SAKHAROV'S TEMPERATURE LIMIT IN A SCHWARZCHILD METRIC MODIFIED BY THE COSMOLOGICAL CONSTANT Λ

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Abstract: In this paper we are going to examine the effect, if any exists, that a modification of the Schwarzchild metric by a lamda term could have on the so called Sakharov's upper temperature limit. It's known that Zakharov's limit is the maximum possible black body temperature that can occur in our universe.

Introduction: In 1966 Sakharov first came up with an upper temperature bound for the black body radiation by using the thermodynamic properties of the hot matter in an isotropic universe. This upper bound can be written as follows:[1]

$$T \leq \mathrm{T}_{\max} = \frac{c_0}{k_B}\sqrt{\frac{\hbar c^5}{G}} \cong 1.419 \times 10^{32} K \tag{1}$$

where T is the temperature of the black body radiation, k_B = is Boltman's constant, $\hbar$ is Planck's constant, c is the speed of light, and G is the gravitational constant, and c_0 is a constant of the order of unit. This temperature is the maximum temperature of any substance in equilibrium with the radiation field. Following Massa's work [2] we now include a cosmological constant Λ in the suggested metric equation (2) and work out possible differences, if any, in the original definition of Sakharov's limit.

To start our analysis we first write out the appropriate metric modified by the cosmological constant Λ:

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda}{3}r^2\right)dt^2 - \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 - r^2 d\Omega^2 \quad (2)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Next, we consider black body radiation which is filling a spherical box of radius r and let the radiation field have a temperature T which would imply an energy density $\varepsilon = \sigma T^4$, where σ is the Stefan-Boltzman constant. Then, the total energy of the radiation field is given by:

$$E_{rad} = \frac{4\pi\,\varepsilon}{3}r^3 \quad (3)$$

(2) would therefore imply a mass equal to:

$$M = \left(\frac{4\pi}{3}\right)\left(\frac{\pi}{c^2}\right)\mathrm{r}^3 \quad (4)$$

Further we assume that the gravitational field in the" box " can be described by the already defined metric. Then the time component of the metric g_{44} can be written as follows:

$$g_{44} = 1 - \frac{2GM}{rc^2} - \frac{\Lambda}{3}r^2 \quad (5)$$

where:Λ is the cosmological constant and all the other variables have their usual meanings. Writing (4) and substituting for M we have:

$$g_{44} = 1 - \frac{8\pi\,\mathrm{G}\varepsilon}{3c^4}r^2 - \frac{\Lambda}{3}r^2 \quad (6)$$

For relativity to make sense we must have that $g_{44} \geq 0$ [4] and that further implies that:

$$\varepsilon \leq \frac{3}{8\pi}\left(\frac{c^4}{Gr^2}\right)\left(1-\frac{\Lambda}{3}r^2\right) \tag{7}$$

From (5) we can obtain:

$$\sigma \mathrm{T}^4 = \frac{3}{8\pi}\left(\frac{c^4}{Gr^2}\right)\left(1-\frac{\Lambda}{3}r^2\right) \tag{8}$$

and finally we obtain for the temperature:

$$T^4 \leq \frac{3\mathrm{c}^4}{8\pi\,\mathrm{G}\sigma\,\mathrm{r}^2}\left(1-\frac{\Lambda}{3}r^2\right) \tag{9}$$

Based on the basic quantum theory [5] now we can use the fact that the number of photons in a spherical box of radius r is defined by $N = (r\, T\, k_B \,/\, \hbar\, c)^3$ [1]. Since $N \geq 1$ we also have that:

$$r \geq \frac{\hbar c}{k_B T} \tag{10}$$

Finally, substituting (9) into (8) we obtain the expression for the temperature:

$$T^4 \leq \frac{3\mathrm{c}^4}{8\pi\,\mathrm{G}\,\sigma}\left(\frac{k_B T}{\hbar c}\right)^2\left[1-\frac{\Lambda}{3}\left(\frac{\hbar c}{k_B T}\right)^2\right] \tag{11}$$

Next, making use of the fact that $\sigma = \dfrac{\pi^2 k_B}{15\hbar^3 c^3}$, the above relation takes the form:

$$T^4 \leq \left(\frac{15}{8\pi^3}\right)\left(\frac{c^5\hbar}{G}\right)\left(\frac{T^2}{k_B}\right)\left[1-\frac{\Lambda}{3}\left(\frac{\hbar c}{k_B T}\right)^2\right] \tag{12}$$

Rearranging (11) and solving for the tempewrature T we have the following solutions:

$$T_{1,2} = \left[\frac{3A\left(1 \pm \sqrt{1-\frac{4\Lambda B}{3A^2}}\right)}{6}\right]^{1/2} \tag{13}$$

where:

$$B = \left(\frac{15}{8\pi^{3}}\right)\left(\frac{c^5\hbar}{k^2{}_B G}\right)\left(\frac{\hbar^2 c^2}{k^2{}_B}\right) = a\left(\frac{E^2{}_P}{k^2{}_B}\right)\left(\frac{\hbar^2 c^2}{k^2{}_B}\right) \tag{14}$$

and:

$$A = \left(\frac{15}{8\pi^{3}}\right)\left(\frac{c^5\hbar}{Gk^2{}_B}\right) = a\left(\frac{E_P{}^2}{k_B{}^2}\right) \tag{15}$$

If we finally simplify (12) in terms of all symbols we obtain for Zakharov's temperature limit:

$$T \leq \frac{1}{\sqrt{2}}\left(\frac{45}{8\pi^{3}}\right)^{1/2}\frac{1}{k_B}\left(\frac{c^5\hbar}{G}\right)^{1/2}\left[1 \pm \sqrt{1-\frac{32\pi^3}{405}\Lambda\left(\frac{\hbar c}{E_P}\right)^2}\right]^{1/2} \tag{16}$$

$$T \leq \frac{0.425}{\sqrt{2}} \frac{1}{k_B} \left(\frac{c^5 \hbar}{G} \right)^{1/2} \left[1 \pm \sqrt{1 - 2.446 \Lambda \left(\frac{\hbar c}{E_P} \right)^2 } \right]^{1/2} \quad (17)$$

Where E_P = is the Planck energy = $\sqrt{\frac{c^5 \hbar}{G}} = 1.959 \times 10^{16} erg$ (18)

If we now substitute for the cosmological constant $\Lambda = 10^{-54}$ cm^{-2} [6] and the rest of the constants we obtain that the numerical value of the Zakharov limit :

$$T_{Sch(\Lambda)} = \frac{1}{\sqrt{2}} \times 6.030 \times 10^{31} \left[1 \pm \sqrt{1 - 6.359 \times 10^{-118}} \right] \cong 8.528 \times 10^{31} K \quad (19)$$

It is now evident that the existence of an an extra term which contributes to Zakharov's limit, namely the term into the square bracket, in the RHS of equation (16). This term can be thought as a correction factor to Zakharov's limit due to the existance of the cosmological constant Λ in the metric equation This correction factor is very small even if the cosmological constant Λ is positive or negative as we shall see in (19), and of the order of 10^{-118} which is zero for any practical purposes. The total change finally becomes $T_{Sch(\Lambda)}$ =0.60 T_{max}. Morever if we now take the cosmological constant to its exteme quantum value $\Lambda_{max} = 10^{66}$ cm^{-2}[7] [1] the quantity in the square root is a negative number. This might suggest a problem at the quantum value of the cosmological constant since the temperature can not be a complex number. Alternatively it may be that a different treatment is required taking the quantum metric effects into account, in spite the fact that Boltzman's law which we used to derive the result, is independent of quantum mechanics. We anticipate that quantum curvature effects are taking place and they might affect the temperature in some way. There is of course a second root to equation (13) which makes (19) equal to zero. We will say that this root is of non physical significance since Zakharov's limit is a very high temperature. There is a value of the cosmological constant Λ for which the square root in (18) is always positive. For this, Λ must satisfy:

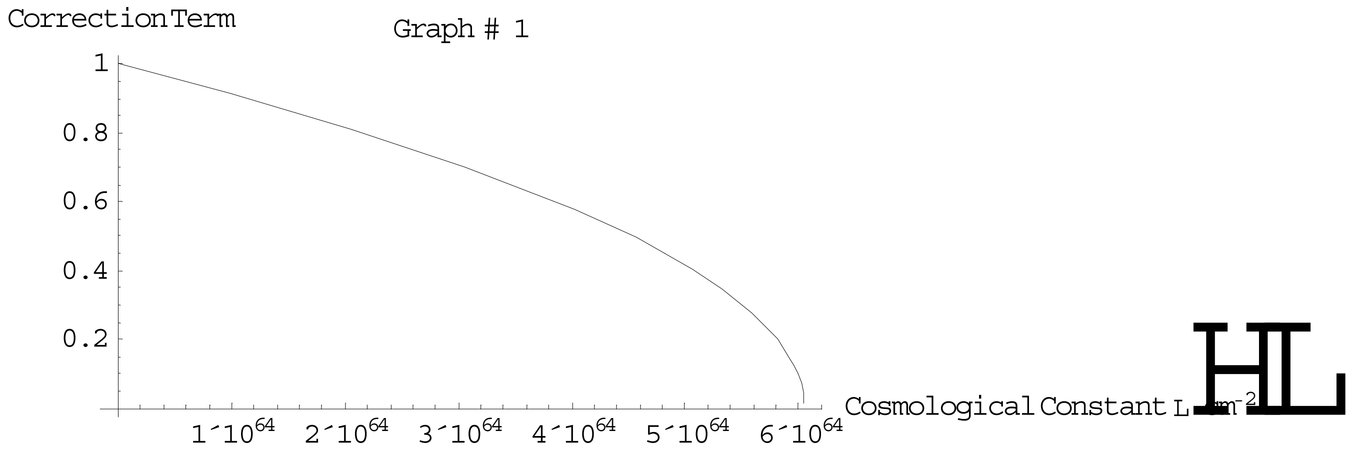
$$\Lambda \leq 0.408\left(\frac{\mathrm{E_P}}{\hbar \mathrm{c}}\right)^2 \leq 1.046\times 10^{65}\, cm^{-2} \tag{20}$$

As an example and if we assume a moment during which Λ = 10^{59} cm^{-2}, which would be early in the history of the universe, then the effect on Zakharov's limit would have the following possible values:

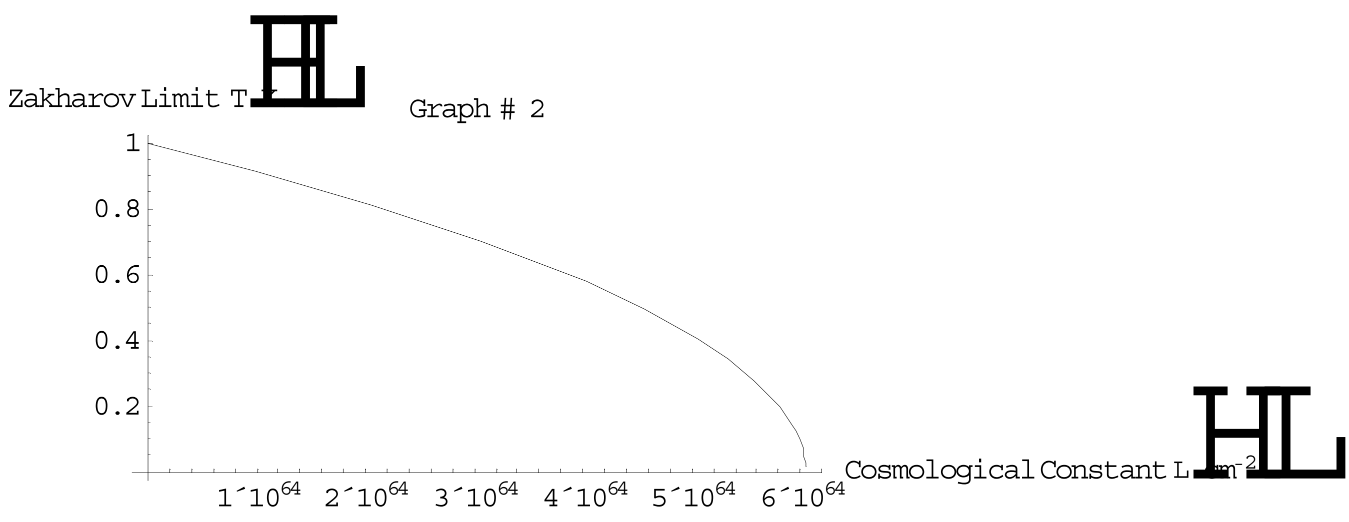
$$T_{Sch(\Lambda)Zakharov} = \left(\frac{0.181}{\sqrt{2}}\right)\frac{1}{k_B}\left(\frac{c^5\hbar}{G}\right)^{1/2}\sqrt{1+0.9999} = 1.80\times 10^{-1}\, T_{Max}$$

$$T_{Sch(\Lambda)Zakharov} = \left(\frac{0.181}{\sqrt{2}}\right)\frac{1}{k_B}\left(\frac{c^5\hbar}{G}\right)^{1/2}\sqrt{1-0.9999} = 1.27\times 10^{-3}\, T_{Max} \tag{21}$$

Graph # 1 shows a plot of the correction term for different values of the cosmological constant. We can see that Zakharov's limit occurs at higher values of the cosmological constant and most likely at Λ = 10^{66} cm^{-2}. After that, quickly drops to lower values of Λ leaving the temperature unaffected. The lower values of Λ are onthe right hand side of the Λ axis.



Similrarly graph # 2 shows the change of the Zakharov's limit itself with the variation of the cosmological constant Λ.



Conclusions

We can now say that if the Schwarzchild metric is modified by the cosmological constant Λ, positive or negative, the so called Zakharov's temperature limit changes by a factor which depends on the square root of the square root of the cosmological constant itself and other fudamental constants of physics like $\hbar$ and c. If Λ is equal in magnitute to the presently accepted value of the cosmological constant, then the correction to Zakharov's limit is extremely small, where as for higher values of the cosmological constant close to the quantum limit, this difference is noticeable especially when Λ becomes smaller than a particular limit. Thus, we conclude that if the Schwarzchild metric gets modified by the cosmological constant Zakharov's temperature, noticeable changes to smaller values if Λ becomes smaller or equal to $10^{65}\ cm^{-2}$

References:

[1] Sakharov, S., A., JETP Letters 3, 288, 1966.

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[6] Islam J., N., An Introduction to Mathematical Cosmology, p: 74-75

[7] Sivaram C., Astrophysics and Space Science, 127, 133-137, 1986.