New Seiberg-Witten Fields Maps Through Weyl Symmetrization and The Pure Geometric Extension of The Standard Model

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Abstract

A unified description of a symmetrized and anti-symmetrized Moyal star product of the noncommutative infinitesimal gauge transformations is presented and the corresponding Seiberg-Witten maps are derived. Moreover, the noncommutative covariant derivative, field strenght tensor as well as gauge transformations are shown to be consistently constructed not on the enveloping but on the Lie and/or Poisson algebra. As an application, a pure geometric extension of the standard model is shown explicitly.

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1 Introduction

In the last few years, noncommutative geometry becomes the focus of interest in theoretical physics and for models building$[1] - [9]$. There are several motivations to speculate that the space-time becomes non commutative at very short distances when quantum gravity becomes relevant and even better, if we believe that the extra dimension approach can push the non commutativity scale lower. Moreover, in string theories, the noncommutative gauge theory appears as a certain limit in the presence of a background field$[11]$.

One approach to the non commutative geometry, is the one based on the deformation of the space-time. If fields are assumed to be Lie algebra valued and allow for the closure of the Lie algebra valued noncommutative transformation gauge parameters, it turns out only $U(N)$ structure groups are conceivable as well as the corresponding gauge transformations must be in the fundamental representation of this group. This is unsuitable to build realistic models for the electroweak and strong interactions and even the $U(1)$ case, charges are quantized and it will be impossible to describe quarks (the choice of charges introduced in the theory is very restricted to ± 1 or $0)[7]$. The matching of the noncommutative action to the ordinary one, requires first to map the noncommutative space-time coordinates to the ordinary ones by introducing a star product$[10]$, than the noncommutative fields are mapped to commutative ones by means of the Seiberg-Witten mapps$[1] - [6]$, $[11]$. The latter has the remarkable property that ordinary gauge transformations induce noncommutative ones. In this case, the low energy

1

action is local in the sense that there is no UV/IR mixing. However, the basic assumption is that the noncommutative fields are not Lie algebra valued but are in the enveloping algebra and allows to consider $SU(N)$ groups.

In section2, we define a new deformed Moyal-Weyl ordering product in the sense of a Weyl symmetrization using star product Lie or Poisson brackets with some golden rules to preserve the invariance of the action and the closure of the algebra and construct noncommutative gauge transformations, covariant derivative, field strength tensor and Seiberg-Witten maps fields without recourse to the envelopping algebra approach. As an application, we have constructed in section3 a pure geometric extension of the standard model. Finally, in section4, we draw our conclusions.

2   The Formalism

In ordinary quantum mechanics (Q.M), to find out the quantum equivalent of a classical observable $F(x,p)$ depending on the canonical variables x and p , one has to go through the symmetrization procedure using the so called Weyl-ordering. In this scheme, and as an example, the Weyl ordering $(x^n p)_w$ of any monomial of the form $x^n p$ is given by:

$$(x^n p)_w = \frac{1}{n+1} \sum_{l=0}^{n} x^{n-l} p\, x^l \tag{1}$$

By analogy to Q.M, one can define a Moyal-Weyl ordering $(f * h)_w$ of any two functions f and h on a

noncommutative space-time as follows:

$$(f * h)_w \equiv \widetilde{f * g} \tag{2}$$

where

$$\widetilde{f * g} \equiv 2 \, [f,g]_\eta^* \tag{3}$$

with

$$[f,g]_\eta^* \equiv \frac{1}{2}(f * g + \eta g * f) \tag{4}$$

where $\eta = \pm$. It is worth to mention that sometimes we use notations like in eq.(2) (with curly bracket on the top) like in the definition of the "w" (Moyal-Weyl) ordering etc...But if we want to get more

simplified forms (compact) like in eq.(12) etc..., it is better to use notations of eq.(3)(with generalized $\eta-$commutators). Of course both notations are equivalent.

Now, let us consider the two matrices valued functions $\Lambda = T_a \Lambda^a$ and $V_\mu = T_a V_\mu^a$ as elements of a Lie algebra $\mathcal{G}$ of a Lie group G $(SU(N), U(N),$etc..)and ψ a matter multiplet in a certain group representation such that it transforms as:

$$\delta_\Lambda \psi = i T_a \Lambda^a \psi; \qquad \delta_\Lambda \overline{\psi} = -i T_a \Lambda^a \overline{\psi} \tag{5}$$

Here Λ^a is the group transformation parameter and T_a are the group generators.

In the noncommutative space-time, one can generalize the ordinary gauge transformations (eq.(5)) by respecting the Weyl ordering as:

$$\delta_{\widehat{\Lambda}}\widehat{\psi} = \left(iT^a\left[\widehat{\Lambda}^a\overset{*}{,}\widehat{\psi}\right]_\eta\right)_w = \frac{1}{2}iT_a\overbrace{\widehat{\Lambda}^a * \widehat{\psi}};$$

$$\delta_{\widehat{\Lambda}}\widehat{\overline{\psi}} = \left(-iT^a\left[\widehat{\overline{\psi}}\overset{*}{,}\widehat{\Lambda}^a\right]_\eta\right)_w = -\frac{1}{2}iT_a\overbrace{\widehat{\overline{\psi}} * \widehat{\Lambda}^a} \tag{6}$$

The $\widehat{\Lambda}^a$ and $\widehat{\psi}$ are the gauge transformation parameter and matter field in the noncommutative space-time respectively. Now, we impose the following golden rules:

$$(T_aT_b\widehat{\Lambda}^a * \widehat{\Sigma}^b)_w = T_aT_b\overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b}$$

$$(T_aT_b\widehat{\Sigma}^b * \widehat{\Lambda}^a)_w = T_aT_b\overbrace{\widehat{\Sigma}^b * \widehat{\Lambda}^a} \tag{7}$$

$$(T_aT_b\widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b)_w = T_aT_b\overbrace{\widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b}$$

and

$$(T_aT_b\widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a)_w = T_bT_a\overbrace{\widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a}$$

where

$$\overbrace{A * B * C} \equiv (A * B * C + \eta C * B * A) \tag{8}$$

Notice that, in eqs.(7) we have to respect the order of the indices a,b,...before the symmetrization procedure take place.Moreover, the generators indices must be in the same order as those of $\widehat{\Sigma}$ and $\widehat{\Lambda}$.Now using eq.(6) and the golden rules of eqs.(7) , one can show easily that (see AppendixA):

$$\delta_{\widehat{\Sigma}}(\delta_{\widehat{\Lambda}}\widehat{\psi}) = \left(-T_aT_b\left[\widehat{\Lambda}^a,[\widehat{\Sigma}^b,\widehat{\psi}]^*_\eta\right]^*_\eta\right)_w$$

$$= \frac{1}{4}\left(-T_aT_b\overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} *\widehat{\psi} - T_bT_a\eta\overbrace{\widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a} - T_aT_b\eta\overbrace{\widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b} - T_aT_b\eta^2\widehat{\psi} * \overbrace{\widehat{\Sigma}^b * \widehat{\Lambda}^a}\right)(9)$$

and

$$\delta_{\widehat{\Lambda}}(\delta_{\widehat{\Sigma}}\widehat{\psi}) = \left(-T_bT_a\left[\widehat{\Sigma}^b,[\widehat{\Lambda}^a,\widehat{\psi}]^*_\eta\right]^*_\eta\right)_w$$

$$= \frac{1}{4}\left(-T_bT_a\overbrace{\widehat{\Sigma}^b * \widehat{\Lambda}^a} *\widehat{\psi} - T_aT_b\eta\overbrace{\widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b} - T_bT_a\eta\overbrace{\widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a} - T_bT_a\eta^2\widehat{\psi} * \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b}\right)(10)$$

Using eqs. (3) and (4) together with the relations:

$$\widehat{\widehat{\Sigma}^b * \widehat{\Lambda}^a} * \widehat{\psi} = \eta \,\widehat{\widehat{\Lambda}^a * \widehat{\Sigma}^b} * \widehat{\psi}$$

$$\widehat{\psi} * \widehat{\widehat{\Lambda}^a * \widehat{\Sigma}^b} = \eta \widehat{\psi} * \widehat{\widehat{\Sigma}^b * \widehat{\Lambda}^a}$$

$$\widehat{\widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a} = \eta \,\widehat{\widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b} \tag{11}$$

and after straightforward simplifications, we obtain:

$$\left[\delta_{\widehat{\Lambda}}, \delta_{\widehat{\Sigma}}\right] \widehat{\psi} = [T_a, T_b]_{-\eta} \left[\left[\widehat{\Lambda}^a, \widehat{\Sigma}^b\right]^*_\eta, \widehat{\psi}\right]^*_\eta \tag{12}$$

Thus, the algebra is closed and the gauge parameters $\widehat{\Sigma}$ and $\widehat{\Lambda}$ are not elements of the envelopping Lie algebra $\mathcal{G}$.

Concerning the covariant derivative $\widehat{\mathcal{D}}_\mu \psi$ of a matter field , its expression can be generalized easily in the noncommutative case as follows:

$$\widehat{\mathcal{D}}_\mu \widehat{\psi} = \partial_\mu \widehat{\psi} - iT_a \left[\widehat{V}^a_\mu, \widehat{\psi}\right]^*_\eta \tag{13}$$

where, $\widehat{V}^a_\mu$ denotes the gauge field in the noncommutative space-time. It is worth to mention that the covariant derivative of eq.(13) transforms as:

$$\begin{aligned}
\delta_{\widehat{\Lambda}}(\widehat{\mathcal{D}}_\mu \widehat{\psi}) &= iT^a \left[\widehat{\Lambda}^a, \partial_\mu \widehat{\psi}\right]^*_\eta + \left(-iT^a iT^b \left[\widehat{\Lambda}^a, \left[\widehat{V}^a_\mu, \widehat{\psi}\right]^*_\eta\right]^*_\eta\right)_w \\
&= iT^a \left[\widehat{\Lambda}^a, \partial_\mu \widehat{\psi}\right]^*_\eta + \frac{1}{4}\left(\begin{array}{c} -iT^a iT^b \widehat{\widehat{\Lambda}^a * \widehat{V}^b_\mu} * \widehat{\psi} - \eta iT^b iT^a \widehat{\widehat{V}^b_\mu * \widehat{\psi}} * \widehat{\Lambda}^a \\ -\eta iT^a iT^b \widehat{\widehat{\Lambda}^a * \widehat{\psi}} * \widehat{V}^b_\mu - \eta^2 iT^a iT^b \widehat{\psi} * \widehat{\widehat{V}^b_\mu * \widehat{\Lambda}^a} \end{array}\right) \tag{14}
\end{aligned}$$

On the other hand, a tedious but direct calculation gives (see Appendix B):

$$\begin{aligned}
\delta_{\widehat{\Lambda}}\left(\widehat{\mathcal{D}}_\mu \widehat{\psi}\right) &= \partial_\mu \left(iT^a \left[\widehat{\Lambda}^a, \widehat{\psi}\right]^*_\eta\right)_w + \left(-iT^b \left[\widehat{V}^b_\mu, \delta_\Lambda \widehat{\psi}\right]^*_\eta - iT^b \left[\delta_\Lambda \widehat{V}^b_\mu, \widehat{\psi}\right]^*_\eta\right)_w \\
&= iT^a \left[\widehat{\Lambda}^a, \partial_\mu \widehat{\psi}\right]^*_\eta + iT^a \left[\partial_\mu \widehat{\Lambda}^a, \widehat{\psi}\right]^*_\eta - iT^b \left[\delta_\Lambda \widehat{V}^b_\mu, \widehat{\psi}\right]^*_\eta \\
&\quad + \frac{1}{4}\left(\begin{array}{c} -iT^b iT^a \widehat{\widehat{V}^b_\mu * \widehat{\Lambda}^a} * \widehat{\psi} - \eta iT^a iT^b \widehat{\widehat{\Lambda}^a * \widehat{\psi}} * \widehat{V}^b_\mu \\ -\eta iT^b iT^a \widehat{\widehat{V}^b_\mu * \widehat{\psi}} * \widehat{\Lambda}^a - \eta^2 iT^b iT^a \widehat{\psi} * \widehat{\widehat{\Lambda}^a * \widehat{V}^b_\mu} \end{array}\right) \tag{15}
\end{aligned}$$

Now, from eqs.(13) and (14), it follows the noncommutative gauge boson transformation law (see Appendix B):

$$\delta_{\widehat{\Lambda}} \widehat{V}_\mu = \partial_\mu \widehat{\Lambda} + i [T_a, T_b]_{-\eta} \left[\widehat{\Lambda}^a, \widehat{V}^b_\mu\right]^*_\eta \tag{16}$$

$(\widehat{V}_\mu = T_a \widehat{V}^a_\mu, \widehat{\Lambda} = T_a \widehat{\Lambda}^a)$.Now, within the infinitesimal noncommutative gauge transformations of eqs.(6) and (16), one can show the invariance of the action I_1 representing the kinetic term of the noncommutative matter fields $\widehat{\psi}$ and their interaction with the noncommutative vector gauge boson $\widehat{V}^b_\mu$:

$$I_1 = i \int d^4x \; \widehat{\overline{\psi}} * \widehat{\not{D}} * \widehat{\psi} \tag{17}$$

where

$$\widehat{\not{D}} = \gamma^\mu \widehat{\mathcal{D}}_\mu$$

(γ^μ stands for Dirac matrices). In fact:

$$\delta_{\widehat{\Lambda}} I_1 = \int d^4x \left\{ -iT^a \left[\widehat{\overline{\psi}}, \widehat{\Lambda}^a \right]_\eta^* * i\gamma^\mu (\partial_\mu \widehat{\psi} - iT^b \left[\widehat{V}_\mu^b, \widehat{\psi} \right]_\eta^*) \right\}_w$$
$$+ \int d^4x \left\{ \widehat{\overline{\psi}} * \left(i\gamma^\mu iT^a \left[\widehat{\Lambda}^a, \partial_\mu \widehat{\psi} \right]_\eta^* + \left(-iT^a iT^b \left[\widehat{\Lambda}^a, \left[\widehat{V}_\mu^a, \widehat{\psi} \right]_\eta^* \right]_\eta^* \right) \right) \right\}_w \tag{18}$$

Direct simplifications give;

$$\delta_{\widehat{\Lambda}} I_1 = \frac{-\eta}{2} \int d^4x (iT^a \widehat{\Lambda}^a * \widehat{\overline{\psi}} * i\gamma^\mu \partial_\mu \widehat{\psi} - \widehat{\overline{\psi}} * i\gamma^\mu iT^a \partial_\mu \widehat{\psi} * \widehat{\Lambda}^a)_w$$
$$+ \int d^4x ((\frac{-i}{2}\eta T^a \widehat{\Lambda}^a * \widehat{\overline{\psi}} * i\gamma^\mu (-\frac{i}{2} T^b V_\mu^b * \widehat{\psi} - \eta \frac{i}{2} T^b \widehat{\psi} * V_\mu^b)$$
$$+ \frac{1}{4} \widehat{\overline{\psi}} * i\gamma^\mu (-iT^a iT^b (\eta \widehat{V}_\mu^b * \widehat{\psi} * \widehat{\Lambda}^a + \eta^2 \widehat{\psi} * \widehat{V}_\mu^b * \widehat{\Lambda}^a)))_w \tag{19}$$

Using the following proprety of the star product:

$$\int d^4x \; f * (g * h) = \int d^4x \; f.(g * h) = \int d^4x \; (g * h).f = \int d^4x \; (g * h) * f \tag{20}$$

one has:

$$\int d^4x (iT^a \widehat{\Lambda}^a * \widehat{\overline{\psi}} * i\gamma^\mu \partial_\mu \widehat{\psi}) = \int d^4x (\widehat{\overline{\psi}} * i\gamma^\mu iT^a \partial_\mu \widehat{\psi}) * \widehat{\Lambda}^a \tag{21}$$

$$\int d^4x \frac{i}{2} T^a \widehat{\Lambda}^a * \widehat{\overline{\psi}} * i\gamma^\mu (-\frac{i}{2} T^b \widehat{V}_\mu^b * \widehat{\psi}) = \frac{1}{4} \int d^4x (\widehat{\overline{\psi}} * i\gamma^\mu (-iT^a iT^b \widehat{V}_\mu^b * \widehat{\psi})) * \widehat{\Lambda}^a \tag{22}$$

and

$$\int d^4x ((\frac{i}{2} T^a \widehat{\Lambda}^a * \widehat{\overline{\psi}} * i\gamma^\mu (\frac{i}{2} T^b \widehat{\psi} * \widehat{V}_\mu^b) = -\frac{1}{4} \int d^4x (\widehat{\overline{\psi}} * i\gamma^\mu (-iT^a iT^b \widehat{\psi} * \widehat{V}_\mu^b)) * \widehat{\Lambda}^a \tag{23}$$

Therefore, we deduce that:

$$\delta_{\widehat{\Lambda}} I_1 = 0 \tag{24}$$

Similarly, one can show that the mass term $I_2 = \int d^4x \; \widehat{\overline{\psi}} * \widehat{\psi}$ is gauge invariant. In fact, taking into account the noncommutative gauge transformation laws of eq.(6), one can write

5

$$\delta_{\widehat{\Lambda}} I_2 = \int d^4x \left(-iT^a \left[\widehat{\overline{\psi}}, \widehat{\Lambda}^a \right]_\eta^* * \widehat{\psi} + \widehat{\overline{\psi}} * iT^a \left[\widehat{\Lambda}^a, \widehat{\psi} \right]_\eta^* \right)$$

$$= \frac{-i\eta}{2} T^a \int d^4x (\widehat{\Lambda}^a * \widehat{\overline{\psi}} * \widehat{\psi} - \widehat{\overline{\psi}} * \widehat{\psi} * \widehat{\Lambda}^a) \tag{25}$$

Again, using the associativity of the star product and the relation in eq.(20) we obtain:

$$\delta_{\widehat{\Lambda}} I_2 = 0 \tag{26}$$

Regarding the noncommutative field strength $\widehat{F}_{\mu\nu}^a$, one can generalize the definition of the curvature tensor such that:

$$\left[\widehat{\mathcal{D}}_\mu, \widehat{\mathcal{D}}_\nu \right] \widehat{\psi} = -iT_a \left[\widehat{F}_{\mu\nu}^a, \widehat{\psi} \right]_\eta^* \tag{27}$$

Using the expression of the noncommutative covariant derivative given by eq.(13), one can show that:

$$\widehat{\mathcal{D}}_\mu \widehat{\mathcal{D}}_\nu \widehat{\psi} = \left(\partial_\mu \widehat{\mathcal{D}}_\nu \widehat{\psi} - iT^a \left[\widehat{V}_\mu^a, \widehat{\mathcal{D}}_\nu \widehat{\psi} \right]_\eta^* \right)_w \tag{28}$$

With the help of eqs.(7) as well as eq.(13) , eq.(28) can be rewritten as:

$$\widehat{\mathcal{D}}_\mu \widehat{\mathcal{D}}_\nu \widehat{\psi} = \partial_\mu \partial_\nu \widehat{\psi} - iT^a \left[\partial_\mu \widehat{V}_\nu^a, \widehat{\psi} \right]_\eta^* - iT^a \left[\widehat{V}_\nu^a, \partial_\mu \widehat{\psi} \right]_\eta^* - iT^a \left[\widehat{V}_\mu^a, \partial_\nu \widehat{\psi} \right]_\eta^*$$

$$\frac{1}{4} \left(\begin{array}{c} -T^aT^b \overbrace{\widehat{V}_\mu^a * \widehat{V}_\nu^b} * \widehat{\psi} - \eta T^aT^b \overbrace{\widehat{V}_\mu^a * \widehat{\psi} * \widehat{V}_\nu^b} \\ -\eta T^bT^a \overbrace{\widehat{V}_\nu^b * \widehat{\psi} * \widehat{V}_\mu^a} - \eta^2 T^aT^b \widehat{\psi} * \overbrace{\widehat{V}_\nu^b * \widehat{V}_\mu^a} \end{array} \right) \tag{29}$$

Using the relations of eqs.(11), we deduce that:

$$\left[\widehat{\mathcal{D}}_\mu, \widehat{\mathcal{D}}_\nu \right] \widehat{\psi} \equiv -iT^a \left[\widehat{F}_{\mu\nu}^a, \widehat{\psi} \right]_\eta^*$$

$$= -iT^a \left[\partial_\mu \widehat{V}_\nu^a - \partial_\nu \widehat{V}_\mu^a, \widehat{\psi} \right]_\eta^* - \frac{1}{4} \left[T^a, T^b \right]_{-\eta} \overbrace{\widehat{V}_\mu^a * \widehat{V}_\nu^b} * \widehat{\psi}$$

$$+ \frac{1}{4} \left[T^a, T^b \right]_{-\eta} \widehat{\psi} * \overbrace{\widehat{V}_\mu^a * \widehat{V}_\nu^b} \tag{30}$$

Finally,

$$\left[\widehat{F}_{\mu\nu}^a, \widehat{\psi} \right]_\eta^* = \left[\partial_\mu \widehat{V}_\nu - \partial_\nu \widehat{V}_\mu, \widehat{\psi} \right]_\eta^* - i \left[T^a, T^b \right]_{-\eta} \left[\left[\widehat{V}_\mu^a, \widehat{V}_\nu^b \right]_\eta^*, \widehat{\psi} \right]_\eta^* \tag{31}$$

or

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{V}_\nu - \partial_\nu \widehat{V}_\mu - i \left[T^a, T^b \right]_{-\eta} \left[\widehat{V}_\mu^a, \widehat{V}_\nu^b \right]_\eta^* \tag{32}$$

6

Notice that one can rewrite eq.(32) as:

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{V}_\nu - \partial_\nu \widehat{V}_\mu - \frac{i}{2} \left(\begin{array}{c} T^a T^b \widehat{V}_\mu^a * \widehat{V}_\nu^b + \eta T^a T^b \widehat{V}_\mu^a \,\bar{*}\, \widehat{V}_\nu^b \\ -\eta T^b T^a \widehat{V}_\nu^b \,\bar{*}\, \widehat{V}_\mu^a - \eta^2 T^b T^a \widehat{V}_\nu^b * \widehat{V}_\mu^a \end{array} \right) \tag{33}$$

or in a more compact form as:

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{V}_\nu - \partial_\nu \widehat{V}_\mu - i \left[\widehat{V}_\mu, \widehat{V}_\nu \right]_-^* - i\eta \left[\widehat{V}_\mu, \widehat{V}_\nu \right]_-^{\bar{*}} \tag{34}$$

where

$$f * g = g \,\bar{*}\, f \tag{35}$$

with

$$f * g = G(f, g, \theta^{\mu\nu}) \tag{36}$$

and

$$f \,\bar{*}\, g = G(f, g, -\theta^{\mu\nu}) \tag{37}$$

Here $G(f, g, \theta^{\mu\nu})$ is a function of f, g and $\theta^{\mu\nu}$. Similarly one can show that:

$$\delta_{\widehat{\Lambda}} \widehat{V}_\mu = \partial_\mu \widehat{\Lambda} + \frac{i}{2} \left(\widehat{\Lambda} * \widehat{V}_\mu + \eta \widehat{\Lambda} \,\bar{*}\, \widehat{V}_\mu - \eta \widehat{V}_\mu \,\bar{*}\, \widehat{\Lambda} - \eta^2 \widehat{V}_\mu * \widehat{\Lambda} \right) \tag{38}$$

Now, from eqs.(34) and (38), one has:

$$\delta_{\widehat{\Lambda}} \widehat{F}_{\mu\nu} = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5 \tag{39}$$

where

$$\Omega_1 = i \left[\widehat{\Lambda}, \partial_\mu \widehat{V}_\nu - \partial_\nu \widehat{V}_\mu \right]_-^* + i\eta \left[\widehat{\Lambda}, \partial_\mu \widehat{V}_\nu - \partial_\nu \widehat{V}_\mu \right]_-^{\bar{*}} \tag{40}$$

$$\Omega_2 = \left[\widehat{\Lambda}, \left[\widehat{V}_\mu, \widehat{V}_\nu \right]_-^* \right]_-^* \tag{41}$$

$$\Omega_3 = \eta \left[\widehat{\Lambda}, \left[\widehat{V}_\mu, \widehat{V}_\nu \right]_-^* \right]_-^{\bar{*}} \tag{42}$$

$$\Omega_4 = \eta \left[\widehat{\Lambda}, \left[\widehat{V}_\mu, \widehat{V}_\nu \right]_-^{\bar{*}} \right]_-^* \tag{43}$$

and

$$\Omega_5 = \left[\widehat{\Lambda}, \left[\widehat{V}_\mu, \widehat{V}_\nu \right]_-^{\bar{*}} \right]_-^{\bar{*}} \tag{44}$$

straightforward simplifications give:

7

$$\delta_{\widehat{\Lambda}} \widehat{F}_{\mu\nu} == i \left[\widehat{\Lambda}, \widehat{F}_{\mu\nu}\right]_{-}^{*} + i\eta \left[\widehat{\Lambda}, \widehat{F}_{\mu\nu}\right]_{-}^{\overline{*}} \tag{45}$$

since

$$
\begin{aligned}
\left[\widehat{\Lambda}, \widehat{F}_{\mu\nu}\right]_{-}^{*} + \eta \left[\widehat{\Lambda}, \widehat{F}_{\mu\nu}\right]_{-}^{\overline{*}} &= T^a T^b \widehat{\Lambda}^a * \widehat{F}_{\mu\nu}^b - T^b T^a \widehat{F}_{\mu\nu}^b * \widehat{\Lambda}^a + \eta T^a T^b \widehat{\Lambda}^a \overline{*} \widehat{F}_{\mu\nu}^b - \eta T^b T^a \widehat{F}_{\mu\nu}^b \overline{*} \widehat{\Lambda}^a \\
&= T^a T^b \widehat{\Lambda}^a * \widehat{F}_{\mu\nu}^b - T^b T^a \widehat{F}_{\mu\nu}^b * \widehat{\Lambda}^a + \eta T^a T^b \widehat{F}_{\mu\nu}^b * \widehat{\Lambda}^a - \eta T^b T^a \widehat{\Lambda}^a * \widehat{F}_{\mu\nu}^b \\
&= \left[T^a, T^b\right]_{-\eta} \left[\widehat{\Lambda}^a, \widehat{F}_{\mu\nu}^b\right]_{\eta}^{*}
\end{aligned}
\tag{46}
$$

therefore

$$\delta_{\widehat{\Lambda}} \widehat{F}_{\mu\nu} = i \left[T^a, T^b\right]_{-\eta} \left[\widehat{\Lambda}^a, \widehat{F}_{\mu\nu}^b\right]_{\eta}^{*} \tag{47}$$

Regarding the gauge invariance of the noncommutative Yang-Mills action I_{YM} defined as:

$$I_{YM} = \int d^4x \, Tr \, (\widehat{F}_{\mu\nu} * \widehat{F}^{\mu\nu}) \tag{48}$$

and with the help of the transformation law of eq. (47), one has:

$$
\begin{aligned}
\delta_{\widehat{\Lambda}} I_{YM} &= iTr\left(\left[T^a, T^b\right]_{-\eta} T^c\right) \int d^4x \left[\widehat{\Lambda}^a, \widehat{F}_{\mu\nu}^b\right]_{\eta}^{*} * \widehat{F}_{\mu\nu}^c \\
&\quad -iTr\left(T^c \left[T^a, T^b\right]_{-\eta}\right) \int d^4x \widehat{F}_{\mu\nu}^c * \left[\widehat{\Lambda}^a, \widehat{F}_{\mu\nu}^b\right]_{\eta}^{*}
\end{aligned}
\tag{49}
$$

Using the fact that:

$$Tr(ABC) = Tr(CBA) \tag{50}$$

(A, B and C are matrices) and the star product property in eq.(20), then

$$Tr\left(\left[T^a, T^b\right]_{-\eta} T^c\right) = Tr\left(T^c \left[T^a, T^b\right]_{-\eta}\right) \tag{51}$$

and

$$\int d^4x \left[\widehat{\Lambda}^a, \widehat{F}_{\mu\nu}^b\right]_{\eta}^{*} * \widehat{F}_{\mu\nu}^c = \int d^4x \widehat{F}_{\mu\nu}^c * \left[\widehat{\Lambda}^a, \widehat{F}_{\mu\nu}^b\right]_{\eta}^{*} \tag{52}$$

Consequently

$$\delta_{\widehat{\Lambda}} I_{YM} = 0 \tag{53}$$

Regarding the Seiberg-Witten maps, if one sets:

$$\widehat{V}_{\mu} = \widehat{V}_{\mu}\left[V\right] = V_{\mu} + \widetilde{V}_{\mu} \tag{54}$$

$$\widehat{\psi} = \widehat{\psi}\left[V, \psi\right] = \psi + \widetilde{\psi} \tag{55}$$

and

$$\widehat{\Lambda} = \widehat{\Lambda}\,[V, \Lambda] = \Lambda + \widetilde{\Lambda} \tag{56}$$

where

$$\delta V_\mu = \partial_\mu \Lambda + i\,[\Lambda, V_\mu] \tag{57}$$

and

$$\delta_\Lambda \psi = i\Lambda\,\psi \tag{58}$$

and uses the transformation laws in eqs.(6) and (16), we obtain:

$$\widetilde{V}_\mu = \frac{1}{4}\theta^{\alpha\beta}\left\{\delta_{\eta,-}\left([F_{\alpha\mu}, V_\beta]_\eta + [V_\beta, \partial_\alpha V_\mu]_\eta\right) + 4\delta_{\eta,+}\left([V_\beta, \partial_\alpha V_\mu]_\eta - \frac{1}{2}[V_\beta, \partial_\mu V_\alpha]_\eta\right)\right\} + O(\theta^2) \tag{59}$$

$$\widetilde{\psi} = \frac{-i}{8}\theta^{\alpha\beta}\left\{\delta_{\eta,-}[V_\alpha, V_\beta]_\eta\,\psi + 8i\delta_{\eta,+}[V_\beta\partial_\alpha\psi + F_{\alpha\beta}\psi]\right\} + O(\theta^2) \tag{60}$$

and

$$\widetilde{\Lambda} = \frac{1}{4}\theta^{\alpha\beta}\left\{\delta_{\eta,-}[V_\beta, \partial_\alpha\Lambda]_\eta + 2\delta_{\eta,+}[V_\beta, \partial_\alpha\Lambda]_\eta\right\} + O(\theta^2) \tag{61}$$

To give further clarifications, it is worth to mention that the paper may give the impression that we have defined in our approach a new "Moyal-Weyl ordering" related to the noncommutative space-time which makes a confusion. In reality, we have just defined new gauge transformations , covariant derivatives, Seiberg-Witten maps through what we have called "Moyal-Weyl ordering". It is just a breach of trust.We mean Weyl symmetrization through Poisson or Lie brackets in the expressions of gauge transformations, definitions of covariant derivative etc..by respecting certain golden rules which are necessary for the invariance of the action and the closure of the algebra and which involve Moyal star product. So, in this paper we did not change the Moyal-Weyl product initially introduced in the noncommutative space-time mathematical formalism and the isomorphism between the classical functions and the corresponding operators still holds. In fact, we can always associate to any function $f(x)$ of the classical commutative space-time an operator denoted by a $W(f)$ and defined by:

$$W(f) = (2\pi)^{-\frac{3}{2}} \int d^4k\; e^{-ik\widehat{x}}\; \widetilde{f}(k) \tag{62}$$

where $\widetilde{f}(k)$ is its Fourrier transform and $\widehat{x}_\mu$ the noncommutative variable.

$$\widetilde{f}(k) = (2\pi)^{-\frac{3}{2}} \int d^4x\; e^{ikx}\; f(x) \tag{63}$$

These operators $W(f), W(g)$, etc..can be multiplied to give other operators. The product operator $W(f).W(g)$ is itself associated to a classical function $h(x) = (f * g)(x)$ such that:

$$W(f)W(g) = W(f * g) \tag{64}$$

where $(f * g)(x)$ is a function of the classical variable x_μ

$$(f * g)(x) = \left[e^{\frac{1}{2}i\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}}\left(f(x)\,g(y)\right)\right]_{x=y} \tag{65}$$

Notice that at the level of the infinitesimal gauge transformations introducing what we have called

9

Moyal-Weyl ordering and meaning the symmetrization by using Poisson or Lie brackets, the operator counterpart in the lagrangian does not change. However for the definition of the covariante derivative it is exactly equivalent to have this symmetrization procedure at the level of the operators counterparts. for example

$$\widehat{\overline{\boldsymbol{\Psi}}}(x) * \gamma^\mu \widehat{\mathcal{D}}_\mu \widehat{\boldsymbol{\Psi}}(x) = \widehat{\overline{\boldsymbol{\Psi}}}(x) * \gamma^\mu \partial_\mu \widehat{\boldsymbol{\Psi}}(x) - iT_a \widehat{\overline{\boldsymbol{\Psi}}}(x) * \gamma^\mu \left[\widehat{\mathcal{A}}_\mu^a(x), \widehat{\boldsymbol{\Psi}}(x) \right]_\eta^* \tag{66}$$

can be defined at the operator level as:

$$\widehat{\overline{\boldsymbol{\Psi}}}(\widehat{x}) \gamma^\mu \widehat{\mathcal{D}}_\mu \widehat{\boldsymbol{\Psi}}(\widehat{x}) = \widehat{\overline{\boldsymbol{\Psi}}}(\widehat{x}) \gamma^\mu \partial_\mu \widehat{\boldsymbol{\Psi}}(\widehat{x}) - iT_a \widehat{\overline{\boldsymbol{\Psi}}}(\widehat{x}) \gamma^\mu \left[\widehat{\mathcal{A}}_\mu^a(\widehat{x}), \widehat{\boldsymbol{\Psi}}(\widehat{x}) \right]_\eta \tag{67}$$

This is always possible because we are dealing with operators and thus, always the ordering ambiguities arise. Now, the choice of a one symmetrization from another depends on what we want to achieve (of course we have to respect certain golden rules if we are dealing also with the Lie generators algebra T^a). Essentially, which is important is the closure of the algebra.where the noncommutative gauge transformations parameters $\widehat{\Lambda}$ are elements of the Lie or Poisson (not the envelopping)algebra of the local gauge group $\mathcal{G}$.

Moreover, one may ask about the necessity and a possible use of the antisymmetrization through star product commutators. If we take for example a$U(1)$gauge theory, where a singlet matter field Φ transforms trivially under the symmetry group as:

$$\delta_\Lambda \Phi(x) = 0 \tag{68}$$

The noncommutative equivalent of this transformation could be generalized to:

$$\widehat{\delta}_{\widehat{\Lambda}} \widehat{\Phi}(x) = i \left[\widehat{\Lambda}, \widehat{\Phi} \right]_-^* . \equiv i \left[\widehat{\Lambda}, \widehat{\Phi} \right] \tag{69}$$

where of course in the limit $\theta \to 0$ we find the classical commutative case. Another important possible application of the antisymmetrisation in defining Moyal-Weyl ordering is the creation of new interactions invariant under new symmetries of a noncommutative gauge group of the $O(\theta)$ or higher and such that at the limit $\theta \to 0$, these interactions will be switched off and disappear. For example, if we take in the commutative space-time the kinetic term of the action I of a Dirac spinor field ψ such that:

$$I = i \int d^4 x \overline{\psi} \gamma^\mu \partial_\mu \psi \tag{70}$$

This action is invariant under infinitesimal local gauge transformations of a certain Lie group G with generators T^a if and only if ψtransforms as a singlet. That is:

$$\delta_\Lambda \psi = \delta_\Lambda \overline{\psi} = 0 \tag{71}$$

Now, in the noncommutative space-time, the action I becomes I^* such that:

$$I^* = i \int d^4 x \widehat{\overline{\psi}} * \gamma^\mu \partial_\mu \widehat{\psi} \tag{72}$$

10

and the transformation laws of eq.(6) (within the antisymmetrization idea and generalization of a singlet state) becomes:

$$
\begin{aligned}
\delta_{\widehat{\Lambda}}\widehat{\psi} &= i\left[\widehat{\Lambda},\widehat{\psi}\right]_{-}^{*} = \frac{i}{2}\left(\widehat{\Lambda}*\widehat{\psi}-\widehat{\psi}*\widehat{\Lambda}\right) \\
&= -\frac{1}{2}\theta^{\mu\nu}\partial_{\mu}\widehat{\Lambda}\partial_{\nu}\widehat{\psi}+O(\theta^2)
\end{aligned}
\tag{73}
$$

Now, the action I^* is no more invariant under these noncommutative transformations of the matter field. To do so, we transform the ordinary derivative ∂_{μ} into a covariant one $\widehat{\mathcal{D}}_{\mu}$ such that:

$$
\widehat{\mathcal{D}}_{\mu}\widehat{\psi} \equiv (\partial_{\mu}\widehat{\psi}-iT^{a}\left[\widehat{V}_{\mu}^{a}{}^{*}{,}\widehat{\psi}\right])_{w} = \partial_{\mu}\widehat{\psi}-iT^{a}\left[\widehat{V}_{\mu}^{a}{}^{*}{,}\widehat{\psi}\right]
\tag{74}
$$

and

$$
\delta_{\widehat{\Lambda}}\widehat{V}_{\mu} = \partial_{\mu}\widehat{\Lambda}+i\left\{T^{a},T^{b}\right\}\left[\widehat{\Lambda}^{a}{}^{*}{,}\widehat{V}_{\mu}^{b}\right]
\tag{75}
$$

where $\widehat{V}_{\mu}$ is the noncommutative gauge boson.Then, the action I^* becomes $I^{*'}$ such that:

$$
I^{*'} = i\int d^{4}x\widehat{\overline{\psi}}*\gamma^{\mu}\widehat{\mathcal{D}}_{\mu}\widehat{\psi}
\tag{76}
$$

Notice that at the limit $\theta\to 0$, and since $\left[\widehat{V}_{\mu}^{a}{}^{*}{,}\widehat{\psi}\right]\to 0$, one has $\widehat{\mathcal{D}}_{\mu}\widehat{\psi}\to\partial_{\mu}\psi$ and we get back the action I. This means that although the interaction (force) between the matter and gauge field is absent in the commutative space-time, It is not (thanks to the antisymmetric transformations and the noncommutative generalization of the singlet state) in the noncommutative space-time. This is the way to generate new interactions within this approach. The most important point is that the order noncommutative parameter θ becomes like a scale for which new physics (interactions) becomes relevant. As a conclusion, if we want to extend any gauge theory and generate models beyond with a pure geometric noncommutative scale θ, we need to consider the antisymmetric noncommutative gauge transformations with the corresponding Seiberg-Witten maps.

3   Applications

Using the previous formalism, we construct a non abelian non commutative gauge theory invariant under the infinitesimal transformations of the gauge Lie group $SU(2)_{L}\times SU(2)_{R}\times U(1)_{Y}$ (left-right model). To keep our idea transparent, we will not consider the higgs and Yukawa sectors. The matter physical states are the doublets:

$$
L=\begin{pmatrix}\nu_{L}\\e_{L}\end{pmatrix}\text{ and }R=\begin{pmatrix}\nu_{R}\\e_{R}\end{pmatrix}
\tag{77}
$$

The isospin generators of the tensor group $SU(2)_{L}\times SU(2)_{R}$, are given by $T_{a}^{(L)}=T_{a}^{(R)}=\frac{\tau_{a}}{2}$ (τ_{a} are the Pauli matrices) and Y is the $U(1)$ hypercharge generator such that:

$$YL = -\frac{1}{2}\left(\begin{array}{c} \nu_L \\ e_L \end{array}\right),$$

$$YR = -\frac{1}{2}\left(\begin{array}{cc} 0 & 0 \\ 0 & 2 \end{array}\right)\left(\begin{array}{c} \nu_R \\ e_R \end{array}\right) \tag{78}$$

and

$$Y^{(L)} - Y^{(R)} = -\frac{1}{2}\tau_3$$

If we de note by $\mathcal{W}_\mu^a$, $\mathcal{B}_\mu^a$ and $\mathcal{A}_\mu$ the gauge potentials of the groups $SU(2)_L$, $SU(2)_R$, and $U(1)_Y$ .respectively,

and using the previous formalism of section2 the noncommutative matter fields covariant derivatives takes the following forms:

$$\widehat{\mathcal{D}}_\mu\widehat{L} = \partial_\mu\widehat{L} - ig_L T_a^{(L)}\left\{\widehat{\mathcal{W}}_\mu^a \overset{*}{,}\widehat{L}\right\} - ig'Y^{(L)}\left\{\widehat{\mathcal{A}}_\mu \overset{*}{,}\widehat{L}\right\} \tag{79}$$

and

$$\widehat{\mathcal{D}}_\mu\widehat{R} = \partial_\mu\widehat{R} - ig_R T_a^{(R)}\left[\widehat{\mathcal{B}}_\mu^a \overset{*}{,}\widehat{R}\right] - ig'Y^{(R)}\left\{\widehat{\mathcal{A}}_\mu \overset{*}{,}\widehat{R}\right\} \tag{80}$$

(here g', g_L and g_R denote the $U(1)_Y$, $SU(2)_L$ and$SU(2)_R$ coupling constants respectively). The left and right symmetrized noncommutative states $\widehat{L}$ and $\widehat{R}$ transform as follows:

$$\delta_{\widehat{\Lambda}_L}\widehat{L} = ig_L T_a^{(L)}\left\{\widehat{\Lambda}_L^a \overset{*}{,}\widehat{L}\right\} + ig'Y^{(L)}\left\{\widehat{\alpha} \overset{*}{,}\widehat{L}\right\} \tag{81}$$

and

$$\delta_{\widehat{\Lambda}_R}\widehat{R} = ig_R T_a^{(R)}\left[\widehat{\Lambda}_R^a \overset{*}{,}\widehat{R}\right] + ig'Y^{(R)}\left\{\widehat{\alpha} \overset{*}{,}\widehat{R}\right\} \tag{82}$$

($\widehat{\Lambda}_L^a, \widehat{\Lambda}_R^a$ and $\widehat{\alpha}$ are infinitesimal noncommutative gauge parameters). Thus, the noncommutative Lagrangian density $\mathfrak{L}_{NC}$ is given by:

$$\mathfrak{L}_{NC} = \overline{\widehat{L}}*i\widehat{\not{D}}\widehat{L} + \overline{\widehat{R}}*i\widehat{\not{D}}\widehat{R} - \frac{1}{2g_L^2}Tr_1\widehat{\boldsymbol{F}}^{\mu\nu}*\widehat{\boldsymbol{F}}_{\mu\nu} - \frac{1}{2g_R^2}Tr_2\widehat{\boldsymbol{G}}^{\mu\nu}*\widehat{\boldsymbol{G}}_{\mu\nu} - \frac{1}{4g'^2}\widehat{\boldsymbol{f}}^{\mu\nu}*\widehat{\boldsymbol{f}}_{\mu\nu} \tag{83}$$

(Tr_1 and Tr_2 represent the trace over the vector space of the fields) where

$$\begin{aligned} \widehat{\boldsymbol{F}}_{\mu\nu} &= \partial_\mu\widehat{\mathcal{W}}_\nu - \partial_\nu\widehat{\mathcal{W}}_\mu - ig_L\left[T_a^{(L)}, T_b^{(L)}\right]\left\{\widehat{\mathcal{W}}_\mu^a \overset{*}{,}\widehat{\mathcal{W}}_\nu^b\right\} \\ \widehat{\boldsymbol{G}}_{\mu\nu} &= \partial_\mu\widehat{\mathcal{B}}_\nu - \partial_\nu\widehat{\mathcal{B}}_\mu - ig_R\left\{T_a^{(R)}, T_b^{(R)}\right\}\left[\widehat{\mathcal{B}}_\mu^a \overset{*}{,}\widehat{\mathcal{B}}_\nu^b\right] \end{aligned} \tag{84}$$

and

$$\widehat{\boldsymbol{f}}_{\mu\nu} = \partial_\mu\widehat{\mathcal{A}}_\nu - \partial_\nu\widehat{\mathcal{A}}_\mu \tag{85}$$

The noncommutative fields $\widehat{L}(x)$, $\widehat{R}(x)$, $\widehat{\mathcal{W}}_\mu^a$, $\widehat{\mathcal{B}}_\mu^a$, $\widehat{\mathcal{A}}_\mu$,are the Seiberg-Witten maps of the classical fields $L(x)$, $R(x)$, $\mathcal{W}_\mu^a$, $\mathcal{B}_\mu^a$, $\mathcal{A}_\mu$,such that:

$$\widehat{L} = L - \frac{i}{8}\theta^{\mu\nu}[\mathcal{W}_\mu, \mathcal{W}_\nu]L + O\left(\theta^2\right)$$

$$\widehat{R} = R - \theta^{\mu\nu}\mathcal{B}_\mu\partial_\nu R + \theta^{\mu\nu}\boldsymbol{G}_{\mu\nu}R + O\left(\theta^2\right)$$

$$\widehat{\mathcal{W}}_\mu = \mathcal{W}_\mu + \frac{1}{4}\theta^{\nu\lambda}[\mathcal{W}_\lambda, \partial_\nu\mathcal{W}_\mu - \boldsymbol{F}_{\nu\mu}] + O\left(\theta^2\right) \qquad (86)$$

$$\widehat{\mathcal{B}}_\mu = \mathcal{B}_\mu - \theta^{\lambda\nu}\{\mathcal{B}_\lambda, \partial_\nu\mathcal{B}_\mu\} + \frac{1}{2}\theta^{\lambda\nu}\{\mathcal{B}_\lambda, \partial_\mu\mathcal{B}_\nu\} + O\left(\theta^2\right)$$

$$\widehat{\mathcal{A}}_\mu = \mathcal{A}_\mu + O\left(\theta^2\right)$$

Straightforward simplifications give:

$$\mathfrak{L}_{NC} = \mathfrak{L} + \mathfrak{L}_L^{(1)} + \mathfrak{L}_R^{(1)} + \mathfrak{L}_g^{(1)} \qquad (87)$$

where $\mathfrak{L}$ is the classical Lagrangian given by:

$$\mathfrak{L} = \overline{L}i\slashed{D}L + \overline{R}i\slashed{D}R - \frac{1}{2g_L^2}Tr_1\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{1}{2g_R^2}Tr_2\boldsymbol{G}^{\mu\nu}\boldsymbol{G}_{\mu\nu} - \frac{1}{4g'^2}\boldsymbol{f}^{\mu\nu}\boldsymbol{f}_{\mu\nu} \qquad (88)$$

and $\mathfrak{L}_L^{(1)}, \mathfrak{L}_R^{(1)}, \mathfrak{L}_g^{(1)}$, represent the contribution of the space-time noncommutativity and have the following expressions:

$$\mathfrak{L}_L^{(1)} = \frac{1}{4}\theta^{\mu\nu}\overline{L}\mathcal{W}_\mu\mathcal{W}_\nu\slashed{D}L + \frac{1}{4}\theta^{\mu\nu}\overline{L}\slashed{D}\mathcal{W}_\mu\mathcal{W}_\nu L + \overline{L}\gamma^\mu\mathcal{W}_\mu^{(1)}L \qquad (89)$$

and

$$\mathfrak{L}_R^{(1)} = -i\theta^{\mu\nu}\partial_\nu\left(\overline{R}\right)\mathcal{B}_\mu\slashed{D}R - i\theta^{\mu\nu}\overline{R}\slashed{D}\mathcal{B}_\mu\partial_\nu R + g_R\theta^{\mu\nu}\overline{R}i\partial_\mu\slashed{\mathcal{B}}\partial_\nu R$$
$$+ \theta^{\mu\nu}\overline{R}i\boldsymbol{G}_{\mu\nu}\slashed{D}R + \theta^{\mu\nu}\overline{R}i\slashed{D}\boldsymbol{G}_{\mu\nu}R \qquad (90)$$

where

$$\mathcal{D}_\mu L = \left(\partial_\mu - ig_L\mathcal{W}_\mu^a\frac{\tau_a}{2} + \frac{i}{2}g'\mathcal{A}_\mu\right)L \qquad (91)$$

$$\mathcal{D}_\mu R = \left(\partial_\mu - ig'\mathcal{A}_\mu Y^{(R)}\right)R \qquad (92)$$

and

$$\mathcal{W}_\mu^{(1)} = \frac{1}{4}\theta^{\alpha\beta}[F_{\alpha\beta}, \mathcal{W}_\beta] + \frac{1}{4}\theta^{\alpha\beta}[\mathcal{W}_\beta, \partial_\alpha\mathcal{W}_\mu] \qquad (93)$$

Notice here that the right currents are of the order θ, thus, they vanish in the classical limits when $\theta \to 0$. Finally, $\mathfrak{L}_g^{(1)}$ has the form:

$$\mathfrak{L}_g^{(1)} = -\frac{1}{g_L^2}Tr_1\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}^{(1)} - \frac{1}{g_R^2}Tr_2\boldsymbol{G}^{\mu\nu}\boldsymbol{G}_{\mu\nu}^{(1)} - \frac{1}{2g'^2}\boldsymbol{f}^{\mu\nu}\boldsymbol{f}_{\mu\nu}^{(1)} \qquad (94)$$

with

$$\mathcal{F}_{\mu\nu}^{(1)} = -\frac{i}{8}\theta^{\alpha\beta}[\mathcal{F}_{\mu\nu}, [\mathcal{W}_\beta, \mathcal{W}_\alpha]] + -\frac{i}{8}\theta^{\alpha\beta}\left(\partial_\mu[\mathcal{W}_\nu, [\mathcal{W}_\beta, \mathcal{W}_\alpha]] - \partial_\nu[\mathcal{W}_\mu, [\mathcal{W}_\beta, \mathcal{W}_\alpha]]\right) \qquad (95)$$

$$\boldsymbol{G}_{\mu\nu}^{(1)} = \partial_\mu \mathcal{B}_\nu^{(1)} - \partial_\nu \mathcal{B}_\mu^{(1)} - i\left[\mathcal{B}_\mu^{(1)}, \mathcal{B}_\nu\right] - i\left[\mathcal{B}_\mu, \mathcal{B}_\nu^{(1)}\right] \tag{96}$$

and

$$\boldsymbol{f}_{\mu\nu}^{(1)} = 0 \tag{97}$$

with

$$\mathcal{B}_\mu^{(1)} = -\theta^{\lambda\nu}\{\mathcal{B}_\lambda, \partial_\nu \mathcal{B}_\mu\} + \frac{1}{2}\theta^{\lambda\nu}\{\mathcal{B}_\lambda, \partial_\mu \mathcal{B}_\nu\} \tag{98}$$

The electroweak currents $\mathfrak{L}_{(\mathbf{currents})}^{NC}$ can be deduced directly from the previous Lagrangian to get:

$$\mathfrak{L}_{(\mathbf{currents})}^{NC} = \mathfrak{L}_{(\mathbf{currents})}^{NC(L)} + \mathfrak{L}_{(\mathbf{currents})}^{NC(R)} \tag{99}$$

where

$$\mathfrak{L}_{(\mathbf{currents})}^{NC(L)} = \overline{L} i\gamma^\mu \left(-ig_L \mathcal{W}_\mu + \frac{i}{2}g'\mathcal{A}_\mu\right) L + \overline{L}\gamma^\mu \mathcal{W}_\mu^{(1)} L +$$

$$\frac{1}{4}\theta^{\alpha\beta}\overline{L}\mathcal{W}_\alpha \mathcal{W}_\beta \gamma^\mu \left(\partial_\mu - ig_L \mathcal{W}_\mu + \frac{i}{2}g'\mathcal{A}_\mu\right) L + \frac{1}{4}\theta^{\alpha\beta}\overline{L}\gamma^\mu \left(\partial_\mu - ig_L \mathcal{W}_\mu + \frac{i}{2}g'\mathcal{A}_\mu\right)\mathcal{W}_\alpha \mathcal{W}_\beta L \tag{100}$$

and

$$\mathfrak{L}_{(\mathbf{currents})}^{NC(R)} = \overline{R} i\gamma^\mu \left(-ig'\mathcal{A}_\mu Y^{(R)}\right) R - i\theta^{\alpha\beta}\overline{R}\gamma^\mu \left(\partial_\mu - ig'\mathcal{A}_\mu Y^{(R)}\right)\mathcal{B}_\alpha \partial_\beta R$$

$$+ g_R \theta^{\mu\nu}\overline{R} i\partial_\mu \mathcal{B}\partial_\nu R - i\theta^{\mu\nu}\partial_\nu \left(\overline{R}\right)\mathcal{B}_\mu \gamma^\mu \left(\partial_\mu - ig'\mathcal{A}_\mu Y^{(R)}\right) R. \tag{101}$$

(L and R stand for left and right). From the above expressions, one can deduce the neutral and charged currents. Regarding the neutral electroweak currents, the Lagrangian $\mathfrak{L}_{(\mathbf{n.c})}^{NC}$ has as expression:

$$\mathfrak{L}_{(\mathbf{n.c})}^{NC} = \mathfrak{L}_{(\mathbf{n.c})} + \mathfrak{L}_{(\mathbf{n.c})}^{(1)} \tag{102}$$

where $\mathfrak{L}_{(\mathbf{n.c})}$ is the classical electroweak neutral current given by

$$\mathfrak{L}_{(\mathbf{n.c})} = g_L J_\mu^3 \mathcal{W}^{3\mu} + \frac{1}{2}g' J_\mu^Y \mathcal{A}_\mu = e J_\mu^{e.m} \mathbf{A}^\mu + \frac{g_L}{\cos\theta_W} J_\mu^0 \mathcal{Z}^\mu \tag{103}$$

with

$$
\begin{aligned}
J_\mu^Y &= -\left(\overline{\nu}_L \gamma^\mu \nu_L + \overline{e}_L \gamma^\mu e_L + 2\overline{e}_R \gamma^\mu e_R\right) \\
J_\mu^{e.m} &= \overline{e}_L \gamma^\mu e_L + \overline{e}_R \gamma^\mu e_R = \overline{e}\gamma^\mu e \\
J_\mu^3 &= \frac{1}{2}\left(\overline{\nu}_L \gamma^\mu \nu_L - \overline{e}_L \gamma^\mu e_L\right) \\
J_\mu^L &= -\left(\overline{\nu}_L \gamma^\mu \nu_L + \overline{e}_L \gamma^\mu e_L\right)
\end{aligned} \tag{104}
$$

and

$$J_\mu^0 = J_\mu^3 - \sin^2\theta_w J_\mu^{e.m}$$

The fields $\mathcal{Z}^\mu$, $\mathbf{A}^\mu$, are defined through the Weiberg angle θ_w rotation as follows:

14

$$\mathcal{B}^{\mu} = \cos\theta_w \mathcal{Z}^{\mu} + \sin\theta_w \mathbf{A}^{\mu} \tag{105}$$

and

$$\mathcal{A}^{\mu} = -\sin\theta_w \mathcal{Z}^{\mu} + \cos\theta_w \mathbf{A}^{\mu}$$

The term $\mathfrak{L}^{(1)}_{(\mathbf{n.c})}$ is the pure noncommutative neutral electroweak current and has the form:

$$\mathfrak{L}^{(1)}_{(\mathbf{n.c})} = \mathfrak{L}^{(1)L}_{(\mathbf{n.c})} + \mathfrak{L}^{(1)R}_{(\mathbf{n.c})} \tag{106}$$

with

$$\mathfrak{L}^{(1)L}_{(\mathbf{n.c})} = \frac{i}{8} \left(g_L\right)^3 \theta^{\alpha\beta} J^{L\mu} \mathcal{W}^3_{\alpha} \left(\mathcal{W}^+_{\beta} \mathcal{W}^-_{\mu} - \mathcal{W}^-_{\beta} \mathcal{W}^+_{\mu}\right) \tag{107}$$

and

$$\mathfrak{L}^{(1)R}_{(\mathbf{n.c})} = \frac{i}{2} g_R \theta^{\alpha\beta} \left[\left(\bar{\nu}_R \gamma^{\mu} \partial_{\beta} \nu_R - \bar{e}_R \gamma^{\mu} \partial_{\beta} e_R \right) \mathbf{G}^3_{\alpha\mu} - \frac{1}{2} \left(\bar{\nu}_R \slashed{\partial} \nu_R - \bar{e}_R \slashed{\partial} e_R \right) \mathbf{G}^3_{\alpha\beta} \right] \tag{108}$$

Notice here that the right neutral currents are of the order θ, and vanish in the commutative classical limit when $\theta \to 0$.

Regarding the left charged electroweak currents, the Lagrangian $\mathfrak{L}^{NCL}_{(\mathbf{c.c})}$ takes the form:

$$\mathfrak{L}^{NCL}_{(\mathbf{c.c})} = \mathfrak{L}^L_{(\mathbf{c.c})} + \mathfrak{L}^{(1)L}_{(\mathbf{c.c})} \tag{109}$$

where $\mathfrak{L}^L_{(\mathbf{c.c})}$ is the classical charged electroweak current which has the following expression:

$$\mathfrak{L}^L_{(\mathbf{c.c})} = \frac{g_L}{\sqrt{2}} \left(\bar{\nu}_L \gamma^{\mu} e_L \mathcal{W}^+_{\mu} - \bar{e}_L \gamma^{\mu} \nu_L \mathcal{W}^-_{\mu}\right) = \frac{g_L}{\sqrt{2}} \left(J^+_{\mu} \mathcal{W}^{+\mu} - J^-_{\mu} \mathcal{W}^{-\mu}\right) \tag{110}$$

with

$$J^+_{\mu} = \left(J^-_{\mu}\right)^+ = \bar{\nu}_L \gamma^{\mu} e_L \tag{111}$$

and $\mathfrak{L}^{(1)L}_{(\mathbf{c.c})}$ is the left charged electroweak current given by:

$$\mathfrak{L}^{(1)L}_{(\mathbf{c.c})} = \frac{i}{4} \frac{\left(g_L\right)^2}{\sqrt{2}} g' \theta^{\alpha\beta} \mathcal{A}^{\mu} \mathcal{W}^3_{\alpha} \left(J^+_{\mu} \mathcal{W}^+_{\beta} - J^-_{\mu} \mathcal{W}^-_{\beta}\right) + \frac{1}{2} \frac{\left(g_L\right)^2}{\sqrt{2}} \theta^{\alpha\beta} \left[\tilde{J}_+ \mathcal{W}^+_{\beta} - \tilde{J}_- \mathcal{W}^-_{\beta}\right] \mathcal{W}^3_{\alpha} \tag{112}$$

with

$$\tilde{J}_+ = \left(\tilde{J}_-\right)^+ = \bar{\nu}_L \slashed{\partial} e_L \tag{113}$$

and

$$\mathcal{W}^{\pm}_{\mu} = \frac{1}{\sqrt{2}} \left(\mathcal{W}^1_{\mu} \pm i \mathcal{W}^2_{\mu}\right) \tag{114}$$

For the right charged currents, the lagrangian $\mathfrak{L}^{NCR}_{(\mathbf{c.c})}$ is given by:

$$\mathfrak{L}^{NCR}_{(\mathbf{c.c})} = \mathfrak{L}^R_{(\mathbf{c.c})} + \mathfrak{L}^{(1)R}_{(\mathbf{c.c})} \tag{115}$$

with

$$\mathfrak{L}^R_{(\mathbf{c.c})} = 0..... \tag{116}$$

15

and

$$\mathfrak{L}_{(\mathbf{c.c})}^{(1)R} = \frac{g_R}{\sqrt{2}} i g' \theta^{\alpha\beta} \overline{\nu}_R \gamma^\mu \left(\partial_\alpha \mathcal{W}_\beta^+ \mathbf{A}_\mu + \mathcal{W}_\beta^+ \partial_\alpha \mathcal{A}_\mu + \partial_\alpha \mathcal{W}_\beta^- \mathcal{A}_\mu + \mathcal{W}_\beta^- \partial_\alpha \mathcal{A}_\mu \right) e_R$$

$$+ \frac{g_R}{\sqrt{2}} g' \theta^{\alpha\beta} \left(\overline{\nu}_R \gamma^\mu \partial_\alpha e_R \mathcal{W}_\beta^+ \mathcal{A}_\mu + \partial_\alpha \left(\overline{e}_R \right) \gamma^\mu \nu_R \mathcal{W}_\beta^- \mathcal{A}_\mu \right) \tag{117}$$

$$+ \frac{g_R}{\sqrt{2}} i \theta^{\alpha\beta} \left(\overline{e}_R \gamma^\mu \partial_\beta \nu_R \mathcal{W}_{\alpha\mu}^- + \overline{\nu}_R \gamma^\mu \partial_\beta e_R \mathcal{W}_{\alpha\mu}^+ \right) - \frac{g_R}{\sqrt{2}} \frac{i}{2} \theta^{\alpha\beta} \left(\overline{e}_R \slashed{\partial} \nu_R \mathcal{W}_{\alpha\beta}^- + \overline{\nu}_R \slashed{\partial} e_R \mathcal{W}_{\alpha\beta}^+ \right)$$

where

$$\mathcal{W}_{\alpha\beta}^\pm = \partial_\alpha \mathcal{W}_\beta^\pm - \partial_\beta \mathcal{W}_\alpha^\pm. \tag{118}$$

Again, notice that the charged right currents are of the order θ, and vanish in the classical limit when $\theta \to 0$.

3.1   Conclusions

Throught this work, in a unified description and in order to avoid gauge fields transformations that are not Lie or Poisson algebra valued, we have defined a new Weyl ordering using Moyal star product (symmetrization through star Poisson or Lie brackets) together with some golden rules. Based on this new approach, the noncommutative covariant derivative, curvature tensor, Seiberg-Witten maps fields and the corresponding gauge transformations as well as an invariant action are constructed.

The most important idea of this approach is the generalization of the singlet notion under non-commutative gauge transformations and the introduction of a geometric way to create new interactions, extend and enlarge a gauge theory namely the standard model. The physical application was done for the left-right extension of the standard model and the corresponding charged and neutral currents were derived. The right sector is shown to have a pure noncommutative space-time origin.

Appendix A

The matter field noncommutative gauge transformation is given by:

$$\delta_{\widehat{\Lambda}} \widehat{\psi} = \left(iT^a \left[\widehat{\Lambda}^a \overset{*}{,} \widehat{\psi} \right]_\eta \right)_w = iT^a \left[\widehat{\Lambda}^a \overset{*}{,} \widehat{\psi} \right]_\eta \tag{A1}$$

thus

$$\delta_{\widehat{\Sigma}} \left(\delta_{\widehat{\Lambda}} \widehat{\psi} \right) = \left(iT^a \left[\widehat{\Lambda}^a \overset{*}{,} \delta_{\widehat{\Sigma}} \widehat{\psi} \right]_\eta \right)_w = \left(iT^a \left[\widehat{\Lambda}^a \overset{*}{,} iT^b \left[\widehat{\Sigma}^b \overset{*}{,} \widehat{\psi} \right]_\eta \right]_\eta \right)_w \tag{A2}$$

$$= \left(iT^a iT^b \left[\widehat{\Lambda}^a \overset{*}{,} \left[\widehat{\Sigma}^b \overset{*}{,} \widehat{\psi} \right]_\eta \right]_\eta \right)_w = \frac{1}{2} \left(iT^a iT^b \left(\left[\widehat{\Lambda}^a \overset{*}{,} \widehat{\Sigma}^b * \widehat{\psi} + \eta \widehat{\psi} * \widehat{\Sigma}^b \right]_\eta \right) \right)_w$$

$$= \frac{1}{4} \left(iT^a iT^b \left(\widehat{\Lambda}^a * \widehat{\Sigma}^b * \widehat{\psi} + \eta \widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a + \eta \widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b + \eta^2 \widehat{\psi} * \widehat{\Sigma}^b * \widehat{\Lambda}^a \right) \right)_w$$

In what follows, we denote by:

$$\overbrace{A * B} \equiv 2\left[A, B\right]^*_\eta = A * B + \eta B * A \tag{A3}$$

Using the following golden rules:

$$iT_a iT_b \widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b \;\; \rightarrow \;\; \left(iT_a iT_b \widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b\right)_w \equiv iT_a iT_b \overbrace{\widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b} \tag{A4}$$
$$= \;\; iT_a iT_b \left(\widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b + \eta \widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a\right)$$

$$iT_a iT_b \widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a \;\; \rightarrow \;\; \left(iT_a iT_b \widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a\right)_w \equiv iT_b iT_a \left(\overbrace{\widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a}\right)$$
$$= \;\; iT_b iT_a \left(\widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a + \eta \widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b\right) \tag{A5}$$

$$iT_a iT_b \widehat{\Lambda}^a * \widehat{\Sigma}^b * \widehat{\psi} \;\; \rightarrow \;\; \left(iT_a iT_b \widehat{\Lambda}^a * \widehat{\Sigma}^b * \widehat{\psi}\right)_w \equiv iT_a iT_b \left(\overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} * \widehat{\psi}\right)$$
$$= \;\; iT_a iT_b \left(\widehat{\Lambda}^a * \widehat{\Sigma}^b * \widehat{\psi} + \eta \widehat{\Sigma}^b * \widehat{\Lambda}^a * \widehat{\psi}\right) \tag{A6}$$

$$iT_a iT_b \widehat{\Sigma}^b * \widehat{\Lambda}^a * \widehat{\psi} \;\; \rightarrow \;\; \left(iT_a iT_b \widehat{\Sigma}^b * \widehat{\Lambda}^a * \widehat{\psi}\right)_w \equiv iT_a iT_b \left(\overbrace{\widehat{\Sigma}^b * \widehat{\Lambda}^a} * \widehat{\psi}\right)$$
$$= \;\; iT_b iT_a \left(\widehat{\Sigma}^b * \widehat{\Lambda}^a * \widehat{\psi} + \eta \widehat{\Lambda}^a * \widehat{\Sigma}^b * \widehat{\psi}\right) \tag{A7}$$

$$iT_a iT_b \widehat{\psi} * \widehat{\Lambda}^a * \widehat{\Sigma}^b \;\; \rightarrow \;\; \left(iT_a iT_b \widehat{\psi} * \widehat{\Lambda}^a * \widehat{\Sigma}^b\right)_w \equiv iT_a iT_b \left(\widehat{\psi} * \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b}\right)$$
$$= \;\; iT_a iT_b \left(\widehat{\psi} * \widehat{\Lambda}^a * \widehat{\Sigma}^b + \eta \widehat{\psi} * \widehat{\Sigma}^b * \widehat{\Lambda}^a\right) \tag{A8}$$

$$iT_a iT_b \widehat{\psi} * \widehat{\Sigma}^b * \widehat{\Lambda}^a \;\; \rightarrow \;\; \left(iT_a iT_b \widehat{\psi} * \widehat{\Sigma}^b * \widehat{\Lambda}^a\right)_w \equiv iT_a iT_b \left(\widehat{\psi} * \overbrace{\widehat{\Sigma}^b * \widehat{\Lambda}^a}\right)$$
$$= \;\; iT_b iT_a \left(\widehat{\psi} * \widehat{\Sigma}^b * \widehat{\Lambda}^a + \eta \widehat{\psi} * \widehat{\Lambda}^a * \widehat{\Sigma}^b\right) \tag{A9}$$

we obtain:

$$\delta_{\widehat{\Sigma}}\left(\delta_{\widehat{\Lambda}}\widehat{\psi}\right) = \frac{1}{4}\left(\begin{array}{c} iT^a iT^b \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} * \widehat{\psi} + \eta iT^b iT^a \overbrace{\widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a} \\ +\eta iT^a iT^b \overbrace{\widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b} + \eta^2 iT^a iT^b \widehat{\psi} * \overbrace{\widehat{\Sigma}^b * \widehat{\Lambda}^a} \end{array} \right) \tag{A10}$$

Similarly:

$$\delta_{\widehat{\Lambda}}\left(\delta_{\widehat{\Sigma}}\widehat{\psi}\right) = \frac{1}{4}\left(\begin{array}{c} iT^b iT^a \overbrace{\widehat{\Sigma}^b * \widehat{\Lambda}^a} * \widehat{\psi} + \eta iT^a iT^b \overbrace{\widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b} \\ +\eta iT^b iT^a \overbrace{\widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a} + \eta^2 iT^b iT^a \widehat{\psi} * \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} \end{array} \right) \tag{A11}$$

we deduce that:

$$\delta_{\widehat{\Lambda}}\left(\delta_{\widehat{\Sigma}}\widehat{\psi}\right) - \delta_{\widehat{\Sigma}}\left(\delta_{\widehat{\Lambda}}\widehat{\psi}\right) = \left[\delta_{\widehat{\Lambda}}, \delta_{\widehat{\Sigma}}\right]\widehat{\psi}$$

$$= \frac{1}{4}\left(\begin{array}{c} iT^b iT^a \overbrace{\widehat{\Sigma}^b * \widehat{\Lambda}^a} * \widehat{\psi} - iT^a iT^b \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} * \widehat{\psi} \\ +\eta^2 iT^b iT^a \widehat{\psi} * \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} - \eta^2 iT^a iT^b \widehat{\psi} * \overbrace{\widehat{\Sigma}^b * \widehat{\Lambda}^a} \end{array} \right) \tag{A12}$$

using the relations

$$\begin{aligned} \overbrace{\widehat{\Sigma}^b * \widehat{\Lambda}^a} * \widehat{\psi} &= \eta \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} * \widehat{\psi} \\ \widehat{\psi} * \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} &= \eta \widehat{\psi} * \overbrace{\widehat{\Sigma}^b * \widehat{\Lambda}^a} \\ \overbrace{\widehat{\Sigma}^b * \widehat{\psi} * \widehat{\Lambda}^a} &= \eta \overbrace{\widehat{\Lambda}^a * \widehat{\psi} * \widehat{\Sigma}^b} \end{aligned} \tag{A13}$$

and $\eta^2 = 1$, we obtain:

$$\left[\delta_{\widehat{\Lambda}}, \delta_{\widehat{\Sigma}}\right]\widehat{\psi} = \frac{1}{4}\left(\begin{array}{c} \eta iT^b iT^a \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} * \widehat{\psi} - iT^a iT^b \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} * \widehat{\psi} \\ +iT^b iT^a \widehat{\psi} * \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} - \eta iT^a iT^b \widehat{\psi} * \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} \end{array} \right)$$

$$= \frac{1}{4}\left(\left[T^a, T^b\right]_{-\eta} \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} * \widehat{\psi} + \eta(T^a T^b - \eta T^b T^a)\widehat{\psi} * \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} \right)$$

$$= \frac{1}{4}\left[T^a, T^b\right]_{-\eta}\left(\overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} * \widehat{\psi} + \eta \widehat{\psi} * \overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b} \right)$$

$$= \frac{1}{2}\left[T^a, T^b\right]_{-\eta}\left[\overbrace{\widehat{\Lambda}^a * \widehat{\Sigma}^b}, \widehat{\psi}\right]_{\eta}^* = \left[T^a, T^b\right]_{-\eta}\left[\left[\widehat{\Lambda}^a, \widehat{\Sigma}^b\right]_{\eta}^*, \widehat{\psi}\right]_{\eta}^* \tag{A14}$$

Appendix B

We define the noncommutative covariant derivative as

$$\widehat{\mathcal{D}}_\mu \widehat{\psi} \equiv \left(\partial_\mu \widehat{\psi} - iT^a \left[\widehat{V}_\mu^a, \widehat{\psi} \right]_\eta^* \right)_w = \partial_\mu \widehat{\psi} - iT^a \left[\widehat{V}_\mu^a, \widehat{\psi} \right]_\eta^* \tag{B1}$$

It is a covariant derivative in the sense:

$$\delta_{\widehat{\Lambda}} \left(\widehat{\mathcal{D}}_\mu \widehat{\psi} \right) = \left(iT^a \left[\widehat{\Lambda}^a, \partial_\mu \widehat{\psi} \right]_\eta^* \right)_w + \left(iT^a \left[\widehat{\Lambda}^a, -iT^b \left[\widehat{V}_\mu^a, \widehat{\psi} \right]_\eta^* \right]_\eta^* \right)_w$$

$$= iT^a \left[\widehat{\Lambda}^a, \partial_\mu \widehat{\psi} \right]_\eta^* + \left(-iT^a iT^b \left[\widehat{\Lambda}^a, \left[\widehat{V}_\mu^a, \widehat{\psi} \right]_\eta^* \right]_\eta^* \right)_w$$

$$= iT^a \left[\widehat{\Lambda}^a, \partial_\mu \widehat{\psi} \right]_\eta^* + \frac{1}{2} \left(-iT^a iT^b \left[\widehat{\Lambda}^a, \widehat{V}_\mu^b * \widehat{\psi} + \eta \widehat{\psi} * \widehat{V}_\mu^b \right]_\eta^* \right)_w$$

$$= iT^a \left[\widehat{\Lambda}^a, \partial_\mu \widehat{\psi} \right]_\eta^* + \frac{1}{4} \left(-iT^a iT^b \left(\begin{array}{c} \widehat{\Lambda}^a * \widehat{V}_\mu^b * \widehat{\psi} + \eta \widehat{V}_\mu^b * \widehat{\psi} * \widehat{\Lambda}^a \\ +\eta \widehat{\Lambda}^a * \widehat{\psi} * \widehat{V}_\mu^b + \eta^2 \widehat{\psi} * \widehat{V}_\mu^b * \widehat{\Lambda}^a \end{array} \right) \right)_w \tag{B2}$$

Using the golden rules eqs.(A4)-(A9) we get:

$$\delta_{\widehat{\Lambda}} \left(\widehat{\mathcal{D}}_\mu \widehat{\psi} \right) = iT^a \left[\widehat{\Lambda}^a, \partial_\mu \widehat{\psi} \right]_\eta^* + \frac{1}{4} \left(\begin{array}{c} -iT^a iT^b \overbrace{\widehat{\Lambda}^a * \widehat{V}_\mu^b} * \widehat{\psi} - \eta i T^b i T^a \overbrace{\widehat{V}_\mu^b * \widehat{\psi}} * \widehat{\Lambda}^a \\ -\eta i T^a i T^b \overbrace{\widehat{\Lambda}^a * \widehat{\psi}} * \widehat{V}_\mu^b - \eta^2 i T^a i T^b \widehat{\psi} * \overbrace{\widehat{V}_\mu^b * \widehat{\Lambda}^a} \end{array} \right) \tag{B3}$$

Moreover, a direct calculation and using the fact that:

$$\left[\delta_{\widehat{\Lambda}}, \partial_\mu \right] = 0 \tag{B4}$$

gives:

$$\delta_{\widehat{\Lambda}} \left(\widehat{\mathcal{D}}_\mu \widehat{\psi} \right) = \partial_\mu \left(iT^a \left[\widehat{\Lambda}^a, \widehat{\psi} \right]_\eta^* \right)_w + \left(-iT^b \left[\widehat{V}_\mu^b, \delta_\Lambda \widehat{\psi} \right]_\eta^* - iT^b \left[\delta_\Lambda \widehat{V}_\mu^b, \widehat{\psi} \right]_\eta^* \right)_w$$

$$= iT^a \left[\widehat{\Lambda}^a, \partial_\mu \widehat{\psi} \right]_\eta^* + iT^a \left[\partial_\mu \widehat{\Lambda}^a, \widehat{\psi} \right]_\eta^* + \left(\begin{array}{c} -iT^b \left[\widehat{V}_\mu^b, iT^a \left[\widehat{\Lambda}^a, \widehat{\psi} \right]_\eta^* \right]_\eta^* \\ -iT^b \left[\delta_\Lambda \widehat{V}_\mu^b, \widehat{\psi} \right]_\eta^* \end{array} \right)_w$$

$$= iT^a \left[\widehat{\Lambda}^a, \partial_\mu \widehat{\psi} \right]_\eta^* + iT^a \left[\partial_\mu \widehat{\Lambda}^a, \widehat{\psi} \right]_\eta^* + \frac{1}{2} \left(-iT^b iT^a \left[\widehat{V}_\mu^b, \widehat{\Lambda}^a * \widehat{\psi} + \eta \widehat{\psi} * \widehat{\Lambda}^a \right]_\eta^* \right)_w$$

$$-iT^b \left[\delta_\Lambda \widehat{V}_\mu^b, \widehat{\psi} \right]_\eta^*$$

$$= iT^a \left[\widehat{\Lambda}^a, \partial_\mu \widehat{\psi} \right]_\eta^* + iT^a \left[\partial_\mu \widehat{\Lambda}^a, \widehat{\psi} \right]_\eta^* - iT^b \left[\delta_\Lambda \widehat{V}_\mu^b, \widehat{\psi} \right]_\eta^*$$

$$+ \frac{1}{4} \left(-iT^b iT^a \left(\widehat{V}_\mu^b * \widehat{\Lambda}^a * \widehat{\psi} + \eta \widehat{\Lambda}^a * \widehat{\psi} * \widehat{V}_\mu^b + \eta \widehat{V}_\mu^b * \widehat{\psi} * \widehat{\Lambda}^a + \eta^2 \widehat{\psi} * \widehat{\Lambda}^a * \widehat{V}_\mu^b \right) \right)_w \tag{B5}$$

again, with the help of the golden rules of eqs.(A4)-(A9) we deduce that:

$$\delta_{\widehat{\Lambda}}\left(\widehat{\mathcal{D}}_\mu\widehat{\psi}\right) = iT^a\left[\widehat{\Lambda}^a,\partial_\mu\widehat{\psi}\right]^*_\eta + iT^a\left[\partial_\mu\widehat{\Lambda}^a,\widehat{\psi}\right]^*_\eta - iT^b\left[\delta_\Lambda\widehat{V}^b_\mu,\widehat{\psi}\right]^*_\eta$$
$$+\frac{1}{4}\left(\begin{array}{l} -iT^biT^a\overbrace{\widehat{V}^b_\mu*\widehat{\Lambda}^a}*\widehat{\psi} - \eta iT^aiT^b\overbrace{\widehat{\Lambda}^a*\widehat{\psi}}*\widehat{V}^b_\mu \\ -\eta iT^biT^a\overbrace{\widehat{V}^b_\mu*\widehat{\psi}}*\widehat{\Lambda}^a - \eta^2 iT^biT^a\widehat{\psi}*\overbrace{\widehat{\Lambda}^a*\widehat{V}^b_\mu} \end{array}\right) \tag{B6}$$

using relations (A13) we obtain:

$$-iT^b\left[\delta_{\widehat{\Lambda}}\widehat{V}^b_\mu,\widehat{\psi}\right]^*_\eta = -iT^a\left[\partial_\mu\widehat{\Lambda}^a,\widehat{\psi}\right]^*_\eta + \frac{1}{4}\left([T^a,T^b]_{-\eta}\overbrace{\widehat{\Lambda}^a*\widehat{V}^b_\mu}*\widehat{\psi} + \eta[T^a,T^b]_{-\eta}\widehat{\psi}*\overbrace{\widehat{\Lambda}^a*\widehat{V}^b_\mu}\right)$$
$$= -iT^a\left[\partial_\mu\widehat{\Lambda}^a,\widehat{\psi}\right]^*_\eta + \frac{1}{2}\left([T^a,T^b]_{-\eta}\left[\overbrace{\widehat{\Lambda}^a*\widehat{V}^b_\mu},\widehat{\psi}\right]^*_\eta\right) \tag{B7}$$

thus

$$-iT^b\delta_{\widehat{\Lambda}}\widehat{V}^b_\mu = -iT^a\partial_\mu\widehat{\Lambda}^a + \frac{1}{2}[T^a,T^b]_{-\eta}\overbrace{\widehat{\Lambda}^a*\widehat{V}^b_\mu} \tag{B8}$$

or

$$\delta_{\widehat{\Lambda}}\widehat{V}_\mu = \partial_\mu\widehat{\Lambda} + \frac{i}{2}[T^a,T^b]_{-\eta}\overbrace{\widehat{\Lambda}^a*\widehat{V}^b_\mu} = \partial_\mu\Lambda + i[T^a,T^b]_{-\eta}\left[\widehat{\Lambda}^a,\widehat{V}^b_\mu\right]^*_\eta \tag{B9}$$

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