_5\right)\left(-if_1+f_2\right)\left(M_{Q^2Q^3}+2M_{Q^1Q^2}\right) \\
\sigma_{Q_{13}} &= \left(if_4+f_5\right)\left(if_3-3f_8\right)M_{Q^1Q^3}-\left(if_6+f_7\right)\left(if_1+f_2\right)\left(M_{Q^1Q^3}+2M_{Q^3Q^2}\right) \\
\sigma_{Q_{23}} &= \left(if_6+f_7\right)\left(3f_8-if_3\right)M_{Q^2Q^3}-\left(if_4+f_5\right)\left(-if_1+f_2\right)\left(M_{Q^2Q^3}+2M_{Q^1Q^2}\right) \\
\sigma_{Q_{33}} &= 2\left[\left(f_6^2+f_7^2\right)M_{Q^2Q^3}+\left(f_4^2+f_5^2\right)M_{Q^1Q^3}\right]
\end{aligned} \tag{68}$$

with

$$M_{Q^iQ^j} \equiv M_{Q^i}M_{Qi}^*-M_{Q^j}M_{Q^j}^* \tag{69}$$

One can show that τ_Q^2 takes the form:

$$\tau_Q^2=\tau_Q^{2\prime}+\sigma_Q=diag\left(\tau_{1Q}^{2\prime},\tau_{2Q}^2\right)+\sigma_Q \tag{70}$$

where

$$\tau_{1Q}^{2\prime}=\begin{pmatrix} \zeta_{11}^{\prime 1} & \zeta_{12}^{\prime 1} & \zeta_{13}^{\prime 1} \\ \zeta_{21}^{\prime 1} & \zeta_{22}^{\prime 1} & \zeta_{23}^{\prime 1} \\ \zeta_{31}^{\prime 1} & \zeta_{32}^{\prime 1} & \zeta_{33}^{\prime 1} \end{pmatrix} \tag{71}$$

with

$$\begin{aligned}
\zeta_{11}^{\prime 1} &= \zeta_{11}^2 \\
\zeta_{22}^{\prime 1} &= \zeta_{22}^2 \\
\zeta_{33}^{\prime 1} &= \zeta_{33}^2 \\
\zeta_{21}^{\prime 1} &= \left(k_{12}^*\left(k_{11}^*-k_{22}\right)-k_{23}k_{13}^*\right)M_{\{Q^2Q^1\}} \\
\zeta_{12}^{\prime 1} &= \left(k_{12}\left(k_{11}-k_{22}^*\right)-k_{23}^*k_{13}\right)M_{\{Q^1Q^2\}} \\
\zeta_{31}^{\prime 1} &= \left(k_{13}^*\left(k_{11}^*-k_{33}\right)+k_{12}^*k_{23}^*\right)M_{\{Q^3Q^1\}} \\
\zeta_{13}^{\prime 1} &= \left(k_{13}\left(k_{11}-k_{33}^*\right)+k_{12}k_{23}\right)M_{\{Q^1Q^3\}} \\
\zeta_{32}^{\prime 1} &= \left(k_{23}^*\left(k_{22}^*-k_{33}\right)-k_{12}^*k_{13}\right)M_{\{Q^3Q^2\}} \\
\zeta_{23}^{\prime 1} &= \left(k_{23}\left(k_{22}-k_{33}^*\right)-k_{12}^*k_{13}\right)M_{\{Q^2Q^3\}}
\end{aligned} \tag{72}$$

and

$$M_{\{Q^iQ^j\}} \equiv M_{Q^i}M_{Qi}^*+M_{Q^j}M_{Q^j}^* \tag{73}$$

Thus, τ_Q^2 can be rewritten as:

$$\tau_Q^2=diag\left(\tau_{1Q}^{1\prime},\tau_{2Q}^2\right)\bmod\sigma_Q\left(\mathbf{\Omega}^1\mathbf{a}\right)$$

Now, using the conditions of eqs.(16), (17), (18), we can show that:

$$\begin{aligned} \mathbf{r}^0 a &= \widehat{\pi}\,(a) \\ \mathbf{r}^1 a &= \widehat{\pi}\left(\Omega^1 a\right) \\ \mathbf{j}^0 a &= 0 \\ \mathbf{j}^1 a &= 0 \end{aligned} \tag{74}$$

and

$$\mathbf{j}^2 a = \widehat{\pi}\left(\Im^2 a\right) \oplus (\{\widehat{\pi}(a), \widehat{\pi}(a)\} \oplus diag\,(\mathbb{R}\mathbf{I}_{18}, \mathbb{R}\mathbf{I}_{18})) \ni J_2 \oplus diag\,(A_Q + \Delta_Q, A_l + \Delta_l) \oplus diag\,(J_Q, J_l) \tag{75}$$

We remind that the ideal $\widehat{\pi}\left(\Im^2 \boldsymbol{a}\right)$ is given as a set of elements $j_2 \in J_2 = \widehat{\pi}\left(\Im^2 \boldsymbol{a}\right)$ of the form:

$$j_2 = \sum_{\alpha, z \geq 0} \left[\widehat{\pi}\,(a_\alpha^z)\,, ... \left[\widehat{\pi}\left(a_\alpha^1\right), \left[\mathcal{M}^2, \widehat{\pi}\left(a_\alpha^0\right)\right]\right] ...\right] \tag{76}$$

where

$$0 = \sum_{\alpha, z \geq 0} \left[\widehat{\pi}\,(a_\alpha^z)\,, ... \left[\widehat{\pi}\left(a_\alpha^1\right), \left[-i\mathcal{M}, \widehat{\pi}\left(a_\alpha^0\right)\right]\right] ...\right] \tag{77}$$

If we use the parametrization:

$$\sum_\alpha c_\alpha^1 \otimes b_{1\alpha} = \phi_i \in \Lambda^0 \otimes \mathbb{C} \tag{78}$$

$$\sum_\alpha c_\alpha^1 \otimes a_{1\alpha} = \sum_\alpha c_\alpha^1 \otimes i f_{0\alpha} = iA_0 \in \Lambda^1 \otimes u(1) \tag{79}$$

and

$$\begin{aligned} \sum_\alpha c_\alpha^1 \otimes a_{3\alpha} &= \mathbf{A} \in \Lambda^1 \otimes su(3) \\ &= \begin{pmatrix} \sum_\alpha c_\alpha^1 \otimes i\,(f_{3\alpha} + f_{8\alpha}) & \sum_\alpha c_\alpha^1 \otimes i\,(f_{1\alpha} - i f_{2\alpha}) & \sum_\alpha c_\alpha^1 \otimes i\,(f_{4\alpha} - i f_{5\alpha}) \\ \sum_\alpha c_\alpha^1 \otimes i\,(f_{1\alpha} + i f_{2\alpha}) & -\sum_\alpha c_\alpha^1 \otimes i\,(f_{3\alpha} - f_{8\alpha}) & \sum_\alpha c_\alpha^1 \otimes i\,(f_{6\alpha} - i f_{7\alpha}) \\ \sum_\alpha c_\alpha^1 \otimes i\,(f_{4\alpha} + i f_{5\alpha}) & \sum_\alpha c_\alpha^1 \otimes i\,(f_{6\alpha} + i f_{7\alpha}) & -2\sum_\alpha c_\alpha^1 \otimes i f_{8\alpha} \end{pmatrix} \\ &= \begin{pmatrix} i\,(A_3 + A_8) & i(A_1 - iA_2) & i(A_4 - iA_5) \\ i(A_1 + iA_2) & -i\,(A_3 - A_8) & i(A_6 - iA_7) \\ i(A_4 + iA_5) & i(A_6 + iA_7) & -2iA_8 \end{pmatrix} \end{aligned} \tag{80}$$

The connection ρ (see eq.(19)), has the following bloc diagonal matrix form:

$$\rho = \begin{pmatrix} \rho_Q & 0 \\ 0 & \rho_l \end{pmatrix} \tag{81}$$

where

$$\rho_Q = \begin{pmatrix} \rho_Q^{A^1} & \rho_Q^{H^1} \\ \rho_Q^{H^2} & \rho_Q^{A^2} \end{pmatrix} \tag{82}$$

and

$$\rho_l = \begin{pmatrix} \rho_l^{A^1} & \rho_l^{H^1} \\ \rho_l^{H^2} & \rho_l^{A^2} \end{pmatrix} \tag{83}$$

with

$$\rho_Q^{A^1} = \begin{pmatrix} i(A_{38} + \widehat{\alpha} A_0) \otimes \mathbf{I}_3 & iA_{-12} \otimes \mathbf{I}_3 & iA_{-45} \otimes \mathbf{I}_3 \\ iA_{12} \otimes \mathbf{I}_3 & i(-A_{-38} + \widehat{\alpha} A_0) \otimes \mathbf{I}_3 & iA_{-67} \otimes \mathbf{I}_3 \\ iA_{45} \otimes \mathbf{I}_3 & iA_{67} \otimes \mathbf{I}_3 & (-2iA_8 + i\widehat{\alpha} A_0) \otimes \mathbf{I}_3 \end{pmatrix} \tag{84}$$

$$\rho_Q^{A^2} = \begin{pmatrix} i\widehat{\boldsymbol{\beta}} A_0 \otimes \mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & i\widehat{\boldsymbol{\gamma}} A_0 \otimes \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & i\widehat{\boldsymbol{\delta}} A_0 \otimes \mathbf{I}_3 \end{pmatrix} \tag{85}$$

$$\rho_Q^{H^1} = \begin{pmatrix} -i\gamma^5 \phi_4 M_{Q^1} & -i\gamma^5 \overline{\phi}_1 M_{Q^2} & -i\gamma^5 \phi_2 M_{Q^3} \\ i\gamma^5 \phi_1 M_{Q^1} & -i\gamma^5 \phi_5 M_{Q^2} & -i\gamma^5 \overline{\phi}_3 M_{Q^3} \\ i\gamma^5 \overline{\phi}_2 M_{Q^1} & i\gamma^5 \phi_3 M_{Q^2} & -i\gamma^5 \phi_6 M_{Q^3} \end{pmatrix} \tag{86}$$

$$\rho_Q^{H^2} = \begin{pmatrix} -i\gamma^5 \overline{\phi}_4 M^*_{Q^1} & i\gamma^5 \overline{\phi}_1 M^*_{Q^1} & i\gamma^5 \phi_2 M^*_{Q^1} \\ -i\gamma^5 \phi_1 M^*_{Q^2} & -i\gamma^5 \overline{\phi}_5 M^*_{Q^2} & i\gamma^5 \overline{\phi}_3 M^*_{Q^2} \\ -i\gamma^5 \overline{\phi}_2 M^*_{Q^3} & -i\gamma^5 \phi_3 M^*_{Q^3} & -i\gamma^5 \overline{\phi}_6 M^*_{Q^3} \end{pmatrix} \tag{87}$$

$$\rho_l^{A^1} = \begin{pmatrix} iA_{38} \otimes \mathbf{I}_3 & iA_{-12} \otimes \mathbf{I}_3 & iA_{-45} \otimes \mathbf{I}_3 \\ iA_{12} \otimes \mathbf{I}_3 & -iA_{-38} \otimes \mathbf{I}_3 & iA_{-67} \otimes I_3 \\ iA_{45} \otimes \mathbf{I}_3 & iA_{67} \otimes \mathbf{I}_3 & -2iA_8 \otimes \mathbf{I}_3 \end{pmatrix} \tag{88}$$

$$\rho_l^{A^2} = \begin{pmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & -iA_0 \otimes \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & iA_0 \otimes \mathbf{I}_3 \end{pmatrix} \tag{89}$$

$$\rho_l^{H^1} = \begin{pmatrix} -i\gamma^5 \phi_4 M_{L^1} & -i\gamma^5 \overline{\phi}_1 M_{L^2} & -i\gamma^5 \phi_2 M_{L^3} \\ i\gamma^5 \phi_1 M_{L^1} & -i\gamma^5 \phi_5 M_{L^2} & -i\gamma^5 \overline{\phi}_3 M_{L^3} \\ i\gamma^5 \overline{\phi}_2 M_{L^1} & i\gamma^5 \phi_3 M_{L^2} & -i\gamma^5 \phi_6 M_{L^3} \end{pmatrix} \tag{90}$$

and

$$\rho_l^{H^2} = \begin{pmatrix} -i\gamma^5 \overline{\phi}_4 M^*_{L^1} & i\gamma^5 \overline{\phi}_1 M^*_{L^1} & i\gamma^5 \phi_2 M^*_{L^1} \\ -i\gamma^5 \phi_1 M^*_{L^2} & -i\gamma^5 \overline{\phi}_5 M^*_{L^2} & i\gamma^5 \overline{\phi}_3 M^*_{L^2} \\ -i\gamma^5 \overline{\phi}_2 M^*_{L^3} & -i\gamma^5 \phi_3 \otimes M^*_{L^3} & -i\gamma^5 \overline{\phi}_6 M^*_{L^3} \end{pmatrix} \tag{91}$$

Here $A_{\pm ij} \equiv (A_i \pm iA_j)$. It is very important to mention that the Higgs multiplets representations of $su(3)$ are uniquely determined by the fermionic mass matrix. In fact, the degrees of freedom of the scalar particles ϕ_i, $i = \overline{1,6}$ are dictated and imposed by NAG and which looks like the $3-3-1$ economical model where there are just two Higgs triplets[34, 35]. Moreover, this minimal scalar sector (as it will be shown later) is able to break the symmetry

$SU(3)_c \otimes SU(3)_L \otimes U(1)_N$ spontaneously to $SU(3)_c \otimes U(1)_{em}$ in one step. The action of the hermitique operators $\widehat{\boldsymbol{\alpha}}$, $\widehat{\boldsymbol{\beta}}$, $\widehat{\boldsymbol{\gamma}}$, $\widehat{\boldsymbol{\delta}}$, $\widehat{\boldsymbol{\alpha}}'$, $\widehat{\boldsymbol{\beta}}'$, $\widehat{\boldsymbol{\gamma}}'$ and $\widehat{\boldsymbol{\delta}}'$ on the scalar matter and vector fields ϕ_i and A_0 respectively is denoted by:

$$\begin{aligned}
\widehat{\boldsymbol{\alpha}}\phi_i &= \widehat{\boldsymbol{\alpha}}'\phi_i = \alpha\phi_i \\
\widehat{\boldsymbol{\beta}}\phi_i &= \widehat{\boldsymbol{\beta}}'\phi_i = \beta\phi_i \\
\widehat{\boldsymbol{\gamma}}\phi_i &= \widehat{\boldsymbol{\gamma}}'\phi_i = \gamma\phi_i \\
\widehat{\boldsymbol{\delta}}\phi_i &= \widehat{\boldsymbol{\delta}}'\phi_i = \delta\phi_i
\end{aligned} \tag{92}$$

$$\left(3\widehat{\boldsymbol{\alpha}} + \widehat{\boldsymbol{\beta}} + \widehat{\boldsymbol{\gamma}} + \widehat{\boldsymbol{\delta}}\right) A_0 = \sqrt{x} A_0 \tag{93}$$

and

$$\left(3\widehat{\boldsymbol{\alpha}}' + \widehat{\boldsymbol{\beta}}' + \widehat{\boldsymbol{\gamma}}' + \widehat{\boldsymbol{\delta}}'\right) A_0 = 0 \tag{94}$$

where α, β, γ and δ are arbitrary real numbers. The parameter x is related to the N charges and is a generation independent (see eq.(112)). Notice also that contrary to the Wulkenhaar construction to the flipped $SU(5) \otimes U(1)$ grand unified theory (GUT) within the framework of NAG [18]where the matrix Lie algebra $su(5)$ is considered as an input and the $u(1)$ part with its representation on the fermions as an algebraic consequence, we have considered in this paper (as it was done in our reference [20])the matrix Lie algebra $su(5) \oplus u(1)$. The reason was that in reference [18], in order to compute the structure of elements belonging to $\mathbf{r}^0\boldsymbol{a}$ after decomposing them into irreducible $su(5)$ representations, the condition $\left[\mathbf{r}^0\boldsymbol{a}, \widehat{\pi}\left(\boldsymbol{a}\right)\right] \subset \widehat{\boldsymbol{\pi}}\left(\boldsymbol{a}\right)$ yields to a block structure where the compatibility with the two conditions $\left\{\mathbf{r}^0\boldsymbol{a}, \widehat{\pi}\left(\boldsymbol{a}\right)\right\} \subset \left\{\widehat{\boldsymbol{\pi}}\left(\boldsymbol{a}\right), \widehat{\boldsymbol{\pi}}\left(\boldsymbol{a}\right)\right\} + \widehat{\boldsymbol{\pi}}\left(\boldsymbol{\Omega^2 a}\right)$ and $\left\{\mathbf{r}^0\boldsymbol{a}, \widehat{\pi}\left(\Omega^1\boldsymbol{a}\right)\right\} \subset \left\{\widehat{\boldsymbol{\pi}}\left(\boldsymbol{a}\right), \widehat{\boldsymbol{\pi}}\left(\boldsymbol{\Omega^1 a}\right)\right\} + \widehat{\boldsymbol{\pi}}\left(\boldsymbol{\Omega^3 a}\right)$ implies that the connection form has a structure with an additional $u(1)$ part with a unique representation on the fermionic Hilbert space. This is not the case in our model.

Let us now compute the anticommutator $\{\widehat{\pi}(a_1,a_3), \widehat{\pi}(a_1,a_3)\}$where:

$$\{\widehat{\pi}(a_1,a_3), \widehat{\pi}(a_1,a_3)\} = diag\left(\{\widehat{\pi}_Q(a_1,a_3), \widehat{\pi}_Q(a_1,a_3)\}, \{\widehat{\pi}_l(a_1,a_3), \widehat{\pi}_l(a_1,a_3)\}\right) \tag{95}$$

Direct simplifications lead to the following form:

$$\{\widehat{\pi}_Q(a_1,a_3), \widehat{\pi}_Q(a_1,a_3)\} = A_Q + \Delta_Q \tag{96}$$

where

$$A_Q = 2\begin{pmatrix}
\widehat{\boldsymbol{\alpha}}\widehat{\lambda}_\alpha^{0(3+8)} \otimes \mathbf{I}_3 & \widehat{\boldsymbol{\alpha}}\left(\widehat{\lambda}_\alpha^{1} - i\widehat{\lambda}_\alpha^{2}\right) \otimes \mathbf{I}_3 & \widehat{\boldsymbol{\alpha}}\left(\widehat{\lambda}_\alpha^{04} - i\widehat{\lambda}_\alpha^{05}\right) \otimes \mathbf{I}_3 & \\
\widehat{\boldsymbol{\alpha}}\left(\widehat{\lambda}_\alpha^{1} + i\widehat{\lambda}_\alpha^{2}\right) \otimes \mathbf{I}_3 & \widehat{\boldsymbol{\alpha}}\widehat{\lambda}_\alpha^{0(3-8)} \otimes \mathbf{I}_3 & \widehat{\boldsymbol{\alpha}}\left(\widehat{\lambda}_\alpha^{06} - i\widehat{\lambda}_\alpha^{07}\right) \otimes \mathbf{I}_3 & \mathbf{0}_9 \\
\widehat{\boldsymbol{\alpha}}\left(\widehat{\lambda}_\alpha^{04} + i\widehat{\lambda}_\alpha^{05}\right) \otimes \mathbf{I}_3 & \widehat{\boldsymbol{\alpha}}\left(\widehat{\lambda}_\alpha^{06} + i\widehat{\lambda}_\alpha^{07}\right) \otimes \mathbf{I}_3 & 4\widehat{\boldsymbol{\alpha}}\widehat{\lambda}_\alpha^{08} \otimes \mathbf{I}_3 & \\
 & \mathbf{0}_9 & & \mathbf{0}_9
\end{pmatrix} \tag{97}$$

and

$$
\Delta_Q = 2\begin{pmatrix}
\left(\widehat{\lambda}^{1}+\widehat{\boldsymbol{\alpha}}^{2}\widehat{\lambda}\right)\otimes\mathbf{I}_3 & \left(\widehat{\lambda}_1-i\widehat{\lambda}_2\right)\otimes\mathbf{I}_3 & \left(\widehat{\lambda}_3-i\widehat{\lambda}_4\right)\otimes\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\left(\widehat{\lambda}_1+i\widehat{\lambda}_2\right)\otimes\mathbf{I}_3 & \left(\widehat{\lambda}^{2}+\widehat{\boldsymbol{\alpha}}^{2}\widehat{\lambda}\right)\otimes\mathbf{I}_3 & \left(\widehat{\lambda}_5-i\widehat{\lambda}_6\right)\otimes\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\left(\widehat{\lambda}_3+i\widehat{\lambda}_4\right)\otimes\mathbf{I}_3 & \left(\widehat{\lambda}_5+i\widehat{\lambda}_6\right)\otimes\mathbf{I}_3 & \left(\widehat{\lambda}^{3}+\widehat{\boldsymbol{\alpha}}^{2}\widehat{\lambda}\right)\otimes\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \widehat{\boldsymbol{\beta}}^{2}\widehat{\lambda}\otimes\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \widehat{\gamma}^{2}\widehat{\lambda}\otimes\mathbf{I}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & 0 & \widehat{\delta}^{2}\widehat{\lambda}\otimes\mathbf{I}_3
\end{pmatrix} \tag{98}
$$

The parameters $\widehat{\lambda}_{\alpha}^{0(3+8)}, \widehat{\lambda}_{\alpha}^{1}, \widehat{\lambda}_{\alpha}^{2}, \widehat{\lambda}_{\alpha}^{04}, \widehat{\lambda}_{\alpha}^{05}, \widehat{\lambda}_{\alpha}^{0(3-8)}, \widehat{\lambda}_{\alpha}^{06}, \widehat{\lambda}_{\alpha}^{07}, \widehat{\lambda}_{\alpha}^{08}, \widehat{\lambda}^{1}, \widehat{\lambda}^{2}, \widehat{\lambda}^{3}, \widehat{\lambda}, \widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3, \widehat{\lambda}_4, \widehat{\lambda}_5$ and $\widehat{\lambda}_6 \in \mathbb{R}$. Similarly, for leptons the anticommutator $\{\pi_l(a_1,a_3),\pi_l(a_1,a_3)\}$ takes the following form :

$$
\{\pi_l(a_1,a_3),\pi_l(a_1,a_3)\} = A_l + \Delta_l \tag{99}
$$

where

$$
A_l = 2\begin{pmatrix}
\widehat{\boldsymbol{\alpha}}'\widetilde{\lambda}_{\alpha'}^{0(3+8)}\otimes\mathbf{I}_3 & \widehat{\boldsymbol{\alpha}}'\left(\widetilde{\lambda}_{\alpha'}^{1}-i\widetilde{\lambda}_{\alpha'}^{2}\right)\otimes\mathbf{I}_3 & \widehat{\boldsymbol{\alpha}}'\left(\widetilde{\lambda}_{\alpha'}^{04}-i\widetilde{\lambda}_{\alpha'}^{05}\right)\otimes\mathbf{I}_3 & \\
\widehat{\boldsymbol{\alpha}}'\left(\widetilde{\lambda}_{\alpha'}^{1}+i\widetilde{\lambda}_{\alpha'}^{2}\right)\otimes\mathbf{I}_3 & \widehat{\boldsymbol{\alpha}}'\widetilde{\lambda}_{\alpha'}^{0(3-8)}\otimes\mathbf{I}_3 & \widehat{\boldsymbol{\alpha}}'\left(\widetilde{\lambda}_{\alpha}^{06}-i\widetilde{\lambda}_{\alpha'}^{07}\right)\otimes\mathbf{I}_3 & \mathbf{0}_9 \\
\widehat{\boldsymbol{\alpha}}\left(\widetilde{\lambda}_{\alpha'}^{04}+i\widetilde{\lambda}_{\alpha'}^{05}\right)\otimes\mathbf{I}_3 & \widehat{\boldsymbol{\alpha}}'\left(\widetilde{\lambda}_{\alpha'}^{06}+i\widetilde{\lambda}_{\alpha'}^{07}\right)\otimes\mathbf{I}_3 & 4\widehat{\boldsymbol{\alpha}}'\widetilde{\lambda}_{\alpha'}^{08}\otimes\mathbf{I}_3 & \\
 & \mathbf{0}_9 & & \mathbf{0}_9
\end{pmatrix} \tag{100}
$$

and

$$
\Delta_l = \begin{pmatrix}
\left(\widetilde{\lambda}^{1}+\widehat{\boldsymbol{\alpha}}'^{2}\widetilde{\lambda}\right)\otimes\mathbf{I}_3 & \left(\widetilde{\lambda}_1-i\widetilde{\lambda}_2\right)\otimes\mathbf{I}_3 & \left(\widetilde{\lambda}_3-i\widetilde{\lambda}_4\right)\otimes\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\left(\widetilde{\lambda}_1+i\widetilde{\lambda}_2\right)\otimes\mathbf{I}_3 & \left(\widetilde{\lambda}^{2}+\widehat{\boldsymbol{\alpha}}'^{2}\widetilde{\lambda}\right)\otimes\mathbf{I}_3 & \left(\widetilde{\lambda}_5-i\widetilde{\lambda}_6\right)\otimes\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\left(\widetilde{\lambda}_3+i\widetilde{\lambda}_4\right)\otimes\mathbf{I}_3 & \left(\widetilde{\lambda}_5+i\widetilde{\lambda}_6\right)\otimes\mathbf{I}_3 & \left(\widetilde{\lambda}^{3}+\widehat{\boldsymbol{\alpha}}'^{2}\widetilde{\lambda}\right)\otimes\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \widehat{\boldsymbol{\beta}}'^{2}\widetilde{\lambda}\otimes\mathbf{I}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \widehat{\boldsymbol{\gamma}}'^{2}\widetilde{\lambda}\otimes\mathbf{I}_3 & \mathbf{0}_3 \\
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \widehat{\boldsymbol{\delta}}'^{2}\widetilde{\lambda}\otimes\mathbf{I}_3
\end{pmatrix} \tag{101}
$$

Here $\widetilde{\lambda}_{\alpha}^{0(3+8)}, \widetilde{\lambda}_{\alpha}^{0(3-8)}, \widetilde{\lambda}_{\alpha}^{08}, \widetilde{\lambda}_{\alpha}^{04}, \widetilde{\lambda}_{\alpha}^{05}, \widetilde{\lambda}_{\alpha}^{06}, \widetilde{\lambda}_{\alpha}^{07}, \widetilde{\lambda}_{\alpha}^{1}, \widetilde{\lambda}_{\alpha}^{2}, \widetilde{\lambda}^{1}, \widetilde{\lambda}^{2}, \widetilde{\lambda}^{3}, \widetilde{\lambda}, \widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\lambda}_3, \widetilde{\lambda}_4, \widetilde{\lambda}_5$ and $\widetilde{\lambda}_6 \in \mathbb{R}$. For generic masse matrices M_{Q_1}, M_{Q_2}, M_{Q_3}, M_{L_1}, M_{L_2}, M_{L_3}(see eqs.(49), (50)), eqs.(16), (17) and(18) have the solution $\mathbf{j}^0 a = 0$ and $\mathbf{j}^1 a = 0$ and

$$
\begin{aligned}
\mathbf{j}^2 a &= \widehat{\pi}\left(\Im^2 a\right) \oplus \left(\{\widehat{\pi}(a),\widehat{\pi}(a)\} + diag\left(\mathbb{R}\mathbf{I}_{18},\mathbb{R}\mathbf{I}_{18}\right)\right) && (102)\\
&\ni j_2 \oplus diag\left(A_Q+\Delta_Q, A_l+\Delta_l\right) \oplus diag\left(J_Q, J_l\right) && (103)
\end{aligned}
$$

where J_Q and J_l have the form:

$$J_Q = diag\left(\left(\begin{array}{c} \lambda_1+\widehat{\boldsymbol{\alpha}}^2\lambda_0, \lambda_2+\widehat{\boldsymbol{\alpha}}^2\lambda_0, \lambda_3+\widehat{\boldsymbol{\alpha}}^2\lambda_0, \lambda_1+\lambda_8+\widehat{\boldsymbol{\beta}}^2\lambda_0, \\ \lambda_2+\lambda_8+\widehat{\boldsymbol{\gamma}}^2\lambda_0, \lambda_3+\lambda_8+\widehat{\boldsymbol{\delta}}^2\lambda_0 \end{array}\right)\otimes \mathbf{I}_3\right) \tag{104}$$

and

$$J_l = diag\left(\left(\begin{array}{c} \nu_1+\widehat{\boldsymbol{\alpha}}'^2\lambda_0, \nu_2+\widehat{\boldsymbol{\alpha}}'^2\lambda_0, \nu_3+\widehat{\boldsymbol{\alpha}}'^2\lambda_0, \nu_1+\nu_8+\widehat{\boldsymbol{\beta}}'^2\lambda_0, \\ \nu_2+\nu_8+\widehat{\boldsymbol{\gamma}}'^2\lambda_0, \nu_3+\nu_8+\widehat{\boldsymbol{\delta}}'^2\lambda_0 \end{array}\right)\otimes \mathbf{I}_3\right) \tag{105}$$

for $j_2 \in \widehat{\pi}\left(\Im^2\boldsymbol{a}\right)$and $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_8, \nu_1, \nu_2, \nu_3 \in \mathbb{R}$. We remind that in the analysis of $\widehat{\pi}\left(\Im^2\boldsymbol{a}\right)$, we must find the space of elements $\widehat{\sigma}\left(\omega^1\right)$, where $\omega^1 \in \Omega^1\mathbf{a}\cap \mathbf{ker}\widehat{\pi}$.For the factorization and elimination of the junks forms, the problem consists of solving the eq.(23) which is equivalent to finding for each given $\tau^2 \in \widehat{\pi}\left(\Omega^2\boldsymbol{a}\right)$ an element $j \in J = j_2 \oplus diag\left(A_Q+\Delta_Q, A_l+\Delta_l\right) \oplus diag\left(J_Q, J_l\right)$ such that

$$Trj'^*\left(\tau^2+j\right) = 0, \forall j \in J \tag{106}$$

Notice that, since J is block diagonal, the off-diagonal blocks $\tau^2_{i,j}$ do not contribute to the trace in eq.(105). Straightforward but lengthy calculation using Mapple package gives the following constraints:

$$\begin{aligned}
\lambda_1 + x_1\widehat{\boldsymbol{\alpha}} + \lambda_0\widehat{\boldsymbol{\alpha}}^2 &= \frac{2}{9}\left(k_{12}k_{12}^* + k_{13}^*k_{13} + k_{11}^*k_{11}\right)TrM_{Q_1}M_{Q_1}^* \\
\lambda_2 + x_2\widehat{\boldsymbol{\alpha}} + \lambda_0\widehat{\boldsymbol{\alpha}}^2 &= \frac{2}{9}\left(k_{12}k_{12}^* + k_{23}^*k_{23} + k_{22}^*k_{22}\right)TrM_{Q_2}M_{Q_2}^* \\
\lambda_3 + 4x_3\widehat{\boldsymbol{\alpha}} + \lambda_0\widehat{\boldsymbol{\alpha}}^2 &= \frac{2}{9}\left(k_{13}^*k_{13} + k_{23}^*k_{23} + k_{33}^*k_{33}\right)TrM_{Q_3}M_{Q_3}^*
\end{aligned}$$

$$\begin{aligned}
z_1 + y_1\widehat{\boldsymbol{\alpha}} &= \frac{1}{9}\left(-k_{23}^*k_{13} + k_{12}^*\left(-k_{22}+k_{11}^*\right)\right)TrM_1 \\
z_2 + y_2\widehat{\boldsymbol{\alpha}} &= \frac{1}{9}\left(k_{12}^*k_{23}^* + k_{13}^*\left(-k_{33}+k_{11}^*\right)\right)TrM_2 \\
z_3 + y_3\widehat{\boldsymbol{\alpha}} &= \frac{1}{9}\left(-k_{12}k_{13}^* + k_{23}^*\left(-k_{33}+k_{22}^*\right)\right)TrM_3
\end{aligned}$$

$$\begin{aligned}
\lambda_1 + \lambda_8 + \lambda_0\widehat{\boldsymbol{\beta}}^2 &= \frac{1}{9}\left(k_{12}k_{12}^* + k_{13}^*k_{13} + k_{11}^*k_{11}\right)TrM_{Q_1}M_{Q_1}^* \qquad (107) \\
\lambda_2 + \lambda_8 + \lambda_0\widehat{\boldsymbol{\gamma}}^2 &= \frac{1}{9}\left(k_{12}k_{12}^* + k_{23}^*k_{23} + k_{22}^*k_{22}\right)TrM_{Q_2}M_{Q_2}^* \\
\lambda_3 + \lambda_8 + \lambda_0\widehat{\boldsymbol{\delta}}^2 &= \frac{1}{9}\left(k_{13}^*k_{13} + k_{23}^*k_{23} + k_{33}^*k_{33}\right)TrM_{Q_3}M_{Q_3}^*
\end{aligned}$$

$$
\begin{aligned}
\nu_1 + x_1\widehat{\boldsymbol{\alpha}}' + \lambda_0\widehat{\boldsymbol{\alpha}}'^2 &= \frac{2}{9}\left(k'_{12}k'^*_{12} + k'^*_{13}k'_{13} + k'^*_{11}k'_{11}\right)TrM_{L_1}M^*_{L_1} \\
\nu_2 + x_2\widehat{\boldsymbol{\alpha}}' + \lambda_0\widehat{\boldsymbol{\alpha}}'^2 &= \frac{2}{9}\left(k'_{12}k'^*_{12} + k'^*_{23}k'_{23} + k'^*_{22}k'_{22}\right)TrM_{L_2}M^*_{L_2} \\
\nu_3 + 4x_3\widehat{\boldsymbol{\alpha}}' + \lambda_0\widehat{\boldsymbol{\alpha}}'^2 &= \frac{2}{9}\left(k'^*_{13}k'_{13} + k'^*_{23}k'_{23} + k'^*_{33}k'_{33}\right)TrM_{L_3}M^*_{L_3}
\end{aligned}
$$

$$
\begin{aligned}
z_1 + y_1\widehat{\boldsymbol{\alpha}}' &= \frac{1}{9}\left(k'^*_{12}\left(-k'^*_{11} + k'_{22}\right) + k'^*_{23}k'_{13}\right)TrW_1 \\
z_2 + y_2\widehat{\boldsymbol{\alpha}}' &= \frac{1}{9}\left(k'^*_{13}\left(-k'^*_{11} + k'_{33}\right) + k'^*_{12}k'^*_{23}\right)TrW_2 \\
z_3 + y_3\widehat{\boldsymbol{\alpha}}' &= \frac{1}{9}\left(k'^*_{23}\left(-k'^*_{22} + k'_{33}\right) + k'_{12}k'^*_{13}\right)TrW_3
\end{aligned}
$$

$$
\begin{aligned}
\nu_1 + \nu_8 + \lambda_0\widehat{\boldsymbol{\beta}}'^2 &= \frac{1}{9}\left(k'_{12}k'^*_{12} + k'^*_{13}k'_{13} + k'^*_{11}k'_{11}\right)TrM_{L_1}M^*_{L_1} \\
\nu_2 + \nu_8 + \lambda_0\widehat{\boldsymbol{\gamma}}'^2 &= \frac{1}{9}\left(k'_{12}k'^*_{12} + k'^*_{23}k'_{23} + k'^*_{22}k'_{22}\right)TrM_{L_2}M^*_{L_2} \\
\nu_3 + \nu_8 + \lambda_0\widehat{\boldsymbol{\delta}}'^2 &= \frac{1}{9}\left(k'^*_{13}k'_{13} + k'^*_{23}k'_{23} + k'^*_{33}k'_{33}\right)TrM_{L_3}M^*_{L_3}
\end{aligned}
$$

where:

$$
\begin{aligned}
x_1 &= \widehat{\lambda}^{0(3+8)}_{\alpha}, x_2 = \widehat{\lambda}^{0(3-8)}_{\alpha}, x_3 = \widehat{\lambda}^{08}_{\alpha} \\
y_1 &= \widehat{\lambda}^{1}_{\alpha} + i\widehat{\lambda}^{2}_{\alpha}, y_2 = \widehat{\lambda}^{04}_{\alpha} + i\widehat{\lambda}^{05}_{\alpha}, y_3 = \widehat{\lambda}^{06}_{\alpha} + i\widehat{\lambda}^{07}_{\alpha} \\
z_1 &= \widehat{\lambda}_1 + i\widehat{\lambda}_2, z_2 = \widehat{\lambda}_3 + i\widehat{\lambda}_4, z_3 = \widehat{\lambda}_5 + i\widehat{\lambda}_6 \\
M_1 &= M_{\{Q_2Q_1\}}, M_2 = M_{\{Q_1Q_3\}}, M_3 = M_{\{Q_3Q_2\}} \\
W_1 &= M_{\{L_2L_1\}}, W_2 = M_{\{L_1L_3\}}, W_3 = M_{\{L_3L_2\}} \qquad (108)
\end{aligned}
$$

Thus, we have found $j \in J = j_2 \oplus diag\left(A_Q + \Delta_Q, A_l + \Delta_l\right) \oplus diag\left(J_Q, J_l\right)$ or equivalently where the relations of eqs.(106) hold such that the representative $e\left(\theta\right)$ is ortogonal to $J \equiv \mathbf{J}^2 g$ and the junks forms are eliminated. It is wotrth to mention and since we are dealing with the same scalar and vector bosons while considering quarks or leptons, one has to have for consistency:

$$
\begin{aligned}
\widehat{\lambda}^{0(3+8)}_{\alpha} &= \widetilde{\lambda}^{0(3+8)}_{\alpha}, \widehat{\lambda}^{0(3-8)}_{\alpha} = \widetilde{\lambda}^{0(3-8)}_{\alpha}, \widehat{\lambda}^{08}_{\alpha} = \widetilde{\lambda}^{08}_{\alpha} \qquad (109) \\
\widehat{\lambda}^{1}_{\alpha} + i\widehat{\lambda}^{2}_{\alpha} &= \widetilde{\lambda}^{1}_{\alpha} + i\widetilde{\lambda}^{2}_{\alpha}, \widehat{\lambda}^{04}_{\alpha} + i\widehat{\lambda}^{05}_{\alpha} = \widetilde{\lambda}^{04}_{\alpha} + i\widetilde{\lambda}^{05}_{\alpha}, \widehat{\lambda}^{06}_{\alpha} + i\widehat{\lambda}^{07}_{\alpha} = \widetilde{\lambda}^{06}_{\alpha} + i\widetilde{\lambda}^{07}_{\alpha} \\
\widehat{\lambda}_1 + i\widehat{\lambda}_2 &= \widetilde{\lambda}_1 + i\widetilde{\lambda}_2, \widehat{\lambda}_3 + i\widehat{\lambda}_4 = \widetilde{\lambda}_3 + i\widetilde{\lambda}_4, \widehat{\lambda}_5 + i\widehat{\lambda}_6 = \widehat{\lambda}_5 + i\widehat{\lambda}_6
\end{aligned}
$$

3.2.1 Bosonic action

Now, if we denote by:

$$\widetilde{M}_{Q_iQi} = M_{Q_i}M^*_{Q_i} - \frac{1}{3}trM_{Q_i}M^*_{Q_i} \tag{110}$$

$$\widetilde{M}_{L_iL_i} = M_{L_i}M^*_{Li} - \frac{1}{3}trM_{L_i}M^*_{L_i}$$

and after use of the relations of eqs.(106), the expression of the matrix ρ (see eqs.(81) – (91)) and a Wick rotation, one can show that the bosonic action S_B takes the form:

$$S_B = \frac{1}{36g_0^2}\int_X dx\, tr\left(e(\theta)^2\right) = \frac{1}{36g_0^2}\int_X dx\ (\pounds_0 + \pounds_1 + \pounds_2) \tag{111}$$

Here g_0 is the $U(1)_N$ gauge coupling constant. After straightforward but tedious

simplifications, we obtain the following expressions:

$$\begin{aligned}
(\pounds_0) \quad = \quad & \frac{1}{18g_0^2}\{Tr\left(\left(\widetilde{M}_{Q_1Q1}\right)^2 + \left(\widetilde{M}_{L_1L_1}\right)^2\right)[\overline{\phi}_1\phi_1 + \overline{\phi}_2\phi_2 + \left(\overline{\phi}_4 - 1\right)(\phi_4 - 1)]^2) \quad (112)\\
& +2Tr\left(\left(\widetilde{M}_{Q_1Q1}\widetilde{M}_{Q_2Q_2} + \widetilde{M}_{L_1L_1}\widetilde{M}_{L_2L_2}\right)\right)\left|\left(\phi_4 - \overline{\phi}_5\right)\phi_1 - \overline{\phi}_2\overline{\phi}_3\right|^2\\
& +Tr\left(\left(\left(\widetilde{M}_{Q_2Q_2}\right)^2 + \left(\widetilde{M}_{L_2L_2}\right)^2\right)\right)\left(\overline{\phi}_1\phi_1 + \overline{\phi}_3\phi_3 + \left(\overline{\phi}_5 - 1\right)(\phi_5 - 1)\right)^2\\
& +2Tr\left(\left(\widetilde{M}_{Q_3Q_3}\widetilde{M}_{Q_2Q_2} + \widetilde{M}_{L_2L_2}\widetilde{M}_{L_3L_3}\right)\right)\left|\left(\phi_5 - \overline{\phi}_6\right)\phi_3 - \overline{\phi}_2\overline{\phi}_1\right|^2\\
& +Tr\left(\left(\widetilde{M}_{Q_3Q_3}\right)^2 + \left(\widetilde{M}_{L_3L_3}\right)^2\right)\left(\overline{\phi}_2\phi_2 + \overline{\phi}_3\phi_3 + \left(\overline{\phi}_6 - 1\right)(\phi_6 - 1)\right)^2\\
& +2Tr\left(\left(\widetilde{M}_{Q_3Q_3}\widetilde{M}_{Q_1Q1} + \widetilde{M}_{L_3L_3}\widetilde{M}_{L_1L_1}\right)\right)\left|\left(\phi_6 - \overline{\phi}_4\right)\phi_2 - \overline{\phi}_3\overline{\phi}_1\right|^2\}
\end{aligned}$$

$$\begin{aligned}
(\pounds_1) \quad = \quad & \frac{1}{18g_0^2}\{[tr\left(\left|d\phi_1 + (-i((\beta-\alpha)A_0 - A_{38}))\phi_1 + iA_{-67}\overline{\phi}_2 + iA_{12}(\phi_4+1)\right|^2\right) \quad (113)\\
& +tr(\left|d\phi_2 + (-i(2A_8 + (\beta-\alpha)A_0))\phi_2 + iA_{-67}\overline{\phi}_1 + iA_{-45}\left(\overline{\phi}_4+1\right)\right|^2)\\
& +tr(\left|d\phi_4 + iA_{-45}\overline{\phi}_2 + iA_{-12}\phi_1 + (-iA_{38} + (\beta-\alpha)A_0)(\phi_4+1)\right|^2]Tr\left(M_{\{Q_1Q_1\}} + M_{\{L_1L_1\}}\right)\\
& +[tr(\left|d\phi_1 + (-i((\gamma-\alpha)A_0 - A_{38}))\phi_1 + \overline{\phi}_3(-iA_{45}) + iA_{12}\left(\overline{\phi}_5+1\right)\right|^2)\\
& +tr(\left|d\phi_3 - iA_{45}\overline{\phi}_1 + (-i(2A_8 + (\gamma-\alpha)A_0))\phi_3 - iA_{67}(\phi_5+1)\right|^2)\\
& +tr(\left|d\phi_5 + iA_{12}\overline{\phi}_1 + iA_{-67}\phi_3 + (i(A_{-38} - (\gamma-\alpha)A_0))(\phi_5+1)\right|^2)]Tr\left(M_{\{Q_2Q_2\}} + M_{\{L_2L_2\}}\right)\\
& +[tr\left(|d\phi_6 + iA_{45}\phi_2 + iA_{67}\phi_3 + (i((\delta-\alpha)A_0 - 2A_8))(\phi_6+1)|^2\right)\\
& +tr(\left|d\phi_3 + iA_{-12}\overline{\phi}_2 + (i(A_{-38} + (\delta-\alpha)A_0))\phi_3 + iA_{67}\left(\overline{\phi}_6+1\right)\right|^2)\\
& +tr(\left|d\phi_2 + (i((\delta-\alpha)A_0 + A_{38}))\phi_2 + iA_{-12}\overline{\phi}_3 + iA_{-45}(\phi_6+1)\right|^2)]Tr\left(M_{\{Q_3Q_3\}} + M_{\{L_3L_3\}}\right)
\end{aligned}$$

and

$$(\pounds_2) \quad = \quad \frac{1}{3g_0^2} tr\left[\left(dA_3 + +\frac{1}{2}\{A_3, A_3\}\right)^2 + \left(dA_8 + +\frac{1}{2}\{A_8, A_8\}\right)^2\right] + +\frac{1}{36g_0^2} xtr\,(dA_0)^2$$
$$+tr\left(\sum_{i=1}^{6} d\,(A_i \pm iA_{i+1}) + \frac{1}{2}\sum_{i=1}^{6}\{(A_i \pm iA_{i+1}), (A_i \pm iA_{i+1})\}\right)^2 \qquad (114)$$

where

$$x = \left(3\alpha^j + \beta^j + \gamma^j + \delta^j\right)^2 = \sum (N \text{ charges})^2, \quad \forall j = 1, 3 \qquad (115)$$

3.2.2 Fermionic action

In what follows, the quarks (resp. leptons) wave functions Ψ_{Q^j} (resp.Ψ_{L^j}) are

denoted by:

$$\Psi_{Q^j} : (Q^j_{1L}, Q^j_{2L}, Q^j_{3L}, Q^j_{1R}, Q^j_{2R}, Q^j_{3R}) \qquad (116)$$

and

$$\Psi_{L^j} (L^j_{1L}, L^j_{2L}, L^j_{3L}, L^j_{1R}, L^j_{2R}, L^j_{3R}) \qquad (117)$$

After a Wick rotation and making the following redefinitions for the gauge fields:

$$A_0 = \frac{ig_0}{\sqrt{x}} \gamma^\mu W^0_\mu \qquad (118)$$

$$A_{\pm 12} = A_1 \pm A_2 = ig\gamma^\mu (W^1_\mu \mp iW^2_\mu) = ig\gamma^\mu W^\pm_\mu \qquad (119)$$

$$A_{\pm 45} = A_4 \pm iA_5 = ig\gamma^\mu (W^4_\mu \mp iW^5_\mu) = ig\gamma^\mu V^\pm_\mu \qquad (120)$$

$$A_{\pm 67} = A_6 \pm A_7 = ig\gamma^\mu (-W^6_\mu \mp iW^7_\mu) = ig\gamma^\mu U^{\pm\pm}_\mu \qquad (121)$$

$$A_8 = ig\gamma^\mu W^8_\mu \qquad (122)$$

and

$$A_3 = ig\gamma^\mu W^3_\mu \qquad (123)$$

we get the following fermionic interactions (the vector gauge bosons and scalar fields (Yukawa terms) are denoted by $\mathcal{L}_{int}$ and $\mathcal{L}_{Yukawa}$ respectively): For quarks,

$$\begin{aligned}\mathcal{L}_{int} &= \mathcal{L}^{Q}_{int} = \sum_{j=1}^{3}\{\frac{-ig_0}{\sqrt{x}}[\overline{Q}^{j}_{1R}\beta^{j}\gamma^{\mu}W^{0}_{\mu}Q^{j}_{1R} + \overline{Q}^{j}_{2R}\gamma^{j}\gamma^{\mu}W^{0}_{\mu}Q^{j}_{2R} + \overline{Q}^{j}_{3R}\delta^{j}\gamma^{\mu}W^{0}_{\mu}Q^{j}_{3R}] \\ &+ig[-\overline{Q}^{j}_{1L}\gamma^{\mu}\left(W^{3}_{\mu} + W^{8}_{\mu} + \alpha^{j}W^{0}_{\mu}\right)Q^{j}_{1L} + \overline{Q}^{j}_{2L}\gamma^{\mu}\left(W^{3}_{\mu} - W^{8}_{\mu} - \alpha^{j}W^{0}_{\mu}\right)Q^{j}_{2L} \\ &+\overline{Q}^{j}_{3L}\gamma^{\mu}\left(2W^{8}_{\mu} - \alpha^{j}W^{0}_{\mu}\right)Q^{j}_{3L} + \overline{Q}^{j}_{1R}\gamma^{\mu}W^{-}_{\mu}Q^{j}_{2L} - \overline{Q}^{j}_{2R}\gamma^{\mu}W^{+}_{\mu}Q^{j}_{1L} \\ &-\overline{Q}^{j}_{3R}\gamma^{\mu}V^{-}_{\mu}Q^{j}_{1L} + \overline{Q}^{j}_{1R}\gamma^{\mu}V^{+}_{\mu}Q^{j}_{3L} + \overline{Q}^{j}_{3R}\gamma^{\mu}U^{--}_{\mu}Q^{j}_{2L} \\ &-\overline{Q}^{j}_{2R}\gamma^{\mu}U^{++}_{\mu}Q^{j}_{3L}]\}\end{aligned} \tag{124}$$

and

$$\begin{aligned}\mathcal{L}_{Yukawa} &= \mathcal{L}^{Q}_{Yukawa} = \frac{1}{\sqrt{2}}\sum_{j}[\frac{1}{\sqrt{Tr\left(M_{\{Q_1Q_1\}} + M_{\{L_1L_1\}}\right)}}m_{Q^{j}_{1}}\left(\overline{Q}^{j}_{1L}\phi_4 Q^{j}_{1R} - \overline{Q}^{j}_{2L}\phi_1 Q^{j}_{1R} - \overline{Q}^{j}_{3L}\phi^{*}_{2}Q^{j}_{1R} + c.c\right) \\ &+\frac{1}{\sqrt{Tr\left(M_{\{Q_2Q_2\}} + M_{\{L_2L_2\}}\right)}}m_{Q^{j}_{2}}\left(\overline{Q}^{j}_{1L}\phi^{*}_{1}Q^{j}_{2R} - \overline{Q}^{j}_{2L}\phi_5 Q^{j}_{2R} - \overline{Q}^{j}_{3L}\phi_3 Q^{j}_{2R} + c.c\right) \\ &+\frac{1}{\sqrt{Tr\left(M_{\{Q_3Q_3\}} + M_{\{L_3L_3\}}\right)}}m_{Q^{j}_{3}}\left(-\overline{Q}^{j}_{1L}\phi_2 Q^{j}_{3R} - \overline{Q}^{j}_{2L}\phi^{*}_{3}Q^{j}_{3R} + \overline{Q}^{j}_{3L}\phi_6 Q^{j}_{3R} + c.c\right)]\end{aligned} \tag{125}$$

Here '$c.c$' means complex conjugate and g denotes the $SU(3)_L$ gauge couplings constant. Similarly for leptons one has:

$$\begin{aligned}&\mathcal{L}_{int} = \mathcal{L}^{L}_{int} = \sum_{j=1}^{3}\{\frac{-ig_0}{\sqrt{x}}[-\overline{L}^{j}_{2R}\gamma^{\mu}W^{0}_{\mu}L^{j}_{2R} + \overline{L}^{j}_{3R}\gamma^{\mu}W^{0}_{\mu}L^{j}_{3R}] + ig[-\overline{L}^{j}_{1L}\gamma^{\mu}\left(W^{3}_{\mu} + W^{8}_{\mu}\right)L^{j}_{1L} \\ &+\overline{L}^{j}_{2L}\gamma^{\mu}\left(W^{3}_{\mu} - W^{8}_{\mu}\right)L^{j}_{2L} + 2\overline{L}^{j}_{3L}\gamma^{\mu}W^{8}_{\mu}L^{j}_{3L} + \overline{L}^{j}_{1R}\gamma^{\mu}W^{-}_{\mu}L^{j}_{2L} - \overline{L}^{j}_{2R}\gamma^{\mu}W^{+}_{\mu}L^{j}_{1L} \\ &-\overline{L}^{j}_{3R}\gamma^{\mu}V^{+}_{\mu}L^{j}_{1L} + \overline{L}^{j}_{1R}\gamma^{\mu}V^{-}_{\mu}L^{j}_{3L} + \overline{L}^{j}_{3R}\gamma^{\mu}U^{++}_{\mu}L^{j}_{2L} - \overline{L}^{j}_{2R}\gamma^{\mu}U^{--}_{\mu}L^{j}_{3L}]\}\end{aligned} \tag{126}$$

and

$$\begin{aligned}\mathcal{L}_{Yukawa} &= \mathcal{L}^{L}_{Yukawa} = \frac{1}{\sqrt{2}}\sum_{j}[\frac{m_{L^{j}_{1}}}{\sqrt{Tr\left(M_{\{Q_1Q_1\}} + M_{\{L_1L_1\}}\right)}}\left(\overline{L}^{j}_{1L}\phi_4 L^{j}_{1R} - \overline{L}^{j}_{2L}\phi_1 L^{j}_{1R} - \overline{L}^{j}_{3L}\phi^{*}_{2}L^{j}_{1R} + c.c\right) \\ &+\frac{m_{L^{j}_{2}}}{\sqrt{Tr\left(M_{\{Q_2Q_2\}} + M_{\{L_2L_2\}}\right)}}\left(\overline{L}^{j}_{1L}\phi^{*}_{1}L^{j}_{2R} - \overline{L}^{j}_{2L}\phi_5 L^{j}_{2R} - \overline{L}^{j}_{3L}\phi_3 L^{j}_{2R} + c.c\right) \\ &+\frac{m_{L^{j}_{3}}}{\sqrt{Tr\left(M_{\{Q_3Q_3\}} + M_{\{L_3L_3\}}\right)}}\left(-\overline{L}^{j}_{1L}\phi_2 L^{j}_{3R} - \overline{L}^{j}_{2L}\phi^{*}_{3}L^{j}_{3R} + \overline{L}^{j}_{3L}\phi_6 L^{j}_{3R} + c.c\right)]\end{aligned} \tag{127}$$

4 NAS mass, couplings and mixing angles

4.1 Scalar bosons

In order to read correctly the lagrangian density of eq.(111), let us first make the following scalar fields redefinitions:

$$\Phi_i = \frac{3}{\sqrt{2}} g_0 \Omega_{\widetilde{\Phi}_i} \widetilde{\Phi}_i, \quad i = \overline{1,3} \tag{128}$$

and

$$\Phi_i - 1 = \frac{3}{\sqrt{2}} g_0 \Omega_{\widetilde{\Phi}_i} \widetilde{\Phi}_i, \quad i = \overline{4,5} \tag{129}$$

If we set:

$$\begin{aligned}
\Omega_{\widetilde{\Phi}_1} &= \Omega_{\overline{\widetilde{\Phi}}_1} = \left[Tr\left(M_{\{Q_1Q_1\}} + M_{\{Q_2Q_2\}} + M_{\{L_1L_1\}} + M_{\{L_2L_2\}}\right)\right]^{\frac{-1}{2}} \\
\Omega_{\widetilde{\Phi}_2} &= \Omega_{\overline{\widetilde{\Phi}}_2} = \left[Tr\left(M_{\{Q_1Q_1\}} + M_{\{Q_3Q_3\}} + M_{\{L_1L_1\}} + M_{\{L_3L_3\}}\right)\right]^{\frac{-1}{2}} \\
\Omega_{\widetilde{\Phi}_3} &= \Omega_{\overline{\widetilde{\Phi}}_3} = \left[Tr\left(M_{\{Q_2Q_2\}} + M_{\{Q_3Q_3\}} + M_{\{L_2L_2\}} + M_{\{L_3L_3\}}\right)\right]^{\frac{-1}{2}} \\
\Omega_{\widetilde{\Phi}_4} &= \Omega_{\overline{\widetilde{\Phi}}_4} = \left[Tr\left(M_{\{Q_1Q_1\}} + M_{\{L_1L_1\}}\right)\right]^{\frac{-1}{2}} \\
\Omega_{\widetilde{\Phi}_5} &= \Omega_{\overline{\widetilde{\Phi}}_5} = \left[Tr\left(M_{\{Q_2Q_2\}} + M_{\{L_2L_2\}}\right)\right]^{\frac{-1}{2}} \\
\Omega_{\widetilde{\Phi}_6} &= \Omega_{\overline{\widetilde{\Phi}}_6} = \left[Tr\left(M_{\{Q_3Q_3\}} + M_{\{L_3L_3\}}\right)\right]^{\frac{-1}{2}}
\end{aligned} \tag{130}$$

then, the lagrangian density $\mathcal{L}_0$ of eq.(111) takes the following form:

$$\mathcal{L}_0 = \frac{9}{8} g_0^2 \sum_{j=1}^{6} \Theta_j \tag{131}$$

where

$$\begin{aligned}
\Theta_1 &= \varpi_1 \left(\chi(\widetilde{\Phi}_1, \overline{\widetilde{\Phi}}_1) + \frac{9}{2}\chi(\widetilde{\Phi}_2, \overline{\widetilde{\Phi}}_2) + \chi(\widetilde{\Phi}_4, \overline{\widetilde{\Phi}}_4)\right)^2 \\
\Theta_2 &= 2\varpi_2 \left|\chi(\widetilde{\Phi}_4, \widetilde{\Phi}_1) - \chi(\overline{\widetilde{\Phi}}_5, \widetilde{\Phi}_1) - \chi(\overline{\widetilde{\Phi}}_2, \overline{\widetilde{\Phi}}_3)\right|^2 \\
\Theta_3 &= \varpi_3 \left(\chi(\widetilde{\Phi}_1, \overline{\widetilde{\Phi}}_1) + \chi(\widetilde{\Phi}_3, \overline{\widetilde{\Phi}}_3) + \chi(\widetilde{\Phi}_5, \overline{\widetilde{\Phi}}_5)\right)^2 \\
\Theta_4 &= 2\varpi_4 \left|\chi(\widetilde{\Phi}_5, \widetilde{\Phi}_3) - \chi(\overline{\widetilde{\Phi}}_6, \widetilde{\Phi}_3) - \chi(\overline{\widetilde{\Phi}}_2, \overline{\widetilde{\Phi}}_1)\right|^2 \\
\Theta_5 &= \varpi_5 \left(\chi(\widetilde{\Phi}_2, \overline{\widetilde{\Phi}}_2) + \chi(\widetilde{\Phi}_3, \overline{\widetilde{\Phi}}_3) + \chi(\widetilde{\Phi}_6, \overline{\widetilde{\Phi}}_6)\right)^2 \\
\Theta_6 &= 2\varpi_6 \left|\chi(\widetilde{\Phi}_6, \widetilde{\Phi}_2) - \chi(\overline{\widetilde{\Phi}}_4, \widetilde{\Phi}_2) - \chi(\widetilde{\Phi}_3, \widetilde{\Phi}_1)\right|^2
\end{aligned} \tag{132}$$

$$\chi(A,B) = \Omega_A \Omega_B A.B \tag{133}$$

and

$$\begin{aligned}
\varpi_1 &= \Omega^{-2}_{\widetilde{\Phi}_4} \\
\varpi_2 &= Tr\left(\widetilde{M}_{Q_1Q_1}\widetilde{M}_{Q_2Q_2} + \widetilde{M}_{L_1L_1}\widetilde{M}_{L_2L_2}\right) \\
\varpi_3 &= \Omega^{-2}_{\widetilde{\Phi}_5} \\
\varpi_4 &= Tr\left(\widetilde{M}_{Q_2Q_2}\widetilde{M}_{Q_3Q_3} + \widetilde{M}_{L_2L_2}\widetilde{M}_{L_3L_3}\right) \\
\varpi_5 &= \Omega^{-2}_{\widetilde{\Phi}_6} \\
\varpi_6 &= Tr\left(\widetilde{M}_{Q_3Q_3}\widetilde{M}_{Q_1Q_1} + \widetilde{M}_{L_3L_3}\widetilde{M}_{L_1L_1}\right)
\end{aligned} \tag{134}$$

Notice that $\mathcal{L}_0$ looks like the minimal scalar field potential with two triplets Higgs bosons. If we choose the vaccum expectation values of the generic scalar fields as:

$$\begin{aligned}
\left\langle \widetilde{\Phi}_j \right\rangle &= 0, \qquad j = \overline{1,3} \\
\left\langle \widetilde{\Phi}_j \right\rangle &= v_{j-3}, \qquad j = \overline{4,6}
\end{aligned} \tag{135}$$

the usual analysis shows that this set of VeV breaks the symmetry in one single

step:

$$SU(3)_c \otimes SU(3)_L \otimes U(1)_N \rightarrow SU(3)_c \otimes U(1)_Q \tag{136}$$

For the particular value $v_2 = 0$ or $v_1 = 0$, the symmetry breaking chain becomes:

$$SU(3)_c \otimes SU(3)_L \otimes U(1)_N \rightarrow^{v_3} SU(3)_c \otimes SU(3)_L \otimes U(1)_N \rightarrow^{v_1 \, or \, v_2} SU(3)_c \otimes U(1)_Q \tag{137}$$

After the Higgs mechanism, the scalar bosons masses become:

$$\begin{aligned}
M^2_{\widetilde{\Phi}_1} &= \frac{9}{4} g_0^2 \Omega^2_{\widetilde{\Phi}_1} \left(\Omega^2_{\widetilde{\Psi}_1} v_1^2 \varpi_1 - 2\Omega_{\widetilde{\Psi}_1}\Omega_{\widetilde{\Psi}_2} v_1 v_2 \varpi_2 + \Omega^2_{\widetilde{\Psi}_2} v_2^2 \varpi_3 \right) \\
M^2_{\widetilde{\Phi}_2} &= \frac{9}{4} g_0^2 \Omega^2_{\widetilde{\Phi}_2} \left(\Omega^2_{\widetilde{\Psi}_1} v_1^2 \varpi_1 - 2\Omega_{\widetilde{\Psi}_1}\Omega_{\widetilde{\Psi}_3} v_1 v_3 \varpi_6 + \Omega^2_{\widetilde{\Psi}_3} v_3^2 \varpi_5 \right) \\
M^2_{\widetilde{\Phi}_3} &= \frac{9}{4} g_0^2 \Omega^2_{\widetilde{\Phi}_3} \left(\Omega^2_{\widetilde{\Psi}_2} v_2^2 \varpi_3 - 2\Omega_{\widetilde{\Psi}_2}\Omega_{\widetilde{\Psi}_3} v_2 v_3 \varpi_4 + 2\Omega^2_{\widetilde{\Psi}_3} v_3^2 \varpi_5 \right) \\
M^2_{\widetilde{\Psi}_1} &= \frac{27}{4} g_0^2 v_1^2 \Omega^2_{\widetilde{\Psi}_1} \\
M^2_{\widetilde{\Psi}_2} &= \frac{27}{4} g_0^2 v_2^2 \Omega^2_{\widetilde{\Psi}_2} \\
M^2_{\widetilde{\Psi}_3} &= \frac{27}{4} g_0^2 v_3^2 \Omega^2_{\widetilde{\Psi}_3}
\end{aligned} \tag{138}$$

4.2 Vector gauge fields

4.2.1 Charged gauge bosons

In the basis $\{W^{\pm}, V^{\pm}, U^{\pm\pm}\}$, and after the one step spontaneous symmetry breaking (SSB) and Higgs mechanism, the charged gauge bosons mass matrix M_{CGB} reads:

$$M_{CGB} = g^2 \begin{pmatrix} v_1^2 + v_2^2 & 0 & 0 \\ 0 & v_1^2 + v_3^2 & 0 \\ 0 & 0 & v_2^2 + v_3^2 \end{pmatrix} \tag{139}$$

and consequently, one deduces that:

$$M_{W^{\pm}}^2 = g^2 \left(v_1^2 + v_2^2\right) \tag{140}$$

$$M_{V^{\pm}}^2 = g^2 \left(v_1^2 + v_3^2\right) \tag{141}$$

and

$$M_{U^{\pm}}^2 = g^2 \left(v_2^2 + v_3^2\right) \tag{142}$$

4.2.2 Neutral gauge bosons

For the neutral gauge bosons, and after straightforward simplifications using eq.(112), one gets (after SSB) in the basis $\left\{W_\mu^0, W_\mu^3, W_\mu^8\right\}$the following symmetric non diagonal mass matrix M_{NGB}:

$$M_{NGB} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{pmatrix} \tag{143}$$

where

$$M_{11} = \frac{g^2}{x} \left[(\beta - \alpha)^2 v_1^2 + (\gamma - \alpha)^2 v_2^2 + (\delta - \alpha)^2 v_3^2\right] \tag{144}$$

$$M_{22} = g^2 \left[v_1^2 + v_2^2\right] \tag{145}$$

$$M_{33} = g^2 \left[v_1^2 + v_2^2 + v_3^2\right] \tag{146}$$

$$M_{12} = \frac{g g_0}{\sqrt{x}} \left[(\gamma - \alpha) v_2^2 + (\alpha - \beta) v_1^2\right] \tag{147}$$

$$M_{13} = \frac{-g g_0}{\sqrt{x}} \left[(\beta - \alpha) v_1^2 + (\gamma - \alpha) v_2 + 2 (\delta - \alpha) v_3^2\right] \tag{148}$$

and

$$M_{23} = g^2 \left[v_1^2 - v_2^2\right] \tag{149}$$

If we set $v_1 = v_2$ and in order to have a one vanishing eigenvalue (representing the rest mass square of the photon M_γ^2), one has to have the constraint:

$$-M_{12}^2 M_{33} - M_{13}^2 M_{22} + M_{11} M_{22} M_{33} = 0 \tag{150}$$

leading to the relations:

$$\delta - \alpha = \beta - \gamma \tag{151}$$

and

$$\alpha = \beta \tag{152}$$

The remaining non vanishing eingenvalues denoted by M_1^2 and M_2^2 which can be identified with the mass square of the Z' and Z^0 bosons respectively are given by:

$$M_1^2 = M_{Z'}^2 = \left[\frac{g_0^2}{x} (\gamma - \alpha)^2 + 4g^2 \right] v_3^2 \tag{153}$$

and

$$M_2^2 = M_{Z^0}^2 = \frac{4g^2 \left[\frac{3g_0^2}{x} (\gamma - \alpha)^2 + 2g^2 \right]}{\left[\frac{g_0^2}{x} (\gamma - \alpha)^2 + 4g^2 \right]} v_1^2 \tag{154}$$

Now, if we introduce a mixing angle θ such that:

$$\tan\theta = \frac{g_0}{g\sqrt{x}} \tag{155}$$

and if we require that the Z and $W^{\pm}$ gauge bosons masses are the same as that of the standard model, then, one can show easily that:

$$\tan^2\theta = \frac{4\sin^2\theta_w}{(\gamma - \alpha)^2 \ (6\cos^2\theta_w - 1)} \tag{156}$$

or equivalently

$$\cos\theta_w = \sqrt{\frac{1 + \frac{1}{4} (\gamma - \alpha)^2 \ \tan^2\theta}{1 + \frac{3}{2} (\gamma - \alpha)^2 \tan^2\theta}} \tag{157}$$

where θ_w is the Weinberg mixing angle. From the expression of eq.(155), one deduces that $\cos^2\theta_w \geq \frac{1}{6}$. This lower limit is compatible with the experimental value $\cos^2\theta_w \approx 0.76$. Notice also that if $\sin^2\theta_w \approx \frac{1}{4}$, one gets:

$$\tan\theta \approx \pm\frac{1}{|\gamma - \alpha|}\sqrt{\frac{2}{7}} \tag{158}$$

Regarding the eigenstates related to M_γ^2, $M_{Z'}^2$ and M_Z^2 eigenvalues, one can show that they are given by the following expressions:

$$B_\mu = \Im_B \left[\zeta_{11} W_\mu^0 + \zeta_{12} W_\mu^3 + \zeta_{13} W_\mu^8 \right] \tag{159}$$

$$Z'_\mu = \Im_{Z'} \left[\zeta_{21} W_\mu^0 + \zeta_{22} W_\mu^3 + \zeta_{23} W_\mu^8 \right] \tag{160}$$

and

$$Z_\mu = \Im_Z \left[\zeta_{31} W_\mu^0 + \zeta_{32} W_\mu^3 + \zeta_{33} W_\mu^8\right] \tag{161}$$

where

$$\zeta_{11} = \zeta_{21} = \zeta_{31} = 1 \tag{162}$$

$$\zeta_{12} = \zeta_{13} = \zeta_{33} = -\frac{(\gamma - \alpha)}{2} \tan\theta \tag{163}$$

$$\zeta_{22} = 0 \tag{164}$$

$$\zeta_{23} = 2\cot\theta \tag{165}$$

$$\zeta_{32} = \frac{2\cot\theta}{5(\gamma - \alpha)} + \frac{\gamma - \alpha}{10} \tag{166}$$

$$\Im_B = \left(\zeta_{11}^2 + \zeta_{12}^2 + \zeta_{13}^2\right)^{-1/2} \tag{167}$$

$$\Im_{Z'} = \left(\zeta_{21}^2 + \zeta_{22}^2 + \zeta_{23}^2\right)^{-1/2} \tag{168}$$

and

$$\Im_Z = \left(\zeta_{31}^2 + \zeta_{32}^2 + \zeta_{33}^2\right)^{-1/2} \tag{169}$$

Of course, the above transformations can be shown to be a result of a general rotation with Euler angles $\left(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}\right)$. In fact abbreviating the sine and cosine functions as s and c respectively, the corresponding rotation matrix $R\left(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}\right)$ is:

$$R\left(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}\right) = \begin{pmatrix} c_{\widetilde{\alpha}} c_{\widetilde{\gamma}} - s_{\widetilde{\alpha}} c_{\widetilde{\beta}} s_{\widetilde{\gamma}} & -c_{\widetilde{\alpha}} s_{\widetilde{\gamma}} - s_{\widetilde{\alpha}} c_{\widetilde{\beta}} c_{\widetilde{\gamma}} & s_{\widetilde{\beta}} s_{\widetilde{\alpha}} \\ s_{\widetilde{\alpha}} c_{\widetilde{\gamma}} + c_{\widetilde{\alpha}} c_{\widetilde{\beta}} s_{\widetilde{\gamma}} & -s_{\widetilde{\alpha}} s_{\widetilde{\gamma}} + c_{\widetilde{\alpha}} c_{\widetilde{\beta}} c_{\widetilde{\gamma}} & -s_{\widetilde{\beta}} c_{\widetilde{\alpha}} \\ s_{\widetilde{\beta}} s_{\widetilde{\gamma}} & s_{\widetilde{\beta}} c_{\widetilde{\gamma}} & c_{\widetilde{\beta}} \end{pmatrix} \tag{170}$$

where:

$$\zeta_{12} = \zeta_{13} = \zeta_{33} = -\frac{(\gamma - \alpha)}{2} \tan\theta \tag{171}$$

$$\zeta_{22} = 0 \tag{172}$$

$$\zeta_{23} = 2\cot\theta \tag{173}$$

$$\zeta_{32} = \frac{2\cot\theta}{5(\gamma - \alpha)} + \frac{\gamma - \alpha}{10} \tag{174}$$

$$\tan\widetilde{\gamma} = \zeta_{32} \tag{175}$$

$$\tan\widetilde{\alpha} = -\frac{\zeta_{12}}{\zeta_{23}} \left(1 + \zeta_{23}^2\right)^{1/2} \left(1 + \zeta_{12}^2\right)^{-1/2} \tag{176}$$

and

$$\cot\widetilde{\beta} = \frac{\tan\widetilde{\alpha} \tan\widetilde{\gamma}}{\sqrt{1 - \tan^2\widetilde{\alpha} \tan^2\widetilde{\gamma}}} \tag{177}$$

If for example $\alpha = \frac{2}{3}$ and $\gamma = -\frac{1}{3}$, then, $\widetilde{\alpha}, \widetilde{\beta}$ and $\widetilde{\gamma}$ get the values $\widetilde{\alpha} \approx -15,96^{\circ}$, $\widetilde{\beta} \approx -68,13^{\circ}$ and $\widetilde{\gamma} \approx 54,38^{\circ}$ respectively. As an illustration, we has displayed in fig.1, $\tan\widetilde{\alpha}$ (solid line, denoted by $\tan\alpha$),$\tan\widetilde{\beta}$ (dashed line,denoted by $\tan\beta$) and $\tan\widetilde{\gamma}$ (dotted line,denoted by $\tan\gamma$) as a function of $\tan\theta$. Notice that in the interval of $\theta \in [2.29°, 27.92°]$, $\tan\widetilde{\beta}$ (resp.$\tan\widetilde{\gamma}$) is an increasing (resp.decreasing) function of θ while $\tan\widetilde{\alpha}$ is almost constant.

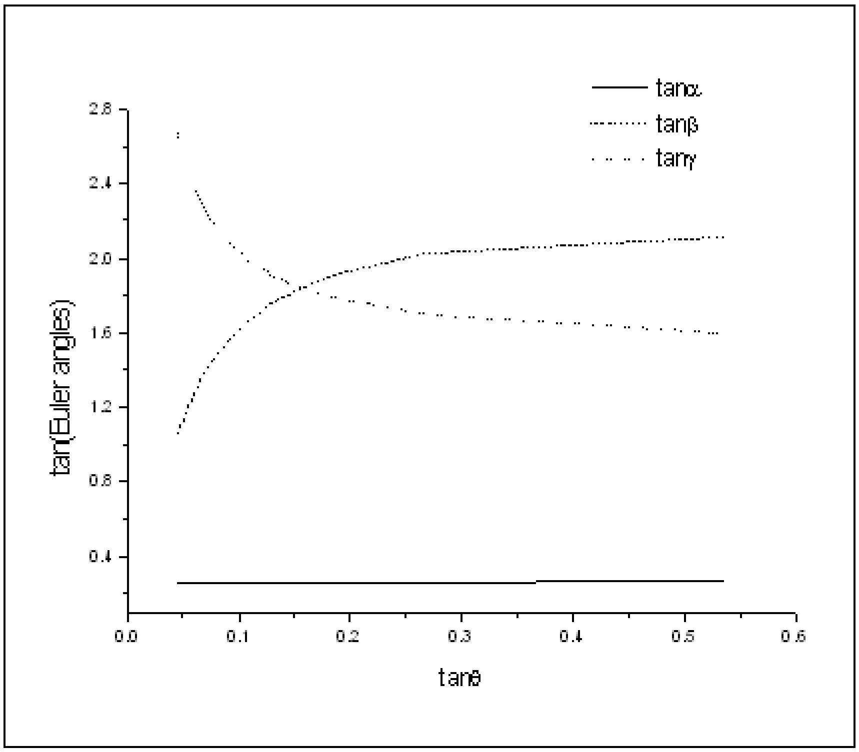

Fig.1 $\tan\widetilde{\alpha}, \tan\widetilde{\beta}$ and $\tan\widetilde{\gamma}$ as a function of $\tan\theta$

4.3 Fermions electromagnetic interaction term

To be more specific and keep our result as clear as possible, let us take the first generation of quarks (u, d, J_1). it is easy to show that the electromagnetic lagrangian density $\pounds_Q^{EM}$ takes the form:

$$\pounds_Q^{EM} = \frac{g_0}{\sqrt{x}}\Omega\left(\beta^1\overline{Q}_1^1\gamma^\mu B_\mu Q_1^1 + \gamma^1\overline{Q}_2^1\gamma^\mu B_\mu Q_2^1 + \delta^1\overline{Q}_3^1\gamma^\mu B_\mu Q_3^1\right) \tag{178}$$

where

$$\Omega = \frac{4\cot\theta\,(1 + 4\cot\theta)\left(1 + \frac{1}{2}\tan^2\theta\right)}{\left(24 + \tan g - 16\cot^2\theta - 4\cot\theta - 5\tan^2\theta\right)} \tag{179}$$

with the constraints

$$\delta^1 - \alpha^1 = \alpha^1 - \gamma^1 = 1 \tag{180}$$

and

$$\beta^1 = \alpha^1 \tag{181}$$

By identifying the u quark charge in eq.(177), one deduces that:

$$\frac{g_0}{\sqrt{x}}\Omega = e \tag{182}$$

and

$$\beta^1 = \frac{2}{3} \tag{183}$$

(e stands for the electric charge) and consequently :

$$\begin{aligned} \gamma^1 &= -\frac{1}{3} \\ \delta^1 &= \frac{5}{3} \end{aligned} \tag{184}$$

Notice that the values of β^1, γ^1 and δ^1 are compatible with the constraints of eq.(179) and correspond exactly to the N charges of the left and right handed first generation quarks of the commutative model[21]. Thus, NAG reproduces uniquely as an output, the values of the particles N charges. Similarly, for the second and third generations of quarks (s, c, J_2) and (b, t, J_3), one can show that:

$$\delta^{2,3} - \alpha^{2,3} = \alpha^{2,3} - \gamma^{2,3} = 1 \tag{185}$$

and

$$\begin{aligned} \beta^2 &= \beta^3 = \alpha^2 = \alpha^3 = -\frac{1}{3} \\ \gamma^2 &= \gamma^3 = +\frac{2}{3} \\ \delta^2 &= \delta^3 = -\frac{4}{3} \end{aligned} \tag{186}$$

The same study can be done for the leptons to obtain:

$$\begin{aligned} \delta'^j - \alpha'^j &= \alpha'^j - \gamma'^j = 1 \\ \beta'^j &= \alpha'^j = 0 \\ \gamma'^j &= -1 \\ \delta'^j &= +1, \forall j = \overline{1,3} \end{aligned} \tag{187}$$

It is very important to notice the remarquable result which is the universality of the following relations :

$$\begin{aligned} \delta^j - \alpha^j &= \alpha^j - \gamma^j = \delta'^j - \alpha'^j = \alpha'^j - \gamma'^j = 1 \\ \beta^j &= \alpha^j = \beta'^j = \alpha'^j, \forall j = \overline{1,3} \end{aligned} \tag{188}$$

independently from the type of particles (leptons or quarks) and generation. More intersting, similar relations hold also for the parameters α, β, γ and δ. This is probably related to the free anomaly of the model. Moreover, the value of the x parameter which is related to the quark N charges and as it is mentioned in subsection3.2, gets a constant value independently of the generation type. In fact, one can check easily (using eqs.(93) – (94)) that $x = 4$ for the quarks and vanishs for leptons.

4.4 $V - A$ fermions-neutral gauge bosons couplings

If we denote by $\pounds^{NC}$ the neutral currents lagrangian density coupled to both $Z^{0\prime}$ and Z^0 massive vector bosons such as:

$$\pounds^{NC} = \pounds_Q^{NC} + \pounds_L^{NC} \tag{189}$$

where

$$\pounds_Q^{NC} = \frac{-g}{2\cos\theta_w} \sum_{i,\ j} \left[\overline{Q}_i^j \gamma^\mu \left(g_{V,Z}^{Q_i^j} - g_{A,Z'}^{Q_i^j}\gamma_5\right) Q_i^j Z_\mu + \overline{Q}_i^j \gamma^\mu \left(g_{V,Z'}^{Q_i^j} - g_{A,Z'}^{Q_i^j}\gamma_5\right) Q_i^j Z'_\mu\right] \tag{190}$$

and

$$\pounds_L^{NC} = \frac{-g}{2\cos\theta_w} \sum_{i,\ j} \left[\overline{L}_i^j \gamma^\mu \left(g_{V,Z}^{L_i^j} - g_{A,Z'}^{L_i^j}\gamma_5\right) L_i^j Z_\mu + \overline{L}_i^j \gamma^\mu \left(g_{V,Z'}^{L_i^j} - g_{A,Z'}^{L_i^j}\gamma_5\right) L_i^j Z'_\mu\right] \tag{191}$$

we deduce the following $V - A$ fermions-neutral gauge bosons couplings $g_{V,Z}^{Q_i^j}$, $g_{V,Z'}^{Q_i^j}$, $g_{A,Z}^{Q_i^j}$, $g_{A,Z'}^{Q_i^j}$, $g_{V,Z}^{L_1^j}$, $g_{V,Z'}^{L_1^j}$, $g_{A,Z'}^{L_1^j}$ and $g_{A,Z'}^{L_2^j}$:

i)For quarks:

$$g_{V,Z}^{Q_1^j} = -Đ^j \left[t_{32}^j + t_{33}^j + \varepsilon_3^j t_{31}^j\right], g_{V,Z'}^{Q_1^j} = Đ^j \left[t_{33}^j - \varepsilon_3^j t_{21}^j\right], g_{A,Z}^{Q_1^j} = -g_{A,Z'}^{Q_1^j} = Đ^j t_{33}^j, \tag{192}$$

$$\begin{aligned} g_{V,Z}^{Q_2^j} &= Đ^j \left[t_{32}^j - t_{33}^j - \varepsilon_4^j t_{31}^j\right], g_{V,Z'}^{Q_2^j} = Đ^j \left[t_{33}^j - \varepsilon_4^j t_{21}^j\right], \\ g_{A,Z}^{Q_2^j} &= Đ^j \left[-t_{32}^j + t_{33}^j + \varepsilon_5^j t_{31}^j\right], g_{A,Z'}^{Q_2^j} = Đ^j \left[-t_{33}^j + \varepsilon_5^j t_{21}^j\right] \end{aligned} \tag{193}$$

and

$$g_{V,Z}^{Q_3^j} = g_{V,Z'}^{Q_3^j} = Đ^j \left[2t_{32}^j + \varepsilon_1^j t_{31}^j\right], g_{A,Z}^{Q_3^j} = g_{A,Z'}^{Q_3^j} = Đ^j \left[-2t_{32}^j + \varepsilon_2^j t_{31}^j\right] \tag{194}$$

ii)For leptons

$$g_{V,Z}^{L_1^j} = -Đ^j \left[t_{32}^j + t_{33}^j\right], g_{V,Z'}^{L_1^j} = Đ^j \left[-t_{32}^j + t_{33}^j\right], g_{A,Z'}^{L_1^j} = -g_{A,Z}^{L_1^j} = -Đ^j t_{33}^j \tag{195}$$

$$
\begin{aligned}
g_{V,Z}^{L_2^j} &= \text{Đ}^j \left[t_{32}^j - t_{33}^j + t_{31}^j \right], g_{V,Z'}^{L_2^j} = \text{Đ}^j \left[t_{33}^j + t_{31}^j \right], \\
g_{A,Z}^{L_2^j} &= \text{Đ}^j \left[-t_{32}^j + t_{33}^j + t_{31}^j \right], g_{A,Z'}^{L_2^j} = \text{Đ}^j \left[-t_{33}^j + t_{21}^j \right]
\end{aligned} \tag{196}
$$

and

$$
g_{V,Z}^{L_3^j} = g_{V,Z'}^{L_3^j} = \text{Đ}^j \left[2t_{33}^j - t_{31}^j \right], g_{A,Z}^{L_3^j} = g_{A,Z'}^{L_3^j} = \text{Đ} \left[-2t_{33}^j - t_{21}^j \right] \tag{197}
$$

where:

$$
\text{Đ}^j = \text{Đ} = \frac{\cos\theta_w}{\chi} \tag{198}
$$

$$
\varepsilon_1^j = -\left(\alpha^j + \delta^j\right) \tan\theta \tag{199}
$$

$$
\varepsilon_2^j = \left(\alpha^j - \delta^j\right) \tan\theta \tag{200}
$$

$$
\varepsilon_3^j = -2\alpha^j \tan\theta \tag{201}
$$

$$
\varepsilon_4^j = -\left(\alpha^j + \gamma^j\right) \tan\theta \tag{202}
$$

$$
\varepsilon_5^j = \left(\alpha^j - \gamma^j\right) \tan\theta \tag{203}
$$

$$
t_{31} = \Im_B \Im_{Z'} \zeta_{12} \zeta_{23} \tag{204}
$$

$$
t_{32} = \Im_B \Im_{Z'} \left(\zeta_{12} - \zeta_{23}\right) \tag{205}
$$

$$
t_{21} = \Im_B \Im_Z \zeta_{12} (\zeta_{32} - \zeta_{12}) \tag{206}
$$

$$
t_{33} = \Im_B \Im_Z \left(\zeta_{32} - \zeta_{12}\right) \tag{207}
$$

and

$$
\chi = \Im_B \Im_{Z'} \Im_Z \left(-\zeta_{23}\zeta_{32} - (\zeta_{12})^2 + \zeta_{12}\zeta_{23} + \zeta_{12}\zeta_{32} \right) \tag{208}
$$

Notice that in NAG and contrary to reference [21](commutative case), the exotic quarks couple to the Z^0 gauge boson not only through vector but axial current as well. Fig.2 represents the vector (solid line,denoted by $g_{V,Z'}^{J_1}$) and axial (dashed line,denoted by $g_{A,Z'}^{J_1}$) couplings of the J_1 exotic quark with the Z$'$ gauge boson as a function of $\tan\theta$. Notice that and contrary to the vector component, the axial coupling is negative and decreases rapidly.

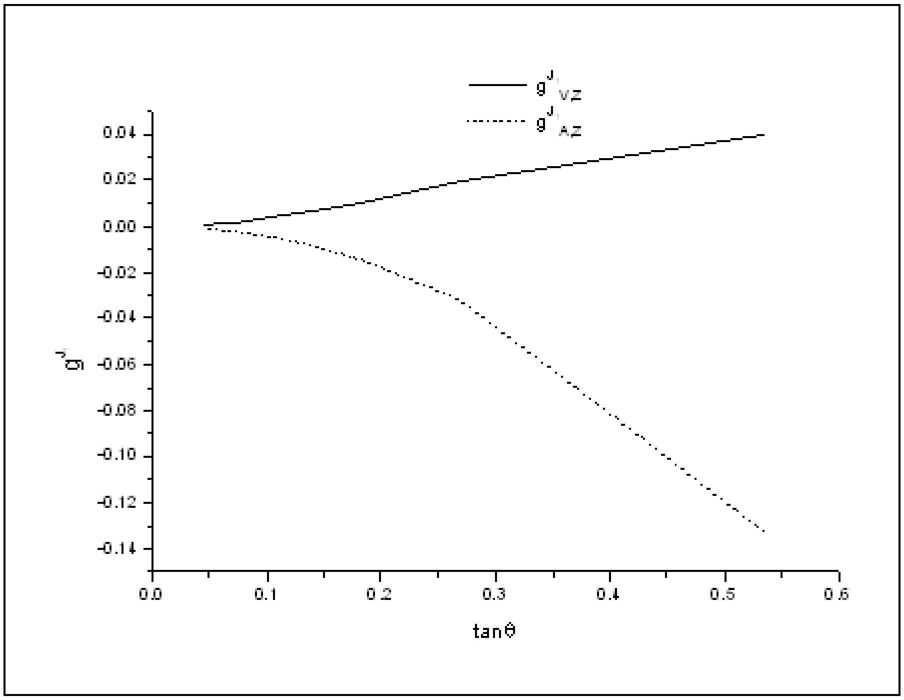

Fig.2 Vector and axial couplings of J_1 exotic quark with Z' boson as a function of $\tan\theta$.

Following reference [21], strong bounds on the masses of the exotic vector bosons coming from flavour changing neutral currents ($FCNC$) induced by $Z^{0\prime}$can be obtained if one considers the contribution to $\overline{K}_L^0 - \overline{K}_S^0$ mass difference due to the exchange of a heavy neutral boson. If one takes for simplicity two family mixing, one has in the lagrangian density a term like:

$$-\frac{g}{4}\frac{M_{Z^0}}{M_W}\cos\theta_c\sin\theta_c\left[\overline{d}\gamma^\mu\left(g^{Q_2^1}_{V,Z} - g^{Q_2^1}_{A,Z'}\gamma_5\right)s - \overline{d}\gamma^\mu\left(g^{Q_1^2}_{V,Z} - g^{Q_1^2}_{A,Z'}\gamma_5\right)s\right]Z'_\mu \tag{209}$$

where Q_2^1 (resp.Q_1^2) denotes the d (resp.s)quark. Then, one at low energies one obtains the effective interaction:

$$\pounds_{eff} = \frac{g^2}{16}\left(\frac{M_{Z^0}}{M_W}\right)^2\frac{\cos^2\theta_c\sin^2\theta_c}{M^2_{Z^{0\prime}}}\left[\overline{d}\gamma^\mu(c_V - c_A\gamma^5)s\right]^2 \tag{210}$$

with

$$c_V = c_A = g^{Q_2^1}_{V,Z} - g^{Q_1^2}_{V,Z} = \text{Ð}\tan\theta \tag{211}$$

We remind the reader that the contribution of the c-quark in the standard model is [36, 37] :

$$\pounds^{SM}_{eff} = -\frac{G_F}{\sqrt{2}}\frac{\alpha_e}{4\pi}\frac{m_c^2}{M_W^2\sin^2\theta_w}\cos^2\theta_c\sin^2\theta_c\left[\overline{d}\gamma^\mu\frac{1}{2}(1-\gamma^5)s\right]^2 \tag{212}$$

where

$$\frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}} \tag{213}$$

and θ_c, α_e and G_F stand for the Cabbibo angle, electromagnetic fine structure coupling constant and Fermi constant respectively. Now, if we assume that any additional contribution to the $\overline{K}_L^0 - \overline{K}_S^0$ mass difference from the $Z^{0\prime}$ boson cannot be much bigger than the contribution of the charmed quark [38], then we get the following lower bound:

$$M_{Z^{0\prime}}^2 \gtrsim \frac{2\pi}{\alpha_e} \frac{M_W^4}{m_c^2} Ð^2 \tan^2\theta \tan^2\theta_w \tag{214}$$

and consequently the VeV v_3 has a lower limit:

$$v_3^2 \gtrsim \frac{\sqrt{2}\pi}{4G_F\alpha_e} \frac{M_W^2}{m_c^2} Ð^2 \tan^2 g\theta \tan g^2\theta_w / \left[\tan^2 g\theta \left(\gamma - \alpha\right)^2 + 4\right] \tag{215}$$

This implies (if $v_3 \gg v_1, v_2$ and using eqs.(140) and (141)) that:

$$M_{V^\pm,U^\pm}^2 \gtrsim \frac{2\pi}{\alpha_e} \frac{M_W^4}{m_c^2} Ð^2 \tan^2\theta \tan^2\theta_w / \left[\tan^2 g\theta \left(\gamma - \alpha\right)^2 + 4\right] \tag{216}$$

For $(\gamma - \alpha)^2 = 1$, one gets $M_{Z^{0\prime}} \approx 21.76TeV$,$v_3 \approx 13.49TeV$ and $M_{V^\pm,U^\pm} \approx 10,51TeV$. Fig.3 (resp.Fig4) displays the variation of the lower bound of $M_{Z^{0\prime}}$ (resp.$M_{V^\pm,U^\pm}$) as a function of $\tan\theta$. Notice that although θ is small, the variations of $M_{Z^{0\prime}}$ (resp.$M_{V^\pm,U^\pm}$) are very important $\sim O(4TeV) - O(14TeV)$ (resp.$\sim O(2TeV) - O(7TeV)$ in the interval $\theta \in \left[18.10^{-4} - 28.10^{-4}\right]$.

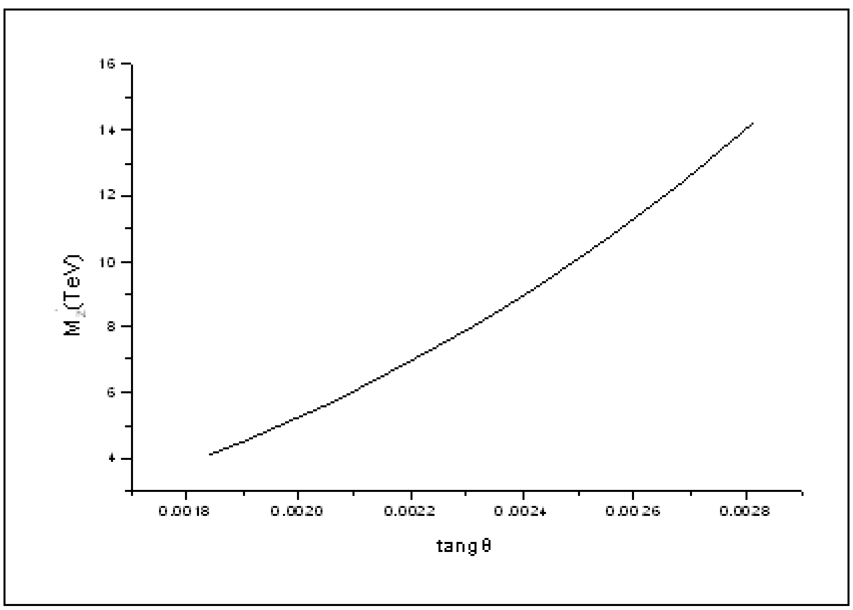

Fig.3 Variation of lower bound of $M_{Z^{0\prime}}$ as function of $\tan\theta$.

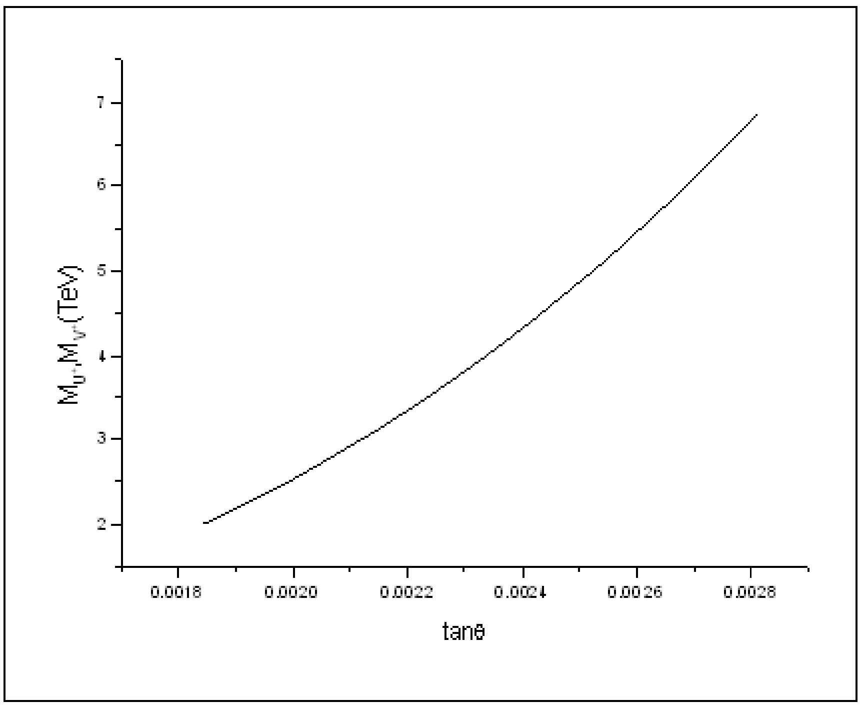

Figure 1: Fig.4 Variation of M_{U^+,V^+} as a function of $\tan\theta$

5 Conclusions

We have reformulated a classical gauge model based on the gauge Lie group $SU(3)_c \otimes SU(3)_L \otimes U(1)_N$ with exotic quarks within the formalism of non-associative geometry NAG associated to an L-cycle. In fact, we have determined the elements of the representation $\widehat{\pi}$ acting on the space $\mathbf{\Omega}^1\mathbf{a}$ and $\Omega^2\boldsymbol{a}$ of the 1-forms and 2-forms respectively and defining the $\widehat{\sigma}: \mathbf{\Omega}^1\mathbf{a} \rightarrow M_F\mathbb{C}$ mapping. The elements of the Junks forms defining the spaces $\mathbf{j}^0\boldsymbol{a}$,$\mathbf{j}^1\boldsymbol{a}$ and $\mathbf{j}^2\ \boldsymbol{a} \subset M_F(\mathbb{C})$ are also determined and its elimination requires some constraints on the various NAG parameters (see eq.(106)). The elements of the $\mathbf{r}^0\ \boldsymbol{a} \subset M_F\mathbb{C}$ and $\mathbf{r}^1\ \boldsymbol{a} \subset M_F(\mathbb{C})$ spaces needed for the construction of a connection form ρ and a curvature θ are also identified and defined. The structure of the connection ρ and the representative $e(\theta)$ of the curvature orthogonal to $\mathbf{J}^2g$ have been constructed and the bosonic as well as fermionic actions are computed after a Wick rotation. Since one of the family of quarks is incorporated differently from the other two, we were obliged to introduce in the representation $\widehat{\pi}_Q(a_1,a_3)$ and $\widehat{\pi}_L(a_1,a_3)$of quarks and leptons respectively the operators $\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\delta}}$ etc...where their action leads to the N charges of the fermionic particles in various generations and are uniquely determined. More interesting is the universality of the constraints imposed on the parameters representing these N charges (see eqs.(187)). This is probably another form of the anomaly cancellation and can be considered as an output of NAG. Furthermore, the number of the scalar bosons are imposed by NAG and it is similar to that of the economical non-supersymmetric $3-3-1$ model with two Higgs triplets. Thus, the degrees of freedom in the scalar sector are dictated by NAG. Regarding the predictions of this formalism, we have obtained a relation between the Weinberg angle θ_w, some free parameters representing the eigenvalues of the operators $\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\boldsymbol{\delta}}$ and $\widehat{\boldsymbol{\gamma}}$ on the scalar particles and the mixing angle θ (see eq.(156)). We have also derived the expressions of the various scalar, charged gauge bosons and neutral gauge bosons masses. In fact, following the argument of reference[21], we have obtained lower limits of the $M_{Z^{0\prime}}$ and $M_{V^\pm,U^\pm}$ as functions of the θ angle. We have also determined the mixing angles between the neutral gauge bosons Z^0 , $Z^{0\prime}$ and the photon B represented by the Euler rotation angles $\widetilde{\alpha}, \widetilde{\beta}$ and $\widetilde{\gamma}$. Concerning, the various expressions of the $V-A$ couplings of the quarks and leptons with the $Z^{0\prime}$ and Z^0 gauge bosons, they are also derived and given explicitly in terms of the mixing angle θ, and the free parameter α, β,etc... It is to be noted that contrary to the commutative case, the exotic quarks couple to the Z^0 gauge boson not only through vector but axial current as well. Finally, all the previous results are valid only at the tree level. As it is argued in our reference [10]in the framework of NCG, there is no satisfactory quantization procedure which has been developed yet treating the gauge and Higgs bosons in an equal footing. Therefore, we expect that the quantum fluctuations may badly violate the resulted tree level NAG constraints, masses and couplings relations. In principle, the change in the quantization rules is needed around certain energy scale and we have to assume that just below such a scale, the standard quantization method makes a good approximation.

Acknowledgement 1 *We are very grateful to the Algerian Ministry of education and research.for the financial support and one of us (N.B.) would like to thank Prof.Thomas Krajewski for useful discussions.*

References

[1] A.Connes and J.Lott, *Particle models and noncommutative geometry*, *Nucl. Phys.* **B18** (*Proc. Suppl.*) (1991) 29.

[2] A.Connes, *Noncommutative geometry*, Academic Press, New York 1994.

[3] T.Schucker and J.M.Zylinski, *Connes's model building kit, J.Geom.Phys.***16** (1995) 207.

[4] F.Lizzi, G.Mangano, G.Miele and G.Sparano, *Constraints on unified gauge theories from noncommutative geometry, Mod. Phys. Lett.* **A11** (1996) 2561.

[5] R.Brout, *Notes on Connes's construction of the standard model, Nucl.Phys.* **65** (Proc.Suppl.) 1998 3.

[6] J.C.Pati and A.Salam, *Lepton number as the fourth color, Phys.Rev.* **D10** (1974) 175.

[7] R.N.Mohapatra and J.C.Pati, *Left-right gauge symmetry and an 'isoconjugate' model of CP-violation, Phys. Rev.* **D11**(1975) 566.

[8] R.N.Mohapatra and G.Senjanovic, *Current algebra predictions for* $D \longrightarrow K_\pi K_\pi L$ *neutrino decay, Phys. Rev.* **D21** (1981) 165.

[9] A.Connes, *Noncommutative geometry and reality*, *J.Math.Phys.* **36** (1995) 6194.

[10] N.Mebarki, M.Harrat and M.Boussahel, *Extended Chamseddine-Frohlich approach to noncommutative geometry and the two doublets Higgs model, Int. J. Mod. Phys.***A22** (2007) 6279.

[11] R.Coquereaux, G.Esposito-Farése and F.Scheck, *Noncommutative geometry and graded algebras in electroweak interactions*, *Int. J. Mod. Phys.***A7** (1992) 6555.

[12] R.Coquereaux, G.Esposito-Farése and G.Vaillant, *Higgs fields as Yang-Mills fields and discrete symmetries, Nucl. Phys.* **B353** (1991) 689.

[13] Y.Okumura, *SO(10) grand unified theory in noncommutative differential geometry on the discrete space $M_4 \otimes Z_N$, Prog.Theor.Phys.* **94** (1995) 607.

[14] A.H.Chamseddine and J.Frohlich, *SO(10) unification in noncommutative geometry, Phys. Rev.* **D 50** (1994) 2893.

[15] R.Wulkenhaar, *Non-associative geometry-unified models based on L-cycles*, PhD Thesis, Leipzig University 1997.

[16] R.Wulkenhaar, *Noncommutative geometry with graded differential Lie algebras, J.Math.Phys.* **38** (1997) 3358.

[17] R.Wulkenhaar, *The standard model within non-associative geometry, Phys. Lett.* **B 390** (1997) 119.

[18] R.Wulkenhaar, *Graded differential Lie algebras and $SU(5) \otimes U(1)$ grand unification, Int.J.Mod.Phys.***A13** (1998) 2627.

[19] R.Wulkenhaar, *SO(10) unification in noncommutative geometry revisited, Int. J. Mod. Phys.***A 14** (1999) 559.

[20] A.Benslama and N.Mebarki, *Left-right gauge model in nonassociative geometry, JHEP* **0002** (2000) 018.

[21] F.Pisano and V.Pleitez, *$SU(3) \otimes U(1)$ model for electroweak interactions, Phys. Rev.* **D 46** (1992) 410.

[22] P.H.Frampton, *Chiral dilepton model and the flavor question, Phys.Rev.Lett.* **69** (1992) 2889.

[23] V.Pleitez and M.D.Tonasse, *Heavy charged leptons in an $SU(3)_L \otimes U(1)_N$ model, Phys.Rev.***D 48** (1993) 2353.

[24] D.Ng, Electroweak theory of $SU(3) \otimes U(1)$, *Phys. Rev.* **D 49,** (1994) 4805.

[25] J.C. Monetro, F.Pisano and V.Pleitez, *Neutral currents and Glashow-Iliopoulos-Maiani mechanism in $SU(3)_L \otimes U(1)_N$ models for electroweak interactions, Phys. Rev.* **D 47** (1993) 2918.

[26] H.N.Long, *$SU(3)_c \otimes SU(3)_L \otimes U(1)_N$ model with right-handed neutrinos, Phys. Rev.* **D 53** (1996) 437.

[27] V.Pleitez, *New fermions and a vectorlike third generation in $SU(3)_c \otimes SU(3)_L \otimes U(1)_N$ models, Phys. Rev.* **D 53** (1996) 514.

[28] D.A.Gutierrez, W.A.Ponce and L.A.Sanchez, *Study of the $SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ model with the minimal scalar sector, Int.J.Mod.Phys.* **A 21** (2006) 2217.

[29] T.Kitabayashi and M.Yasuè, *Two-loop radiative neutrino mechanism in an $SU(3)_L \otimes U(1)_N$ gauge model, Phys. Rev.* **D 63** (2001) 095006.

[30] Y.Okamoto and M.Yasuè, *Radiatively generated neutrino masses in $SU(3)_L \otimes U(1)_N$ gauge models, Phys. Lett.* **B 466** (1999) 267.

[31] T.Kitabayashi, *Comment on neutrino masses and oscillations in an $SU(3)_L \otimes U(1)_N$ model with radiative mechanism.Phys. Rev.* **D 64** (2001) 057301.

[32] T.Kitabayashi and M.Yasuè, *Neutrino oscillations induced by two loop radiative mechanism, Phys. Lett.* **B 490** (2000) 236.

[33] T.Kitabayashi and M.Yasuè, *$S(2L)$ permutation symmetry for left-handed mu and tau families and neutrino oscillations in an $SU(3)_L \otimes U(1)_N$ gauge model, Phys. Rev.* **D 67** (2003) 015006.

[34] P. V. Dong, H. N. Long, and D. T. Nhung, *$SU(3)_c \otimes SU(3)_L \otimes U(1)_X$ model with two Higgs triplets, Phys. Rev.* **D 73** (2006) 035004

[35] W.A.Ponce, Y.Giraldo and L.A.Sanchez, *Phys. Rev.* **D 67**, (2003) 075001.

[36] M.K.Gaillard and B.W.Lee, *Rare decay modes of the K mesons in gauge theories, Phys.Rev.* **D 10** (1974) 897.

[37] R.Shrock and S.B.Treiman, *$K^0 - \overline{K}^0$ Transition amplitude in the MIT bag model., Phys.Rev.* **D 19** (1979) 2148.

[38] R.N.Cahn and H.Harari, *Bounds on the Masses of Neutral Generation Changing Gauge Bosons, Nucl.Phys.* **B 176** (1980) 135