`Derivation of Weinberg's Relation in a Inflationary Universe

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Abstract
We propose a derivation of the empirical Weinberg relation for the mass of an elementary particle and in an inflationary type of universe. Our derivation produces the standard well known Weinberg relation for the mass of an elementary particle, along with an extra term which depends on the inflationary potential, as well as Hubble's constant. The derivation is based on Zeldovich's result for the cosmological constant Λ, in the context of quantum field theory. The extra term can be understood as a small correction to the mass of the elementary particle due to inflation. This term also enables us to calculate, the initial value of the field ϕ_O for two kinds of potentials chosen, which makes Weinberg's relation possible. Closed and flat and open universes give the mass of the particles close to the mass of a pion, 140 MeV/c^2 or as the one also predicted by Weinberg's relation.
Keywords: inflationary cosmology, quantum field theory, elementary particle, mass of the pion, cosmological constant.

1.Introduction

It is a well known result that the mass of an elementary particle m can be obtained as a combination of the fundamental constants of physics namely c, G, H_0 and $\hbar$. [1] namely:

$$m_\pi = \left(\frac{\hbar^2 H_o}{Gc} \right)^{1/3} \tag{1}$$

This is known as the Weinberg's formula and it is purely empirical. In this paper we first offer a simple derivation for it using Einstein's field equations, and in a inflationary model universe with the help of Zeldovich's definition of the cosmological constant Λ. The assumption for using Zeldovich's definition is that there might be a possible relation between the same constants and the definition of the cosmological constant, in the context of quantum field theory, satisfying also the general theory of relativity, and probably implying a relation between microcosm and macrocosm.

2.Theoretical Background

In an inflationary universe, the law of expansion of its radius resembles that of the De-Sitter universe. The Friedmann equation in the vacuum dominated case has as its first solution the equation given by the relation below: [2]

$$R(t) = R_o \exp[H\, t] \tag{2}$$

where R(t) is the radius of the universe at any time t, and R_o is some initial radius, and finally H is not the Hubble's parameter at arbitrary time unless k = 0. For the purpose of our calculation, H_o was taken to be constant. In the concept of the an inflationary cosmological senario H is a function of the inflationary field ϕ, which itself is a function of time. Therefore we have that :[3]

$$H[\varphi(t)] = \sqrt{\frac{8\pi V[\varphi(t)]}{3m^2{}_P}} \tag{3}$$

where $V(\phi)$ is the inflationary potential function of the inflationary field, m_P is the Planck mass, when natural units are used. For the above field, the modified equations in the cosmological sense take the form given below when the energy density ε_o and the pressure p_o of the cosmic fluid can be replaced ε_o with $\varepsilon_o+\varepsilon$ and p_o with p_o+p and p and ε can be defined as follows: [4]

$$\varepsilon^{'} = \frac{\dot{\varphi}^2}{2} + V(\varphi), \quad \mathrm{p}^{'} = \frac{\dot{\varphi}^2}{2} - V(\varphi), \quad \dot{\varphi} = \frac{\partial \varphi}{\partial t} \tag{4}$$

with $\varepsilon^{'}$ and $p^{'}$ are respectively the energy density and pressure due to inflationary field. Using (4) the field equations can be now written as follows:

$$3\left[\frac{\dot{R}(t)}{R(t)}\right]^2 = \frac{8\pi G}{c^2}(\varepsilon_o + \varepsilon) + \Lambda \tag{5}$$

$$2\left[\frac{\ddot{R}(t)}{R(t)}\right] + \left[\frac{\dot{R}(t)}{R(t)}\right]^2 = \Lambda - \frac{8\pi G}{c^2}(p + p_o). \tag{6}$$

It is convenient now to to write the equations above in terms of Hubble's parameter H. Therefore we caan also have:

$$3H^2(t) = \frac{8\pi G}{c^2}(\varepsilon_o + \varepsilon) + \Lambda \tag{7}$$

$$-2q(t)H(t) + H^2(t) = \Lambda - \frac{8\pi G}{c^2}(p + p_o). \tag{8}$$

3. Analysis

Before continuing our analysis, it would be good to elaborate on our assumptions in the context of general relativity theory used in writing down these equations. First, it is assumed that we are dealing with a region which is within the horizon distance at the time under consideration. This region then undergoes rapid expansion, being more or less independent from the rest of the universe. The metric used for this region is a Robertson-Walker type under homogeneous and isotropic space. We also ignore the spatial variation of the field ϕ, which becomes uniform in value all over this region. The value of curvature k = 0 has been used in the line element which of course implies a flat spatial geometry. For simplicity, we let $\varepsilon_o = p_o = 0$. Using (2) we can now subtract (6), (5) and the new values of ε and p we obtain equation:

$$\left[\frac{\ddot{R}(t)}{R(t)}\right]^2 - \frac{\ddot{R}(t)}{R(t)} = \frac{4\pi G}{c^2}\dot{\varphi}^2(t) \tag{9}$$

Using (2) we substitute in (9) for $\dot{R}, \ddot{R}$ we finally obtain:

$$\frac{4\pi G}{c^2}\dot{\varphi}^2 = 0 \tag{10}$$

which implies that:

$$\varphi(t) = \text{const} = \varphi_o . \tag{11}$$

Next from (8) upon substitution of the derivatives $\dot{R}, \ddot{R}$ we obtain:

$$3H^2[\varphi(t)] = \Lambda - \frac{4\pi G}{c^2}\dot{\varphi}^2(t) + \frac{8\pi G}{c^2}V[\varphi(t)] \tag{12}$$

Solving fot the cosmological constant Λ we have:

$$\Lambda = 3H^2(\varphi_o) - \frac{8\pi G V(\varphi_o)}{c^2} \tag{13}$$

4 Using Zeldovich's relation

As a next step, we will make use of Zeldovich's result [5], where he obtains an expression for the cosmological constant Λ from the energy tensor of a polarized vacuum in the quantized theory of fields. Zeldovich gives the following expression:

$$\Lambda = \frac{G^2 m^6 c^2}{\hbar^4} . \tag{14}$$

Substituting in equation (14) in (13) and solving for the mass of the elementary particle m we have:

$$m'_\pi = \left[\frac{\hbar^2 H(\varphi_o)}{Gc}\right]^{1/3}\left[3 - \frac{8\pi G V(\varphi_o)}{c^2 H^2(\varphi_o)}\right]^{1/6} . \tag{15}$$

Observing (14) we see that the first part of the RHS equation is the well known Weinberg's relation modified during inflation by an extra term which involves the inflationary potential $V(\phi_o)$ and Hubble's parameter $H(\phi_o)$, both of them calculated a the initial value ϕ_o of the scalar field ϕ. Somebody can retrieve Weinberg's relation if the second parenthesis in the RHS becomes one. This is possible when the scalar field ϕ takes initial values ϕ_o which can be calculated for the choice of different potentials $V(\phi)$. There woul also be a value of the Hubble parameter given by (16) for the value

$$H(\varphi_o) = \frac{2}{c}\sqrt{\pi G V(\varphi_o)} \tag{16}$$

which the original Weinberg relation can be retrieved.

5 Examine Weinberg's relation for two different inflationary potentials

Next we will examine Weinberg's relation for two different kinds of inflationary potentials, namely a massive scalar field and also a self-interacting scalar field given by:[6]

$$V(\varphi)=\frac{m^2}{2}\varphi^2=\frac{m^2c^4}{2\hbar c}\varphi^2 \qquad (17)$$
$$V(\varphi)=\frac{\lambda}{4}\varphi^4=\frac{\lambda c\hbar}{4}\varphi^4$$

where λ is dimensionless constant and m has the dimensions of mass. From equation (10) we can now use the fact that $\phi(t) = \phi_o$ some initial value of the field. So using(15) and (10) along with (13) we have after substitution in (13) that:

$$m'_{1\pi}=\left(\frac{\hbar^2 H(\varphi_o)}{Gc}\right)^{1/3}\left[3-\frac{8\pi G}{c^2H^2(\varphi_o)}\left(\frac{m^2c^4}{2\hbar c}\varphi^2{}_o\right)\right]^{1/6} \qquad (18)$$

$$m'_{2\pi}=\left(\frac{\hbar^2 H(\varphi_o)}{Gc}\right)^{1/3}\left[3-\frac{8\pi G\lambda c\hbar\varphi^4{}_o}{4c^2H^2(\varphi_o)}\right]^{1/6} \qquad (19)$$

We can now see that the mass of the elementary particle depends on the initial value of the inflationary potential ϕ_o and the Hubble parameter $H(\phi_o)$. If the second square bracket on the RHS becomes equal to one then somebody exactly tetrieves the unmodified or original definition by Weinberg of an elementary particle's mass. This occurs when:

$$\varphi_o=\frac{H(\varphi_o)}{m}\left(\frac{\hbar}{\pi cG}\right)^{1/2} \qquad (20)$$
$$\varphi_o=\left(\frac{H(\varphi_o)c}{\sqrt{\pi\lambda Gc\hbar}}\right)^{1/2}$$

for the first and second potential respectivelly.

6. The case of k = 1 or closed universe

The k = 1 case corresponds to a closed Friedman vacuum dominated universe which evolves according to the law. Therefore the field equations can be written as follows:

$$3\left[\frac{\dot{R}(t)}{R(t)}\right]^2+\frac{3}{\dot{R}^2}=\frac{8\pi G}{c^2}\varepsilon+\Lambda \qquad (21)$$

$$2\frac{\ddot{R}(t)}{R(t)}+\left[\frac{\dot{R}(t)}{R(t)}\right]^2+\frac{1}{\dot{R}^2(t)}=\Lambda-\frac{8\pi G}{c^2}p \qquad (22)$$

Substituting in (21) and (22) as before the expressions for ε and p and subtracting (21) from (22) we obtain the following equation:

$$-\frac{\ddot{R}(t)}{R(t)}+\left[\frac{\dot{R}(t)}{R(t)}\right]^2+\frac{1}{R^2(t)}=\frac{4\pi G}{c^2}\dot{\varphi}^2(t)\,. \tag{23}$$

In the k = 1 case the radius of the universe evolves according to the law:[7] [8]

$$R(t)=\frac{1}{H}\cos(H\ \mathrm{t})\,. \tag{24}$$

If we now substitute (24) into (23) we again obtain that:

$$\dot{\varphi}^2(t)=0 \text{ or } \varphi(\mathrm{t})=\mathrm{const}=\varphi_{\mathrm{o}} \tag{25}$$

Using (23) and substituting into (23) we obtain after simplifying that:

$$\Lambda=H^2(\varphi_o)\left[3+\cosh^{-2}\left[H(\varphi_o)t\right]\right]-\frac{8\pi G}{c^2}V(\varphi_o) \tag{26}$$

But in any cosmological model Ht = 1, and therefore (26) finally becomes:

$$\Lambda=3.420H^2(\varphi_o)-\frac{8\pi G}{c^2}V(\varphi_o)\qquad . \tag{27}$$

Using next Zeldovich's definition of the cosmological constant we obtain:

$$m'_\pi=\left(\frac{\hbar^2H(\varphi_o)}{Gc}\right)^{1/3}\left[3.420-\frac{8\pi GV(\varphi_o)}{c^2H^2(\varphi_o)}\right]^{1/6} \tag{28}$$

which interms of the two potentials become:

$$m'_\pi=\left(\frac{\hbar^2H(\varphi_o)}{Gc}\right)^{1/3}\left[3.420-\frac{4\pi G}{c^2H^2(\varphi_o)}\left(\frac{m^2c^4}{\hbar c}\varphi^2{}_o\right)\right]^{1/6} \tag{29}$$

$$m'_\pi=\left(\frac{\hbar^2H(\varphi_o)}{Gc}\right)^{1/3}\left[3.420-\frac{2\pi G}{c^2H^2(\varphi_o)}\left(\lambda\hbar c\varphi^4{}_o\right)\right]^{1/6} \tag{30}$$

Again as before we can retrieve Weinberg's original relation if for example:

$$\varphi_o=0.192\frac{H(\varphi_o)}{m}\left(\frac{\hbar}{c}\right)^{1/2} \tag{31}$$

$$\varphi_o=1.048\left(\frac{cH(\varphi_o)}{\sqrt{\pi\lambda Gc\hbar}}\right)^{1/2} \tag{32}$$

7. The case of k = -1 or open universe

This case corresponds to an open Friedman vacuum dominated universe which evolves according to the law: [9]

$$R(t)=\frac{1}{H}\sinh[H\,t]\,, \qquad (33)\mathrm{f}$$

following the same steps as before we obtain the following equation for the mass of the elementary particle:

$$m'_\pi=\left(\frac{\hbar^2H(\varphi_o)}{Gc}\right)^{1/3}\left[2.276-\frac{8\pi GV(\varphi_o)}{c^2H^2(\varphi_o)}\right]^{1/6}, \tag{34}$$

and again:

$$m_{\pi}^{'} = \left(\frac{\hbar^2 H(\varphi_o)}{Gc}\right)^{1/3}\left[2.276 - \frac{4\pi G}{c^2 H^2(\varphi_o)}\left(\frac{m^2c^4}{\hbar c}\varphi^2{}_o\right)\right]^{1/6} \quad (35)$$

$$m_{\pi}^{'} = \left(\frac{\hbar^2 H(\varphi_o)}{Gc}\right)^{1/3}\left[2.276 - \frac{2\pi G}{c^2 H^2(\varphi_o)}\left(\lambda\hbar c\varphi^4{}_o\right)\right]^{1/6}. \quad (36)$$

As before Weinberg's relation for the mass of the elementary particle can be retrived if:

$$\varphi_o = 0.318\frac{H(\varphi_o)}{m}\left(\frac{\hbar}{c}\right)^{1/2} \quad (37)$$

$$\varphi_o = 0.893\left(\frac{cH(\varphi_o)}{\sqrt{\pi\lambda\hbar c}}\right)^{1/2} \quad (38)$$

8 Numerical calculations for all cases

To get an estimate for the mass of an elementary particle in different universes and in conjunction with Zeldovich's relation we will use relation (15) for each model universe. For that we choose $H_o = 1/t_o = 1/10^{-35}$ sec^{-1}. First assume that the potential energy density V(0) of the field to be equal the quantum density of matter $\rho_{quantum} = \frac{c^5}{G^2\hbar} = 1.237\times10^{93}\, g/cm^3$, and also as a second value $\rho_{critical}$ = V(0) = 10^{-29} g/cm^{-3} when $H_o = 1/t_o = 10^{-17}$ sec^{-1}. That can be based on a conjecture that is recently proposed that the current expansion of the universe is merely a decayed state of inflation . Therefore we obtain:

Case k = 0

$$\begin{aligned} m'_{\pi} &= 1.2\,01 m_{\pi} = 4.500\times10^{-8}\,g = 2.083\times10^{-3}\,m_{planck} \\ m'_{\pi} &= 1.201 m_{\pi} = 2.992\times10^{-25}\,g \end{aligned} \quad (39)$$

Case k =1

$$\begin{aligned} m'_{\pi} &= 1.227 m_{\pi} = 4.600\times10^{-8}\,g = 2.112\times10^{-3}\,m_{planck} \\ m'_{\pi} &= 1.227 m_{\pi} = 2.172\times10^{-25}\,g \end{aligned} \quad (40)$$

Case k = -1

$$\begin{aligned} m_{\pi}^{'} &= 1.146 m_{\pi} = 4.300\times10^{-8}\,g = 1.975\times10^{-3}\,m_{planck} \\ m_{\pi}^{'} &= 1.146 m_{\pi} = 2.029\times10^{-24}\,g \end{aligned} \quad (41)$$

Here we must note that the mass of the pion is 140 MeV/c^2 = 2.492×10^{-25}g where the one calculated using Weinberg's relation and for $H_o = 10^{17}$ sec^{-1} is 60 Mev/c^2 = 7.120×10^{-26} g [10]. From (39), (40), (41) we can also see that during the inflationary period (15) predicts a mass of the order of $10^{-3}m_{Planck}$ and also a mass which is close to the mass of the pion observed today and shown above. Particles can be created when the field ϕ starts oscillating near the minimum of V(ϕ), its energy its transferred to the particles as a result of these oscillations. The particles then created collide with one another, and approach a state of thermodynamic equilibrium.

Conclusions

In our paper, an expression of Weinberg's empirical relation for the mass an elementary particle has been derived using Einstein's field equations in an vacuum dominated inflationary universe with a nonzero cosmological constant. For that, Zeldovich's definition of the cosmological constant in the theory of quantized vacuum was used. The fact that Weinberg's relation can now be proven via Einstein's field equations makes the result not an empirical one. So Zeldovich's definition of the cosmological constant could point to a possible relation between microcosm and macrocosm and in a more fudamental way. The expression derived for the mass of the elementary particle is now modified, by an extra term which further depends on the inflationary potential $V(\phi_o)$ calculated at an initial value of the inflationary field ϕ.

Two well known inflationary potentials were used, namely a masive scalar field, and a self interacting scalar field. Expressions for the masses were calculated at the initial value of the field ϕ_o. For both fields, expressions for the values ϕ_o were given such that the original and unmodified Weinberg relation can be retrieved. All calculations were done in a flat, k = 0, closed, k = 1, and open k= -1, universe.

Finally estimates for the masses of the elementary particles were given for the three possible model universes during the era when inflation takes place, and also at the present era where the expansion of the universe can be thought as a decayed state of inflation.

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