Shifting assignments between infinite sets

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Abstract

To shift assignments between infinite sets is to create a disturbance within the assignment itself that cannot be removed. An assignment carrying such a disturbance cannot be regarded as static.

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The shifting and reassigning of sets play an important role in set theory and lead to surprising results with infinite sets. Shifts and reassignments are common in proofs such as that of the Banach–Tarski theorem. (The Banach–Tarski theorem, in a nutshell, seems to prove that one can disassemble a ball into six pieces and then reassemble them into two identical balls of the same size as the original.) With infinite sets, shifting and reassigning can produce up to infinite gains and losses, a finding that is still unexplained. This essay attempts to explain the origin of these gains and losses.

A well-known illustration of the assumptions and logic of operations with infinite sets is the *virtual Hotel Infinity*. The Hotel Infinity consists of two infinite sets: an infinite set of hotel rooms and an infinite set of guests. The sets are assigned in a one-to-one relation such that the hotel has no vacancies: Every guest has exactly one room and in each room is exactly one guest. However, even when the hotel is booked up, one can show that by shifting or reassigning, it is possible to create an infinite number of new rooms.

Let the following prerequisites hold in this hotel:
1. The hotel has an infinite number of rooms (infinite set of rooms).
2. The hotel is booked up (one-to-one assignment between guests and rooms).

If a new guest arrives at Hotel Infinity, the clever hotel manager advises the guests as follows: The inhabitant of room #1 should move to room #2, the inhabitant of room #2 should move to room #3, and so on.

Instantly a free room is available for the new guest, room #1. It looks as if a new room is created, since the hotel was previously booked up.

There are multiple variants of this principle that can create up to infinite new spaces (rooms). Since they are all based on the same principle, this easy example is given.

Where is the extra room coming from?

The hotel manager simply shifts all his guests one room ahead, into infinity. This looks easy at first view but is rather problematic during execution. For a better understanding of this shift, the following rules for static assignments are defined:

Rule #1: Every guest is only allowed to stay in a hotel room and nowhere else (i.e., there is no hallway).

Rule #2: In every room only one guest is allowed.

Those rules are a direct consequence of the pre-assessment of a one-to-one assignment between the guests and rooms and I think unchallenged.

In a first breakdown of the problem, defining these rules hints at the origin of these gains and losses. As many rooms as are created, there are as many guests in temporary double occupation of rooms or in a virtual hallway, as will be shown.

There are two possibilities for executing such a shift. One can try with respect to rule #1.

The guest from room #1 enters room #2, followed by the guest from room #2 entering room #3, and so on. You can see that as guest #1 is entering room #2, there are two guests in room #2. Guest #2 soon leaves the room but then enters room #3, such that there are two guests in room #3, and so on. At all times, two guests will be together in one room if the set is infinite. This situation is like a disturbance in assignment that moves toward infinity but never ends. This is no longer a static assignment and will never again be one.

Or one can try with respect to rule #2:

The guest from room #1 leaves room #1 (onto a virtual hallway), followed by the guest from room #2 leaving room #2. The guest from room #1 now enters room #2. The guest from room #3 leaves room #2. The guest from room #2 enters room #3, and so on. In this case there is no double occupancy of hotel rooms but there will always be a guest in the virtual hallway. At no time will all the guests be in a room. Again, the attribute of a static one-to-one assignment is lost.

If performed on infinite sets, there is no way to completely transform a static assignment (an assignment that complies with both rules) into a new static assignment through a shift or reassignment.

The results of such a shift or reassignment are regarded as a new static assignment, but that is not the case. There are always guests in double occupancy or in the virtual hallway. Double occupancy or use of the hallway is temporary only for the single guest, but permanent for an infinite set of guests and rooms; therefore it cannot be considered a static assignment.

A shifted or reassigned infinite set contains a disturbance in assignment that cannot be removed, while a static assignment has a fixed and unmoving assignment. A static assignment can only be transformed into a new static assignment through a shift or reassignment if this process can be completed. A way out would be if all the elements of the whole infinite set were shifted at the same time. All the guests would move from one room to the next together. By doing so, none of the rules are broken (no double occupancy and no use of the hallway). In this case, a rule is needed that is valid for all elements of the set at the same time, which brings one back to the hotel manager's advice to the guests.

It was implicitly assumed that the hotel manager gave the guests advice in the form of a rule, such as, "Move into the next room and pass this rule to the guest you find in that room." This is a serial rule, which cannot be applied to all the guests at the same time; the rule is passed from one guest to the next. All the guests must be informed in front for this advice to be executed at the exact same time. All the guests must receive this information, but the problem is that the very last guest never gets the information, since there is no such thing as a last guest. This situation is just a relocation of the problem. In this case, the shift or reassignment's preparation cannot be completed and will therefore never be executed.

Basically, this means one would have to write out the defined assignment function for all the elements (of the infinite sets). To simply define such a conversion without any proof of existence makes no sense. All examples of gains and losses with infinite sets are based on the assumption that one is able to transform a static assignment between infinite sets into another static assignment between the infinite sets, but the very existence of such assignments does not imply their ability to transform into each other.

In the Banach–Tarski theorem, the cutting lanes are seen as a set of infinite points that are moved or rotated against each other, shifting their assignments. Those rotations or movements are the source of the gains, since they are equal to the shift into infinity described above. A ***static one-to-one relation*** is assumed between the sets of points from the cutting lanes before **and** after the shifting, but this is not the case, as has been shown. Without these virtual movements or shifts, the Banach–Tarski theorem does not hold. One's common sense plays tricks, since one can imagine such different relationships between infinite sets. But to shift or reassign them is a task that cannot be finished and, without completion of the shift or reassignment, there is no such thing as a gain or loss.