

ON SOLVING GENERAL LINEAR EQUATIONS IN THE SET OF NATURAL NUMBERS

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The utility of this article is that it establishes if the number of the natural solutions of a general linear equation is limited or not. We will show also a method of solving, using integer numbers, the equation $ax - by = c$ (which represents a generalization of lemmas 1 and 2 of [4]), an example of solving a linear equation with 3 unknowns in \mathbb{N} , and some considerations on solving, using natural numbers, equations with n unknowns.

Let's consider the equation:

(1) $\sum_{i=1}^n a_i x_i = b$ with all $a_i, b \in \mathbb{Z}$, $a_i \neq 0$, and the greatest common factor $(a_1, \dots, a_n) = d$.

Lemma 1: The equation (1) admits at least a solution in the set of integers, if d divides b .

This result is classic.

In (1), one does not diminish the generality by considering $(a_1, \dots, a_n) = 1$, because in the case when $d \neq 1$, one divides the equation by this number; if the division is not an integer, then the equation does not admit natural solutions.

It is obvious that each homogeneous linear equation admits solutions in \mathbb{N} : at least the banal solution!

PROPERTIES ON THE NUMBER OF NATURAL SOLUTIONS OF A GENERAL LINEAR EQUATION

We will introduce the following definition:

Definition 1: The equation (1) has variations of sign if there are at least two coefficients a_i, a_j with $1 \leq i, j \leq n$, such that $\text{sign}(a_i \cdot a_j) = -1$

Lemma 2: An equation (1) which has sign variations admits an infinity of natural solutions (generalization of lemma 1 of [4]).

Proof: From the hypothesis of the lemma it results that the equation has h no null positive terms, $1 \leq h \leq n$, and $k = n - h$ non null negative terms. We have $1 \leq k \leq n$; it is supposed that the first h terms are positive and the following k terms are negative (if not, we rearrange the terms).

We can then write:

$$\sum_{i=1}^h a_i x_i - \sum_{j=h+1}^n a_j' x_j = b \text{ where } a_j' = -a_j > 0.$$

Let's consider $0 < M = [a_1, \dots, a_n]$, the least common multiple, and $c_i = |M / a_i|$, $i \in \{1, 2, \dots, n\}$.

Let's also consider $0 < P = [h, k]$, the least common multiple, and $h_1 = P / h$ and $k_1 = P / k$.

$$\text{Taking } \begin{cases} x_t = h_1 c_t \cdot z + x_t^0, & 1 \leq t \leq h \\ x_j = k_1 c_j \cdot z + x_j^0, & h+1 \leq j \leq n \end{cases}$$

where $z \in \mathbb{N}$, $z \geq \max \left\{ \left\lceil \frac{-x_t^0}{h_1 c_t} \right\rceil, \left\lceil \frac{x_j^0}{k_1 c_j} \right\rceil \right\} + 1$ with $[\gamma]$ meaning integer part of γ , i.e.

the greatest integer less than or equal to γ , and x_i^0 , $i \in \{1, 2, \dots, n\}$, a particular integer solution (which exists according to lemma 1), we obtain an infinity of solutions in the set of natural numbers for the equation (1).

Lemma 3:

- a) An equation (1) which does not have variations of sign has at maximum a limited number of natural solutions.
- b) In this case, for $b \neq 0$, constant, the equation has the maximum number of solutions if and only if all $a_i = 1$ for $i \in \{1, 2, \dots, n\}$.

Proof: (see also [6]).

- a) One considers all $a_i > 0$ (otherwise, multiply the equation by -1).

If $b < 0$, it is obvious that the equation does not have any solution (in \mathbb{N}).

If $b = 0$, the equation admits only the trivial solution.

If $b > 0$, then each unknown x_i takes positive integer values between 0 and $b / a_i = d_i$ (finite), and not necessarily all these values. Thus the maximum number of

solutions is lower or equal to: $\prod_{i=1}^n (1 + d_i)$, which is finite.

- b) For $b \neq 0$, constant, $\prod_{i=1}^n (1 + d_i)$ is maximum if and only if d_i are

maximum, i.e. iff $a_i = 1$ for all i , where $i = \{1, 2, \dots, n\}$.

Theorem 1. The equation (1) admits an infinity of natural solutions if and only if it has variations of sign.

This naturally follows from the previous results.

Method of solving.

Theorem 2. Let's consider the equation with integer coefficients $ax - by = c$, where a and $b > 0$ and $(a, b) = 1$. Then the general solution in natural numbers of this equation is:

$$\begin{cases} x = bk + x_0 \\ y = ak + y_0 \end{cases} \text{ where } (x_0, y_0) \text{ is a particular integer solution of the equation,}$$

and $k \geq \max\{[-x_0/b], [-y_0/a]\}$ is an integer parameter (generalization of lemma 2 of [4]).

Proof: It results from [1] that the general integer solution of the equation is

$$\begin{cases} x = bk + x_0 \\ y = ak + y_0 \end{cases} \text{ where } (x_0, y_0) \text{ is a particular integer solution of the equation and}$$

$k \in \mathbb{Z}$. Since x and y are natural integers, it is necessary for us to impose conditions for k such that $x \geq 0$ and $y \geq 0$, from which it results the theorem.

WE CONCLUDE!

To solve **in the set of natural numbers** a linear equation with n unknowns we will use the previous results in the following way:

- a) If the equation does not have variations of sign, because it has a limited number of natural solutions, the solving is made by tests (see also [6])
- b) If it has variations of sign and if b is divisible by d , then it admits an infinity of natural solutions. One finds its general integer solution (see [2], [5]);

$$x_i = \sum_{j=1}^{n-1} \alpha_{ij} k_j + \beta_i, \quad 1 \leq i \leq n \text{ where all the } \alpha_{ij}, \beta_i \in \mathbb{Z} \text{ and the } k_j \text{ are integer}$$

parameters.

By applying the restriction $x_i \geq 0$ for i from $\{1, 2, \dots, n\}$, one finds the conditions which must be satisfied by the integer parameters k_j for all j of $\{1, 2, \dots, n-1\}$. (c)

The case $n = 2$ and $n = 3$ can be done by this method, but when n is bigger, the conditions (c) become more and more difficult to find.

Example: Solve in \mathbb{N} the equation $3x - 7y + 2z = -18$.

Solution: In \mathbb{Z} one obtains the general integer solution:

$$\begin{cases} x = k_1 \\ y = k_1 + 2k_2 \\ z = 2k_1 + 7k_2 - 9 \end{cases} \text{ with } k_1 \text{ and } k_2 \text{ in } \mathbb{Z}.$$

From the conditions (c) result the inequalities $x \geq 0, y \geq 0, z \geq 0$. It results that $k_1 \geq 0$ and also:

$$k_2 \geq \lceil -k_1/2 \rceil + 1 \text{ if } -k_1/2 \notin \mathbb{Z}, \text{ or } k_2 \geq -k_1/2 \text{ if } -k_1/2 \in \mathbb{Z};$$

$$\text{and } k_2 \geq \lceil (9-2k_1)/7 \rceil + 1 \text{ if } (9-2k_1)/7 \notin \mathbb{Z}, \text{ or } k_2 \geq (9-2k_1)/7 \text{ if } (9-2k_1)/7 \in \mathbb{Z};$$

$$\text{that is } k_2 \geq \lceil (2-2k_1)/7 \rceil + 2 \text{ if } (2-2k_1)/7 \notin \mathbb{Z}, \text{ or } k_2 \geq (2-2k_1)/7 + 1 \text{ if } (2-2k_1)/7 \in \mathbb{Z}.$$

With these conditions on k_1 and k_2 we have the general solution in natural numbers of the equation.

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