A Derivation of $\pi(n)$ Based on a Stability Analysis of the Riemann-Zeta Function

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The prime-number counting function $\pi(n)$, which is significant in the prime number theorem, is derived by analyzing the region of convergence of the real-part of the Riemann-Zeta function using the unilateral z-transform. In order to satisfy the stability criteria of the z-transform, it is found that the real part of the Riemann-Zeta function must converge to the prime-counting function.

1 Introduction

The Riemann-Zeta function, which is an infinite series in a complex variable s, has been shown to be useful in analyzing nuclear energy levels [1] and the filling of s_1-shell electrons in the periodic table [2]. The following analysis of the Riemann-Zeta function with a z-transform shows the stability zones and requirements for the real and complex variables.

2 Stability with the z-transform

The Riemann-Zeta function is defined as

$$\Gamma(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1}$$

We start by setting the following equality

$$\Gamma(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} e^{-as}. \tag{2}$$

Then by simplifying

$$n^{-s} = e^{-as} = e^{-a(r+j\omega)} \tag{3}$$

and taking natural logarithm of both sides we obtain

$$-s\ln(n) = -as. \tag{4}$$

We then find the constant a such that

$$a = \ln(n). \tag{5}$$

We then apply the unilateral z-transform on (1):

$$\Gamma(s) = \sum_{n=1}^{\infty} n^{-s} z^{-n} = \sum_{n=1}^{\infty} e^{-as} z^{-n} = \sum_{n=1}^{\infty} e^{-a(r+j\omega)} z^{-n}. \tag{6}$$

Substituting (5), the real part of (6) becomes:

$$\mathrm{Re}\,[\Gamma(s)] = \sum_{n=1}^{\infty} e^{-ar} z^{-n} = \sum_{n=1}^{\infty} e^{-r\ln(n)} z^{-n}. \tag{7}$$

In order to find the region of convergence (ROC) of (7), we have to factor (7) to the common exponent $-n$, which requires

$$r = n/\ln(n), \tag{8}$$

which is the same as saying that the real part of $\Gamma(s)$ must converge to the prime-number counting function $\pi(n)$. With (8) satisfied, (7) becomes

$$\mathrm{Re}\,[\Gamma(s)] = \sum_{n=1}^{\infty} (ez)^{-n}. \tag{9}$$

which has a region of convergence (ROC)

$$\mathrm{ROC} = \frac{1}{1 - \frac{1}{ez}}. \tag{10}$$

To be within the region of convergence, z must satisfy the following relation

$$|z| > e^{-1} \quad \text{or} \quad |z| > 0.368. \tag{11}$$

which, places z within the critical strip. It can also be shown that the imaginary part of (6)

$$\mathrm{Im}\,[\Gamma(s)] = \sum_{n=1}^{\infty} e^{-aj\omega} z^{-n} = \sum_{n=1}^{\infty} e^{-j\omega\ln(n)} z^{-n}. \tag{12}$$

converges based on the Fourier series of $\sum e^{-j\omega\ln(n)}$.

3 Conclusions

The prime number-counting function $\pi(n)$ has been derived from a stability analysis of the Riemann-Zeta function using the z-transform. It is found that the real part of the roots of the zeta function correspond to $\pi(n)$ under the conditions of stability dictated by the unit-circle of the z-transform. The distribution of prime numbers has been found to be useful in analyzing electron and nuclear energy levels.

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References

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