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A NOTE ON THE MELLIN CONVOLUTION OF FUNCTIONS AND ITS RELATION TO RIESZ CRITERION AND RIEMANN HYPOTHESIS

Jose Javier Garcia Moreta
Graduate student of Physics at the UPV/EHU (University of Basque country)
In Solid State Physics
Address: Address: Practicantes Adan y Grijalba 2 5 G
P.O 644 48920 Portugalete Vizcaya (Spain)
Phone: (00) 34 685 77 16 53
E-mail: josegarc2002@yahoo.es

- ABSTRACT: In this paper we study how the Mellin convolution of functions f and g $\left(f*g \right)$ and how is related to the Riesz criterion for the Riemann Hypothesis, the idea is to stablish a Fredholm integral equation of First kind for the Riesz function and the Hardy function.

1. MELLIN CONVOLUTION OF FUNCTIONS:

Given two functions f and g , we can define the Mellin Convolution $\left(f*g \right)$ by the two equivalent integral forms

$$\left(f*g \right)=\int_0^\infty \frac{dt}{t} f\left(\frac{x}{t}\right) g(t^{-1})=\int_0^\infty \frac{dt}{t} f\left(xt\right) g(t) \qquad (1.1)$$

This operator is linear $\left(f*(\lambda g+\mu h) \right)=\lambda\left(f*g \right)+\mu\left(f*h \right)$, and associative $\left(f*(g*h) \right)=\left(f*g \right)*h$, another main property is the 'Convolution Theorem' , if we define the Mellin transform of f and g, $\hat{F}(s)=\int_0^\infty dtf(t)t^{s-1}$ and $\hat{G}(s)=\int_0^\infty dtg(t)t^{s-1}$ this theorem tells that the Mellin transform of the Convolution theorem is the product of the Mellin transforms of f and g $M[\left(f*g \right)=\hat{F}(s)\hat{G}(-s)$

Proof: If we take the integral $\int_0^\infty x^{s-1}dx$ to both sides , and introduce the change of variable $z = xt$ inside of the double integral involving x and t.

$$\int_0^\infty (f*g)(x)x^{s-1}dx = \int_0^\infty dt\int_0^\infty dxf(xt)g(t)\frac{dt}{t}x^{s-1} = \hat{F}(s)\int_0^\infty dtg(t)t^{-s-1} \quad (1.2)$$

The last factor is just $\hat{F}(s)\hat{G}(-s)$, so our Convolution Theorem is proved , the main and direct application of this theorem is the solution of the Convolution integral equation $\int_0^\infty dxK(xt)y(t)\frac{dt}{t} = h(x)$ for some given K(xt) and h(x) , then the solution for y(t) can be described by the Mellin-Cahen integral

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\hat{H}(-s)}{\hat{K}(-s)}\frac{ds}{t^s} \qquad \int_0^\infty dtK(t)t^{s-1} = \hat{K}(s) \qquad \int_0^\infty dth(t)t^{s-1} = \hat{H}(s) \quad (1.3)$$

Here 'c' is a real constant chosen in a manner so all the poles of $\frac{\hat{H}}{\hat{K}}(-s)$ lie on the left of the line in the complex plane defined by Re (c) .

2. RIESZ AND HARDY CRITERION AND RIEMANN HYPOTHESIS:

Given an infinite sum $\sum_{n=0}^\infty a_n$ we can define its Borel transform and its 'sum' B(S)

$$B(a_n,x) = \sum_{n=0}^\infty \frac{a_n}{\Psi(n+1)}x^n \qquad \Psi(n+1) = \int_0^\infty dtg(t)t^n \qquad B(S) = \int_0^\infty dtg(t)B(a_n,x) \quad (2.1)$$

(2.1) can give the sum of the series (providing the integral is convergent and well defined) no matter if the series is convergent or divergent , for example it gives the correct value of the Grandi's series $1-1+1-1+1-1+... = -1/2$, and the correct expansion for the exponential integral $\int_x^\infty \frac{e^{-t}}{t}dt$, by evaluation of the divergent series (except when x=0) $\sum_{n=0}^\infty (-x)^n n!$ via the Laplace transform $\int_0^\infty \frac{e^{-t}}{1+xt}dt$, this results with $\Psi(n+1) = n!$ were known to Borel , the general result (2.1) appeared in the paper by Leopold Nachbin as a generalization of Borel series to the case when $B(a_n,x) \neq O(e^{ax})$ (see reference [5]) . Appart from the linearity of the Borel transform and its sum B(S), we have another interesting property S = B(S) , the usual 'sum' in the sense of addition gives the same result as the Borel transform B(S)

Example: $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n$ its Borel transform is $B(a_n, xt) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(xt)^n = e^{-xt}$ and its sum $B(S,x) = \int_0^{\infty} dte^{-t(1+x)} = \frac{1}{1+x}$ S = B (S) are equal just because the power series expansion of $(1+x)^{-1}$ is convergent on the interval $(-1,1)$

We will exploit this property in order to study and give an integral equation for the following Hardy and Riesz criterion for the Riemann Hypothesis

- *Riesz criterion and Riemann Hypothesis:*

Hardy [3] and Riesz provided two different criteria for the truth of Riemann Hypothesis based on the growth of two different power series $\sum_{n=0}^{\infty} a_n x^n$ involving the Riemann zeta:

$$Riesz(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{(n-1)!\zeta(2n)} = O(x^{1/4+\varepsilon}) \qquad H(x) = \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!\zeta(2n+1)} = O(x^{-1/4+\varepsilon}) \quad (2.2)$$

Here ε is any positive real number , and $O(x^{1/4+\varepsilon})$ means that there is a positive constant 'C' so $f(x) \le Cx^{1/4+\varepsilon}$, that is if Riemann Hypothesis is true then $Riesz(x) \le Cx^{1/4+\varepsilon}$ and $H(x) \le Dx^{-1/4+\varepsilon}$ for both positive 'C' and 'D' (see [7]) .

Using the Mellin convolution of two functions $(f * g)$ and the properties of convolution theorem, we can give an integral equation representation for the Riesz and Hardy functions via a Fredholm integral of first kind with Kernel $\left[\sqrt{\frac{x}{t}}\right]$ in the form:

$$1 - e^{-x} = \int_0^{\infty}\frac{dt}{t}\left[\sqrt{\frac{x}{t}}\right]Riesz(t) \qquad \sqrt{x}(e^{-x}-1) = \int_0^{\infty}\frac{dt}{\sqrt{t}}\left[\sqrt{\frac{x}{t}}\right]\left(\frac{H}{2} + t\dot{H}\right) \quad x > 0 \quad (2.3)$$

- With $\dot{H} = \frac{dH}{dt}$,from the representation $\zeta(2s) = s\int_0^1 dt\left[\frac{1}{\sqrt{t}}\right]t^{s-1}$ we have that

$Riesz(t) = O(t^{1/2+\varepsilon})$ due to the 'Prime Number theorem' .In order to give a proof for the Riesz function integral equation inside (2.3) we should recall the properties of the Mellin integral transforms for Riesz function , $M\{Riesz(t)\} = \frac{\Gamma(s+1)}{\zeta(-2s)}$, the inverse of the Floor function [x] and the Riemann Zeta are related by the integral $\zeta(2s) = s\int_0^1 dt\left[\frac{1}{\sqrt{t}}\right]t^{s-1}$ valid for Re(s) > 1 , using the properties and definition of Gamma function for positive 's' $\int_0^{\infty} dte^{-t}t^{s-1} = \Gamma(s)$ and the 'Mellin Convolution theorem' we find $-\Gamma(s)=\frac{\Gamma(s+1)}{\zeta(-2s)}.\frac{\zeta(-2s)}{-s}=-\Gamma(s)$, so we can give an inmediate justification for the first integral equation inside (2.3) for the Riesz function. A proof based on the fact that for a infinite (and convergent) power series its Borel generalized transform and its 'normal' or usual definition of sum are equal so S= B(s) , is the following :

- First we use the known expansion $1-e^{-x}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n!}$ (2.4)

- We use the Mellin integral representation valid for $n \ge 1$ $\zeta(2n)=n\int_0^1 dt\left[\frac{1}{\sqrt{t}}\right]t^{n-1}$

- The Riesz(t) function has the power series $Riesz(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{(n-1)!\zeta(2n)}$, now if we use that S= B(S) since the power series defining the Riesz function is convergent we get the result
$1-e^{-x}=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n!}=\int_0^1\frac{dt}{t}\left[\frac{1}{\sqrt{t}}\right]\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\zeta(2n)}\frac{(xt)^n}{(n-1)!}\right)$ (2.5)

The final expression inside (2.5) is just the Riesz(xt) so with a change of variable y=xt and the fact that for $x<1$, then $[x]=0$, we can see that the series expansion of Riesz(t) solves the integral equation (2.3) for 'x' positive , in order to give a proof for Hardy's series we must use the properties

$$x\frac{dH}{dx}=\sum_{n=1}^{\infty}\frac{(-1)^n nx^n}{n!\zeta(2n+1)} \qquad \frac{\zeta(2n+1)}{n+1/2}=\int_0^1 dt\left[\frac{1}{\sqrt{t}}\right]t^{n-1/2} \quad n\ge 1 \qquad (2.6)$$

In order to prove (2.3) we have used simply the fact $S=B(S)$, in a previous paper [2] we investigated a similar procedure to solve integral equation with Kernel $g(s)=\int_0^{\infty}dtK(st)f(t)$, and applied this to solve and obtain an expression for the Prime counting function $\pi(x)=\sum_{p\le x}1$ via the series expansion $\sum_{n=1}^{\infty}\frac{a_n}{\zeta(n+1)\Gamma(n+1)}(\log x)^n$, since the Prime number theorem imposes a bound to the Prime number counting fucntion $\lim_{x\to\infty}\frac{\pi(x)\log x}{x}=1$ then for big 'n' $a_n\approx\frac{1}{n}$ (formally) , these $\{a_n\}$ come from the expasion of $\log\zeta(s)$ into a power series $\sum_{n=0}^{\infty}a_n s^{-n}$ valid for Re (s) > 1 with $a_n=\int_C\frac{dz}{z^{n+1}}\log\zeta(z)$ for a certain contour 'C' inside complex plane, the proof again is based on the fact S= B(S)

$$\log\zeta(s)=\sum_{n=0}^{\infty}a_n s^{-n}=\int_0^{\infty}\frac{dt}{e^t-1}\left(\sum_{n=0}^{\infty}a_n\frac{(ts^{-1})^n}{\zeta(n+1)n!}\right)=s\int_0^{\infty}\frac{dy}{e^{sy}-1}\left(\sum_{n=0}^{\infty}a_n\frac{y^n}{\zeta(n+1)n!}\right) \quad (2.7)$$

To prove (2.7) we have used the Generalized Borel transform involving the Zeta function $\zeta(n+1)\Gamma(n+1)=\int_0^{\infty}\frac{dt}{e^t-1}t^n$, and an appropiate change of variable, the last expression inside (2.7) is $\pi(e^t)$ due to the Fredholm integral equation of First kind satisfied by the Prime Number counting function $\frac{\log\zeta(s)}{s}=\int_0^{\infty}\frac{dt}{e^{st}-1}\frac{\pi(e^t)}{t}$ as it can be seen in , Apostol [1] (for other issues involving Number theory treated on this paper , this is perhaps the best introductory reference).

An alternative definition of equations (2.3) exists if we take the representation of Riemann Zeta as the Mellin transform involving the fractional part of x

$$\int_0^{\infty}dt\, frac\left(\frac{1}{t}\right)t^{s-1}dt=-\frac{\zeta(s)}{s} \qquad frac(t)=\frac{1}{2}-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k} \quad (2.8)$$

In (2.8) the 'fractional part function' is understood in terms of Fourier Analysis to be proportional to the sawtooth function , its derivative is the distribution $-2\sum_{k=1}^{\infty}\cos(2\pi kx)$, which is zero everywhere except for $x=0,1,2,3,4,......$ using again the property of the Borel resummation algorithm $S=B(S)$ the integral equations for Riesz and Hardy functions become

$$e^{-x}-1=\int_0^{\infty}\frac{dt}{t}\, frac\left(\sqrt{\frac{x}{t}}\right)Riesz(t) \qquad \sqrt{x}(1-e^{-x})=\int_0^{\infty}\frac{dt}{\sqrt{t}}\, frac\left(\sqrt{\frac{x}{t}}\right)\left(\frac{H}{2}+t\dot{H}\right) \quad (2.9)$$

The main advantage of using (2.8) instead of (2.6) is that (2.6) is easier for practical calculations involving the Fourier expansion in order to check if the RH is true so $Riesz(z)=O(z^{1/4+\varepsilon})$, another problem comes as $x\to 0$ for $\left[\frac{1}{\sqrt{x}}\right]$ this problem can be avoided using the Fourier representation for the fractional part of 'x' ,the fact that we can take the Mellin transform of $\left[\frac{1}{\sqrt{x}}\right]$ or $frac\left(\frac{1}{\sqrt{t}}\right)$ in order to define the Borel sum B(S) comes from the 'regularized' integral $\int_0^{\infty}t^{s-1}dt=0$ valid for every s and the definition of the fractional part as $t-[t]$.

According to Wolf [7] the solution to the Riesz function in(2.9) as $x\to\infty$ should be $\mathrm{R}\,ies(x)=Cx^{1/4}\sin\left(\phi-\frac{\gamma_1}{2}\log x\right)$ with $\gamma_1\approx 14.134725....$ (imaginary part of the first non-trivial zero for Riemann Zeta) , $C = 7.775 \bullet 10^{-5}$ and $\phi = -0.54916..$ some computer calculations are still to be made in order to check if this is correct. If we insert this formula inside (2.9) and use the Mellin representation of the Riemann zeta function $\zeta(s) = s\int_0^\infty dt frac\left(\frac{1}{t}\right)t^{s-1}$, since $\zeta\left(\frac{1}{2} \pm i\gamma\right) = 0$ and using Euler's formula for the sine

as $x \to \infty$, $e^{-x} \to 0$ $$1 \approx C\frac{\zeta\left(\frac{1}{2} - i\gamma\right)}{i + 2\gamma}x^{1/4 - i\gamma/2}e^{i\phi} - C\frac{\zeta\left(\frac{1}{2} + i\gamma\right)}{i - 2\gamma}x^{1/4 + i\gamma/2}e^{-i\phi} \approx 0 \quad (2.10)$$

So we get 0 = 1 ¡¡ this apparent contradiction would come from the fact that Wolf's formula is only valid for big 'x' [8] and we have taken the regularized value for the divergent Mellin transform $\int_0^\infty t^{s-1}dt = 0 = \int_0^\infty t^{s-1}H(t)dt$. A final question is , could we reproduce from (2.9) , and using the Mellin transform technique, the equalities ??

$$Riesz(x) = x\sum_{n=1}^\infty \frac{\mu(n)}{n^2}\exp\left(-\frac{x}{n^2}\right) \quad \text{and} \quad xe^{-x} = \sum_{n=1}^\infty Riesz\left(\frac{x}{n^2}\right) \qquad (2.11)$$

Both formulae in (2.11) were known to Riesz , and can be proved in the following way , using the Mellin transform inside (2.9) we get $\Gamma(s) = \frac{\zeta(-2s)}{s}.R(s)$, rearranging terms $s\Gamma(s) = \Gamma(s+1) = \zeta(-2s)R(s)$ or simply $\frac{\Gamma(s+1)}{\zeta(-2s)} = R(s) = \int_0^\infty dtRiesz(t)t^{s-1}$, now using the following representation (Perron formula [1] for Dirichlet series)

$$F(s) = s\int_0^\infty \frac{dt}{x^{s+1}} f(t) \quad F(s)\zeta(s) = s\int_0^\infty \frac{dt}{x^{s+1}}\sum_{n=1}^\infty f\left(\frac{x}{n}\right) \quad \frac{F(s)}{\zeta(s)} = s\int_0^\infty \frac{dt}{x^{s+1}}\sum_{n=1}^\infty \mu(n) f\left(\frac{x}{n}\right) \quad (2.12)$$

And $\frac{1}{\zeta(s)} = s\int_0^\infty \frac{dt}{x^{s+1}}M(t)$, the proof of (2.11) is inmediate , however (2.9) have the advantage of avoid a summation involving the Möebius function defined as follow

$$\mu(n)\begin{cases} 0 & \text{if n has repeated prime factor} \\ (-1)^k & \text{if n is product of k different primes} \end{cases} \quad \text{and} \quad \mu(1) = 1$$

From integral equation (2.9) could we deduce that we have solved RH ?, we do not know the answer, however if there was an extra term inside Riesz function $x^{\sigma/2}\sin\left(C + \frac{u}{2}\log(x)\right)$ so $\zeta(\sigma \pm iu) = 0$, the imaginary part of this root 'u' would yield to a very oscillating function with period $T = \frac{2\pi}{u}$ $u \to \infty$, this fact would be noticeable when solving the integral equation (2.9) , also from the definition for 'x' big of the Riesz function , we have the (approximate) functional equation $g(x) = Riesz\left(\frac{1}{x}\right)$

$$g\left(\frac{1}{x}\right)\frac{1}{\sqrt{x}} = g(x) + O\left(\frac{1}{x^{1/4}}\right) \qquad (2.13)$$

Appendix A: An integral equation involving Gamma function for Kernel $frac\left(\sqrt{\frac{x}{t}}\right)$

The Riesz function, is not the only one that satisfy an integral equation similar to (2.9) , given the logarithm of the Gamma function defined by

$$\log\Gamma(1+z) + \gamma_1(z - A_0) = \sum_{n=2}^{\infty}\frac{(-z)^n}{n}\zeta(n) \qquad \gamma_1 = \int_1^{\infty} dt\left(\frac{1}{[x]} - \frac{1}{x}\right) \qquad (A.1)$$

And A_0 is a Real constant of integration , using again the Borel resummation trick with

$$1 = -\frac{n}{\zeta(n)}\int_0^{\infty}\frac{dt}{t}\, frac\left(\frac{1}{t}\right)t^n \qquad \sum_{n=0}^{\infty}(-z)^n = \frac{1}{1+z} \qquad B(S) = S \qquad (A.2)$$

The first equation in (A.3) is just the representation for fractional function, the last one is the condition that for a summable power series, the Borel transform must be equal to the usual sum , with (A.1) and (A.2) we can obtain formally the integral equation

$$2\log\Gamma(1+\sqrt{z}) + 2\gamma_1\left(\sqrt{z} - A_0\right) = -\int_0^{\infty}\frac{dt}{t}\, frac\left(\sqrt{\frac{z}{t}}\right) f(t) \qquad f(t) = \frac{t}{1+\sqrt{t}} \qquad (A.3)$$

From (A.3) we see that f(t) diverges as $\sqrt{t}$ for big ‘t’ , on the left part we have a function of order $O(z^{1+\varepsilon})$, form the theory of integral equations with Symmetryc Kernels , the solution to the Riesz function can be expressed into an integral form

$$\frac{1}{x}Riesz\left(\frac{1}{x}\right) = y_0(x) + \int_0^{\infty} dtW(x,t)\left(e^{-t} - 1\right) \qquad W(x,t) = frac\left(\sqrt{xt}\right) + \sum_{m=0}^{\infty}\frac{\phi(x)\phi(t)}{\lambda_i^2} \quad (A.4)$$

Here $y_0(x)$ is the solution to the equation $0 = \int_0^{\infty} frac\left(\sqrt{zt}\right)y_0(t)dt$, and $\{\phi_i\}_{i=0}^{\infty}$, $\{\lambda_i\}_{i=0}^{\infty}$ is the set of Orthogonal Eigenfunctions /Eigenvalues of $K(x,t) = K(t,x) = frac\left(\sqrt{xt}\right)$, function W(x,t) is called the Resolvent (or ‘inverse’ operator)

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