GAUSS PLANETARY EQUATIONS IN A NON-SINGULAR GRAVITATIONAL POTENTIAL

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Abstract
We study the effects of a non-singular gravitational potential on satellite orbits by calculating the corresponding changes of its orbital elements, using Gauss' planetary equations. We derive two non-zero expressions for the changes of the argument of the perigee and the mean anomaly, and we compare them to those of the general relativity. Using the GRACE satellite system, we obtain numerical results from which we conclude that the effect of such a potential, on the perigee cannot be separated from that of general relativity. Furthermore, we conclude that the effect on the mean anomaly can probably be observed by today's technology.

Key Words: Gravitation, celestial mechanics Gauss' equations, GRACE, orbital elements.

1 Introduction

A new non-singular gravitational potential appears in the literature that has the following form [1]:

$$V(r) = -\frac{GM}{r} e^{-\frac{\lambda}{r}} \tag{1}$$

where the constant λ appearing in the potential above is defined as follows:

$$\lambda = \frac{GM}{c^2} \tag{2}$$

and G is the Newtonian gravitational constant, M is the mass of the main body that produces the potential, and c is the speed of light. In this paper we wish to investigate the motion of a satellite in such a potential using Gauss' planetary equations of celestial mechanics.

The goal of this contribution is to examine the possibility of validating this non-singular potential by studying its effect on the changes of the orbital elements of a satellite. Various satellite effects can conveniently be expressed as orbital element time rates of change, which are observable by modern geodetic techniques. In general, the well-known Gauss-planetary equations, as they are presented for instance in Blanco and McCuskey [2], link the orbital element time derivatives to their cause, a disturbing (or perturbing) force per unit mass. Here, perturbing force per unit mass implies any deviation of the total acceleration of a central Newtonian field. Accepting that Eq. (1) holds true, we can write $V(r)$ as a central Newtonian acceleration plus other terms that constitute the total disturbing acceleration. These disturbing acceleration components can then be entered separately into the Gauss'-planetary equations to study their effects on the satellite central field (Keplerian) orbit, with the hope that we can see some measurable orbital element time rates of change and thus observationally verify or disprove Eq. (1).

In a treatment developed by Gauss, the perturbing forces acting on a satellite are resolved into a three mutually perpendicular components [2]:

$$\nabla R = \begin{bmatrix} F_X \\ F_Y \\ F_Z \end{bmatrix} \tag{4}$$

where R is a perturbing function, and F_X perpendicular to the orbital plane, positive towards the north pole, F_Y perpendicular to the radius vector in the orbital plane, positive in the direction of increasing longitude, and F_Z is the direction of the radius vector, positive in the direction of increasing radial distance and therefore Gauss' equations can be writes as:

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left[F_Z e \sin f + \frac{a\left(1-e^2\right)}{r}F_Y\right] \tag{5}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left[F_Z \sin f + \left(\frac{e+\cos f}{1+e\cos f} + \cos f\right)F_Y\right] \tag{6}$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nea}\left[-F_Z \cos f + \left(1 + \frac{r}{a\left(1-e^2\right)}\right)F_Y \sin f\right] - \frac{d\Omega}{dt}\cos i \tag{7}$$

$$\frac{di}{dt} = \frac{1}{na\sqrt{1-e^2}}\frac{r}{a}\cos(\omega + f)F_X \tag{8}$$

$$\frac{d\Omega}{dt} = \frac{1}{na\sqrt{1-e^2}}\frac{r}{a}\frac{\sin(\omega + f)}{\sin i}F_X \tag{9}$$

$$\frac{dM}{dt} = n + \frac{1}{na}\left[\frac{2r}{a} - \frac{\left(1-e^2\right)}{e}\cos f\right]F_Z - \frac{\left(1-e^2\right)}{nae}\left[1 + \frac{r}{a\left(1-e^2\right)}\right]F_Y \sin f \tag{10}$$

where, a is the semi-major axis of the orbit, i, e are the inclination and eccentricity of the orbit Ω, the argument of the ascending node, and ω is the argument of the perigee, and M is the mean anomaly of the satellite defined as $M=n(t-T)$ and $n= 2\pi/P = \sqrt{GM}/a^{-3/2}$ and f is the true anomaly the angle between the perigee and the radial vector of the satellite. Equations (5)–(10) are convenient because they allow us for the influences of the three components F_X, F_Y, F_Z to be separately studied. We can see that the influence of F_X consists in changing the orbital orientation or the elements i and Ω. Next F_Y changes the semi-major axis assuming e << 1, and it is important for the satellite's maneuvers.

2 The perturbing function

Next, we obtain an expression the perturbing acceleration per unit mass due to the non-singular potential to be:

$$R_{NS} = -\frac{\partial V(r)}{\partial r} = -\frac{\partial}{\partial r}\left(-\frac{GM}{r}e^{-\frac{\lambda}{r}}\right) \tag{11}$$

which becomes:

$$R_{NS} = -\frac{GM}{r^2}e^{-\frac{\lambda}{r}}\left(1-\frac{\lambda}{r}\right). \tag{12}$$

From Eq.(12) we see that the first term in the RHS is the Newtonian gravity multiplied by the factor $e^{-\lambda/r}(1-\lambda/r)$. This force per unit mass has only a radial component and $F_X = F_Y = 0$ simplifies Gausses' equations considerably. In orbital scenarios since $\lambda << r$ the force function in Eq. (12) can be to first order approximated by:

$$F_Z = R_{NS} = -\frac{GM}{r^2}\left(1-\frac{\lambda}{r}\right)^2 \tag{13}$$

substituting λ with Eq. (2) we obtain

$$R_{NS} = -\frac{GM}{r^2}\left(1-\frac{GM}{rc^2}\right)^2. \tag{14}$$

This is the radial perturbing potential component to be in Gausses' orbital equations. Next, substituting Eq. (14) into Eqs. (5)-(10) we obtain the non zero time rates of change associated with this non-singular disturbing potential to be:

$$\frac{da}{dt} = \frac{2}{nr\sqrt{1-e^2}}\left[-\frac{GM}{r^2}\left(1-\frac{GM}{rc^2}\right)^2 er\sin f\right] \tag{15}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left[-\frac{GM}{r^2}\left(1-\frac{GM}{rc^2}\right)^2 \sin f\right] \tag{16}$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nea}\left[-\frac{GM}{r^2}\left(1-\frac{GM}{rc^2}\right)^2 \cos f\right] \tag{17}$$

$$\frac{dM}{dt} = n - \frac{1}{na}\left[\left[\frac{(1-e^2)\cos f}{e} - \frac{2r}{a}\right]\frac{GM}{r^2}\left(1-\frac{GM}{rc^2}\right)^2\right] \tag{18}$$

3 Solving the orbital equations

To solve eqs. (15)-(18), we evaluate them on the unperturbed Keplerian ellipse, assuming that the orbit does not deviate to much from that of a Keplerian ellipse, and that a Keplerian ellipse constitutes a good approximation. Therefore, we use that:

$$r(f) = \frac{a(1-e^2)}{1+e\cos f} \tag{19}$$

we also use the transformation expressing time in terms of the true anomaly, and therefore we have [3]

$$dt = \frac{1}{n}\left(\frac{r}{a}\right)^2 \frac{df}{\sqrt{1-e^2}} . \tag{20}$$

Substituting Eqs. (19) and (20) we obtain that

$$dt = \frac{1}{n}\left(\frac{\left(1-e^2\right)^{3/2}}{\left(1+e\cos f\right)^2}\right)df . \tag{21}$$

Therefore Eq. (5) becomes:

$$da = -\frac{2GM\, e\sin f}{n^2a^2\left(1-e^2\right)}\left[\left(1-\frac{GM\left(1+e\cos f\right)}{c^2a\left(1-e^2\right)}\right)^2\right]df . \tag{22}$$

To find the change in one revolution we integrate from 0 to 2π and therefore we have:

$$\Delta a = -\int_0^{2\pi}\frac{2GMe\sin f}{n^2a^2\left(1-e^2\right)}\left[\left(1-\frac{GM\left(1+e\cos f\right)}{c^2a\left(1-e^2\right)}\right)^2\right]df = 0 . \tag{23}$$

Similarly Eq. (6) over one revolution gives

$$\Delta e = -\int_0^{2\pi}\frac{GM\sin f}{n^2a^3}\left(1-\frac{GM\left(1+e\cos f\right)}{c^2a\left(1-e^2\right)}\right)^2 df = 0 , \tag{24}$$

similarly

$$\Delta\omega = -\int_0^{2\pi}\frac{GM}{en^2a^3}\left(1-\frac{GM\left(1+e\cos f\right)}{c^2a\left(1-e^2\right)}\right)^2 \cos f df \tag{25}$$

and Eq. (25) becomes

$$\Delta\omega = \frac{2G^2M^2}{c^4a^5n^2\left(1-e^2\right)^2}\left(ac^2\left(1-e^2\right)-GM\right) \tag{26}$$

which could also be written as follows:

$$\Delta\omega = \frac{2\pi G^2M^2}{n^2a^4c^2\left(1-e^2\right)}\left[1-\frac{GM}{c^2a\left(1-e^2\right)}\right]. \tag{27}$$

Equation (27) can be also written as a function of the parameter λ of the non-singular potential in the following way:

$$\Delta\omega = \frac{2\pi\lambda}{a\left(1-e^2\right)}\left[1-\frac{\lambda}{a\left(1-e^2\right)}\right]. \tag{28}$$

To compare we say that general relativity predicts a perigee change that is given by [Taff, 1985]:

$$\Delta\omega_{GR} = \frac{6\pi GM}{c^2a\left(1-e^2\right)} \quad , \tag{29}$$

using Eq. (27) and (28) we obtain that

$$\varDelta\omega_{NS} = \frac{1}{3}\left[1 - \frac{\lambda}{a\left(1-e^2\right)}\right]\varDelta\omega_{GR} \tag{30}$$

Finally Eq. (10) becomes

$$\varDelta M = \int_0^{2\pi}\left[\begin{array}{l}\dfrac{\left(1-e^2\right)^{3/2}}{\left(1+e\cos f\right)^2} \\ -\dfrac{GM\sqrt{1-e^2}}{n^2a^3}\left(1-\dfrac{GM\left(1+e\cos f\right)}{c^2a\left(1-e^2\right)}\right)^2\left(\dfrac{\left(1-e^2\right)\cos f}{e}-\dfrac{2\left(1-e^2\right)}{\left(1+e\cos f\right)}\right)\end{array}\right]df \tag{31}$$

Integrating from 0 to 2π the integral above simplifies to:

$$\varDelta M = -\frac{6\pi G^2M^2}{n^2c^2a^4\left(1-e^2\right)^{1/2}}\left(1-e^2-\frac{GM}{3c^2a}\right) \tag{32}$$

which can be also written as a function of the non-singular potential parameter λ as follows:

$$\varDelta M = -\frac{6\pi\lambda}{a\sqrt{1-e^2}}\left[1-e^2-\frac{\lambda}{3a}\right]. \tag{33}$$

The mean anomaly change that general relativity predicts in a year is given by [Schwarzschild, 1916]

$$\varDelta M = \frac{3\left(GM_p\right)^{3/2}}{c^2a^{5/2}\left(1-e^2\right)^{1/2}}\varDelta t\ . \tag{34}$$

Comparing to Eq. (34) for $\varDelta t$ =1 year we can write that:

$$\varDelta M_{NS} = -2\pi\left(1-e^2-\frac{\lambda}{3a}\right)\varDelta M_{GR}\ . \tag{35}$$

4 Numerical results

To calculate the changes per revolution of the two non-zero orbital elements we use the orbital parameters of the *Gravity Recovery and Climate Experiment – GRACE*. GRACE-A satellite has a= 6876.4816 km, and e = 0.00040989, and therefore n = 0.001100118 rad/s = 15.113 rev/d, i = 89.025446°, ω = 302.414244°, Ω = 354.447149°, M = 80.713591° [*http://www.csr.utexas.edu/grace/newsletter/archive/august2002.html*]. Substituting these values in Eqs. (27) and (31) we obtain the corresponding changes on ω and M due to the non-singular potential to be:

$$\varDelta\omega = 0''.0127762\ /\mathrm{d} \tag{32}$$

$$\varDelta M = -0''.0389686\ /\mathrm{d}. \tag{33}$$

Therefore, for GRACE satellite using Eq. (32) and (33) we calculate an annual change of the perigee to be equal to 4″.66/a and similarly for the mean anomaly we obtain a negative -14″.22/a. Comparing with Eq. (29) we calculate the change of the perigee attributed to general relativity to be 14″.05/a. This is approximately three times larger, than the one predicted by the non-singular potential and most likely it would not be observed. For the mean anomaly, change general relativity predicts a positive change of approximately 14″.15/a. We conclude, that for the current state of technology such a decrease in mean anomaly of 14″.22/a could be easily detected.

5 Conclusions

We used Gauss' planetary equations, in order to validate the non-singular potential given by Eq.(1) using satellite orbit perturbations. We have derived the non-zero orbital changes for the perigee and the mean anomaly, and we have compared them to the ones predicted by general relativity. We conclude that for such a potential the perigee effect will not be easily separated by that of general relativity, where the yearly effect of the mean anomaly could be probably observed with today's technology.

References

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