Inequalities for Integer and Fractional Parts

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Abstract: In this paper we present some new inequalities relative to integer and functional parts.

Theorem 1. If $x > 0$, then $\frac{[x]}{3x+\{x\}} + \frac{\{x\}}{3x+[x]} \geq \frac{4}{15}$, where $[\cdot]$ and $\{\cdot\}$ denote the integer part, and respectively the fractional part.

Proof. In inequality $\frac{a}{a+2b+2c} + \frac{b}{2a+b+2c} + \frac{c}{2a+2b+c} \geq \frac{3}{5}$, we take $a = x,\ b = [x],\ c = \{x\}$.

Theorem 2. If $a, b, c, x > 0$, then

$$\frac{a}{[x]b+\{x\}c} + \frac{b}{[x]c+\{x\}a} + \frac{c}{[x]a+\{x\}b} \geq \frac{3}{x}.$$

Proof. In inequality $\frac{a}{ub+vc} + \frac{b}{uc+va} + \frac{c}{ua+vb} \geq \frac{3}{u+v}$, we take $u = [x]$ and $v = \{x\}$.

Theorem 3. If $x > 0$ and $a \geq 1$, then

$$\frac{[x]}{(a+1)[x]+2\{x\}} + \frac{[x]}{(a+1)\{x\}+2[x]} \leq \frac{2a+1}{(a+1)(a+2)}.$$

Proof. In inequality $\frac{x}{ax+y+z} + \frac{y}{x+ay+z} + \frac{z}{x+y+az} \leq \frac{3}{a+2}$, we take $y = [x]$ and $z = \{x\}$.

Theorem 4. If $x > 0$, then

$$[x]\left(\frac{1}{x[x]+x+1}+\frac{1}{[x]\{x\}+[x]+1}\right)+\{x\}\left(\frac{1}{x[x]+x+1}+\frac{1}{x\{x\}+\{x\}+1}\right)\le 1.$$

Proof. In inequality $\frac{x}{xy+x+1}+\frac{y}{yz+y+1}+\frac{z}{zx+z+1}\le 1$, we take $y=[x]$ and $z=\{x\}$.

Theorem 5. If $x>0$, then

$$\frac{x^3}{[x]\left(3[x]^2+3[x]\{x\}+\{x\}^2\right)}+\frac{x[x]^2}{\{x\}\left([x]^2+[x]\{x\}+\{x\}^2\right)}+\frac{x\{x\}^2}{[x]^2+3[x]\{x\}+3\{x\}^2}\ge\frac{3}{2}$$

Proof. In inequality $\sum\frac{x^2}{y\left(x^2+xy+y^2\right)}\ge\frac{3}{x+y+z}$, we take $y=[x]$ and $z=\{x\}$.

Theorem 6. If $x>0$, $\frac{1}{[x]+2\{x\}}+\frac{1}{2[x]+\{x\}}\ge\frac{1}{x}$.

Proof. In inequality $\sum\frac{a^2+bc}{b+c}\ge a+b+c$, we take $a=x,\ \ b=[x],\ \ c=\{x\}$.

Theorem 7. If $x>0$,

$$\frac{[x]^3}{[x]^2+[x]\{x\}+\{x\}^2}+\frac{\{x\}^3}{3\{x\}^2+3[x]\{x\}+[x]^2}\ge\frac{x\left(3[x]^2-\{x\}^2\right)}{3\left(3[x]^2+3[x]\{x\}+\{x\}^2\right)}$$

Proof. In inequality $\sum\frac{a^3}{a^2+ab+b^2}\ge\frac{a+b+c}{3}$, we take $a=x,\ \ b=[x]$, $c=\{x\}$.

Theorem 8. If $x>0$, then

$$\frac{1}{2[x]^3+4[x]^2\{x\}+4[x]\{x\}^2+\{x\}^3}+\frac{1}{[x]^3+[x]^2\{x\}+[x]\{x\}^2+\{x\}^3}+$$
$$+\frac{1}{[x]^3+4[x]^2\{x\}+4[x]\{x\}^2+2\{x\}^3}\le\frac{1}{x[x]\{x\}}.$$

Proof. In inequality $\sum\frac{1}{a^3+b^3+abc}\le\frac{1}{abc}$, we take $a=x,\ \ b=[x],\ \ c=\{x\}$.

Theorem 9. If $x>1$, then $4\left(\frac{[x]^3}{\{x\}}+\frac{\{x\}^3}{[x]}\right)\ge[x]^2+[x]\{x\}+\{x\}^2$.

Proof. In inequality $\sum\frac{1}{a}\left(-a+b+c\right)^3\ge a^2+b^2+c^2$, we take $a=x,\ \ b=[x]$, $c=\{x\}$.

Theorem 10. If $x>0$, then

$$\frac{x^4}{[x]^2-[x]\{x\}+\{x\}^2}+\frac{x\left([x]^3+\{x\}^3\right)}{[x]^2+[x]\{x\}+\{x\}^2}\geq\frac{3}{2}\left(x^2+[x]\{x\}\right)$$

Proof. In inequality $\sum\frac{a^3}{b^2-bc+c^2}\geq\frac{3\sum ab}{\sum a}$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 11. If $x>0$, then $\left|\frac{[x]\{x\}\left([x]-\{x\}\right)}{x\left(x+[x]\right)\left(x+\{x\}\right)}\right|<1$.

Proof. In inequality $\left|\sum\frac{a-b}{a+b}\right|<1$ we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 12. If $x>0$, then $\sqrt{\frac{[x]}{x+\{x\}}}+\sqrt{\frac{\{x\}}{x+[x]}}>1$.

Proof. In inequality $\sum\sqrt{\frac{x}{y+z}}>2$, we take $y=[x],\ z=\{x\}$.

Theorem 13. If $x>1$, then $3+\frac{\{x\}}{[x]}+\frac{[x]}{\{x\}}\geq 3\sqrt[3]{\frac{\left(x+[x]\right)\left(x+\{x\}\right)}{[x]\{x\}}}$.

Proof. In inequality $\left(\sum a\right)\left(\sum\frac{1}{a}\right)\geq 3\left(1+\sqrt[3]{\frac{\prod(a+b)}{abc}}\right)$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 14. If $x>0$, then $\left(\sqrt{[x]}+\sqrt{\frac{[x]\{x\}}{x}}+\sqrt{\{x\}}\right)^4\geq 32[x]\{x\}$.

Proof. In inequality $\sum\sqrt{xy}\geq 2\sqrt[4]{xyz\sum x}$, we take $y=[x],\ z=\{x\}$.

Theorem 15. If $x>0$, then $\left(x^2+[x]\{x\}\right)^2\geq 6x^2[x]\{x\}$.

Proof. In inequality $\left(\sum xy\right)^2\geq 3xyz\sum x$, we take $y=[x],\ z=\{x\}$.

Theorem 16. If $x>0$, then

$$x^2-x\sqrt{[x]\{x\}}+[x]\{x\}\geq\left([x]\sqrt{\{x\}}+\{x\}\sqrt{[x]}\right)\sqrt{x}.$$

Proof. In inequality $\sum xy\geq\sum x\sqrt{yz}$, we take $y=[x],\ z=\{x\}$.

Theorem 17. If $x>0$, then $\sqrt{[x]\left(x+\{x\}\right)}+\sqrt{\{x\}\left(x+[x]\right)}\leq\left(2\sqrt{2}-1\right)x$.

Proof. In inequality $\sum\sqrt{x(y+z)} \le \sqrt{2}\sum x$, we take $y=[x],\ z=\{x\}$.

Theorem 18. If $x>0$, then $\frac{[x]}{x+\{x\}}+\frac{\{x\}}{x+[x]} \ge \frac{1}{2}$.

Proof. In inequality $\sum\frac{a}{b+c} \ge \frac{3}{2}$, we take $a=x,\ \ b=[x],\ \ c=\{x\}$.

Theorem 19. If $x>0$, then $(x+[x])^3+(x+\{x\})^3 \ge 21x[x]\{x\}+[x]^3+\{x\}^3$.

Proof. In inequality $\sum(x+y)^3 \ge 21xyz+\sum x^3$, we take $y=[x],\ z=\{x\}$.

Theorem 20. If $x>1$, then $\sqrt{\frac{x+[x]}{x+\{x\}}}+\sqrt{\frac{x+\{x\}}{x+[x]}} \le \sqrt{\frac{[x]}{\{x\}}}+\sqrt{\frac{\{x\}}{[x]}}$.

Proof. In inequality $\sqrt{\frac{x+y}{x+z}}+\sqrt{\frac{x+z}{x+y}} \le \frac{y+z}{\sqrt{yz}}$, we take $y=[x],\ z=\{x\}$.

Theorem 21. If $x>0$, then $\frac{x}{x+[x]}+\frac{x}{x+\{x\}} \ge \frac{5}{2}$.

Proof. In inequality $2\sum\frac{1}{x+y} \ge \frac{9}{\sum x}$, we take $y=[x],\ z=\{x\}$.

Theorem 22. If $x>1$, then $\frac{\{x\}}{[x]}+\left(\frac{\{x\}}{[x]}\right)^2+\left(\frac{[x]}{\{x\}}\right)^2+\frac{\{x\}^2}{x^2} \ge \left(\frac{1}{x}+\frac{1}{\{x\}}\right)[x]$.

Proof. In inequality $\sum\frac{x^2}{y^2} \ge \sum\frac{x}{z}$, we take $y=[x],\ z=\{x\}$.

Theorem 23. If $x>0$, then

$$[x]^2-[x]\{x\}+\{x\}^2 \ge \frac{3}{4}\max\left\{[x]^2;([x]-\{x\})^2;\{x\}^2\right\}.$$

Proof. In inequality $\sum x^2-\sum xy \ge \frac{3}{4}\max\left\{(x-y)^2;(y-z)^2;(z-x)^2\right\}$, we take $y=[x],\ z=\{x\}$.

Theorem 24. If $x>0$, then $e^{\{x\}}+e^{[x]} \ge 2+x$.
Proof. In inequality $e^y+e^z \ge 2+y+z$, we take $y=[x],\ z=\{x\}$.

Theorem 25. If $x\in\mathrm{R}$, then $|\sin[x]|+|\sin\{x\}|+|\cos x| \ge 1$.
Proof. In inequality $|\sin a|+|\sin b|+|\cos(a+b)| \ge 1$ we take $a=x,\ \ b=\{x\}$.

Theorem 26. If $x>0$, then $\left(3[x]^2+3[x]\{x\}+\{x\}^2\right)\cdot\left([x]^2+[x]\{x\}+\{x\}^2\right)\cdot$

$\cdot\left(3\{x\}^2+3\{x\}[x]+[x]^2\right)\geq\left(x^2+[x]\{x\}\right)^3$.

Proof. In inequality

$\left(a^2+ab+b^2\right)\cdot\left(b^2+bc+c^2\right)\cdot\left(c^2+ca+a^2\right)\geq\left(ab+bc+ca\right)^3$, we take $a=x,\ \ b=[x],\ \ c=\{x\}$.

Theorem 27. If $x>0$, then

$$\frac{[x]}{\left(3[x]+2\{x\}\right)\left([x]+2\{x\}\right)}+\frac{\{x\}}{\left(3\{x\}+2[x]\right)\left(\{x\}+2[x]\right)}\geq\frac{11}{48x}.$$

Proof. In inequality $\left(\sum x\right)\sum\frac{x}{\left(2x+y+z\right)\left(y+z\right)}\geq\frac{9}{8}$, we take $y=[x],\ z=\{x\}$.

Theorem 28. If $x>0$, then $\frac{[x]^2}{x+[x]}+\frac{\{x\}^2}{x+\{x\}}\geq\frac{x\left(2x^2+3[x]\{x\}\right)}{\left(x+[x]\right)\left(x+\{x\}\right)}$.

Proof. In inequality $\sum\frac{x^2}{\left(x+y\right)\left(x+z\right)}\geq\frac{3}{4}$, we take $y=[x],\ z=\{x\}$.

Theorem 29. If $x>1$, then

$$\sqrt{1+\frac{2\{x\}}{[x]}}+\sqrt{1+\frac{2[x]}{\{x\}}}\geq 1+2\left(\sqrt{\frac{[x]}{x+\{x\}}}+\sqrt{\frac{\{x\}}{x+[x]}}\right).$$

Proof. In inequality $\sum\sqrt{\frac{y+z}{x}}\leq\sqrt{2}\sum\sqrt{\frac{x}{y+z}}$ we take $y=[x],\ z=\{x\}$.

Theorem 30. If $x>1$, then $\frac{\{x\}}{[x]}+\frac{[x]}{\{x\}}\geq 1+2\left(\frac{[x]}{x+\{x\}}+\frac{\{x\}}{x+[x]}\right)$.

Proof. In inequality $\sum\frac{y+z}{x}\geq 4\sum\frac{x}{y+z}$, we take $y=[x],\ z=\{x\}$.

Theorem 31. If $x>0$, then

1). $\min\left\{\left(\sqrt{2}+1\right)\sqrt{x}+\sqrt{\{x\}};\left(\sqrt{2}+1\right)\sqrt{x}+\sqrt{x+[x]}\right\}\geq\sqrt{5\left([x]+2\{x\}\right)}$

2). $\left(\sqrt{2}+1\right)\sqrt{x}+\sqrt{x+\{x\}}\geq\sqrt{5\left(\{x\}+2[x]\right)}$.

Proof. In $\sqrt{a+b+c}+\sqrt{b+c}+\sqrt{c}\geq\sqrt{a+4b+9c}$, we take $a=x,\ \ b=[x],$ $c=\{x\}$, etc.

Theorem 32. If $x \in \mathrm{R}$, then

1). $|\sin x| \le |\sin[x]| + |\sin\{x\}|$

2). $|\cos x| \le |\cos[x]| + |\cos\{x\}|$

Proof. In inequalities $|\sin(a+b)| \le |\sin a| + |\sin b|$ and $|\cos(a+b)| \le |\cos a| + |\cos b|$, we take $a = x,\ b = [x]$.

Theorem 33. If $x > 1$, then $6 + \frac{\{x\}}{[x]} + \frac{[x]}{\{x\}} \ge \left(\sqrt[3]{\frac{x}{[x]}} + \sqrt[3]{\frac{[x]}{\{x\}}} + \sqrt[3]{\frac{\{x\}}{x}} \right)^3$.

Proof. In inequality $3\left(\sum a\right)\left(\sum \frac{1}{a}\right) \ge \left(\sum \sqrt[3]{\frac{a}{b}}\right)$, we take $a = x,\ b = [x],$ $c = \{x\}$.

Theorem 34. If $x > 0$, then $\frac{[x]}{(x + \{x\})^2} + \frac{\{x\}}{(x + [x])^2} \ge \frac{1}{8x}$.

Proof. In inequality $\sum \frac{a}{(b+c)^2} \ge \frac{9}{4\sum a}$, we take $a = x,\ b = [x],\ c = \{x\}$.

Theorem 35. If $x > 0$, then

$$\frac{x[x]}{(x + \{x\})(2x + [x])} + \frac{\{x\}(x + \{x\})}{(x + [x])(2x + \{x\})} \ge \frac{[x] + 5\{x\}}{12x}.$$

Proof. In inequality $\sum \frac{a(a+b)}{(b+c)(2a+b+c)} \ge \frac{3}{4}$, we take $a = x,\ b = [x],$ $c = \{x\}$.

Theorem 36. If $x > 0$, then $\frac{[x]}{2x + \{x\}} + \frac{\{x\}}{2x + [x]} \le \frac{3[x]^2 + 4[x]\{x\} + 3\{x\}^2}{6x}$.

Proof. In inequality $\sum \frac{ab}{a+b+2c} \le \frac{1}{4}\sum a$, we take $a = x,\ b = [x],\ c = \{x\}$.

Theorem 37. If $x > 0$, then $\left([x]^5 - [x]^2 + 3\right)\left(\{x\}^5 - \{x\}^2 + 3\right) \ge \frac{8x^3}{x^5 - x^2 + 3}$.

Proof. In inequality $\prod\left(a^5 - a^2 + 3\right) \ge \left(\sum a\right)^3$, we take $a = x,\ b = [x],\ c = \{x\}$.

Theorem 38. If $x > 0$, then $\frac{(2x + [x])^2}{2[x]^2 + (x + \{x\})^2} + \frac{(2x + \{x\})^2}{2\{x\}^2 + (x + [x])^2} \le 5$.

Proof. In inequality $\sum \frac{(2a+b+c)^2}{2a^2+(b+c)^2} \le 8$, we take $a = x,\ \ b = [x],\ c = \{x\}$.

Theorem 39. If $x > 0$, then

1). $\left(x+\sqrt{x[x]}+\sqrt[3]{x[x]\{x\}}\right)^3 \le 9x^2\left(x+[x]\right)$

2). $\left([x]+\sqrt{[x]\{x\}}+\sqrt[3]{x[x]\{x\}}\right)^3 \le 9x^2[x]$

3). $\left(\{x\}+\sqrt{x\{x\}}+\sqrt[3]{x[x]\{x\}}\right)^3 \le 9x^2\left(x+\{x\}\right)$.

Proof. In inequality $\frac{a+\sqrt{ab}+\sqrt[3]{abx}}{2} \le \sqrt[3]{a\left(\frac{a+b}{2}\right)\left(\frac{a+b+c}{b}\right)}$, we take $a = x,\ \ b = [x],\ c = \{x\}$, etc.

Theorem 40. If $x > 0$, then $7\left(x+[x]\right)^4 + 7\left(x+\{x\}\right)^4 \ge 3x^4 + 4\left([x]^4 + \{x\}^4\right)$.

Proof. In inequality $\sum (a+b)^4 \ge \frac{4}{7}\sum a^4$, we take $a = x,\ \ b = [x],\ c = \{x\}$.

Theorem 41. If $x > 0$, then $\frac{\{x\}^2}{\left(x+\{x\}\right)^2+[x]^2} + \frac{[x]^2}{\left(x+[x]\right)^2+\{x\}^2} \ge \frac{3}{20}$.

Proof. In inequality $\sum \frac{(b+c-a)^2}{(b+c)^2+a^2} \ge \frac{3}{5}$, we take $a = x,\ \ b = [x],\ c = \{x\}$.

Theorem 42. If $x > 0$, then $\sqrt{\frac{2x}{x+[x]}} + \sqrt{\frac{2[x]}{x}} + \sqrt{\frac{2\{x\}}{x+\{x\}}} \le 3$.

Proof. In inequality $\sum \frac{2a}{a+b} \le 3$, we take $a = x,\ \ b = [x],\ c = \{x\}$.

Theorem 43. If $x > 0$, then $\frac{1}{\left(x+[x]\right)^2} + \frac{1}{\left(x+\{x\}\right)^2+\{x\}^2} \ge \frac{5x^2-4[x]\{x\}}{4x^2\left(x^2+[x]\{x\}\right)}$.

Proof. In inequality $\left(\sum xy\right)\left(\sum \frac{1}{(x+y)^2}\right) \ge \frac{9}{4}$, we take $y = [x],\ z = \{x\}$.

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[Published in OCTOGON Mathematical Magazine, Vol. 14, No. 1, pp. 206-211, 2006.]