On N-ary Algebras, Polyvector Gauge Theories in Noncommutative Clifford Spaces and Deformation Quantization

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Abstract

Polyvector-valued gauge field theories in noncommutative Clifford spaces are presented. The noncommutative binary star products are associative and require the use of the Baker-Campbell-Hausdorff formula. An important relationship among the **n**-ary commutators of noncommuting spacetime coordinates $[X^1, X^2, ..., X^n]$ and the poly-vector valued coordinates $X^{123...n}$ in noncommutative Clifford spaces is explicitly derived and is given by $[X^1, X^2, ..., X^n] = n!\ X^{123...n}$. It is argued how the large N limit of **n**-ary commutators of n hyper-matrices $\mathbf{X}_{i_1 i_2 .... i_n}$ leads to Eguchi-Schild p-brane actions when $p+1 = n$. A noncomutative n-ary generalized star product of functions is provided which is associated with the deformation quantization of n-ary structures. Finally, brief comments are made about the mapping of the Nambu-Heisenberg n-ary commutation relations of linear operators into the *deformed* Nambu-Poisson brackets of their corresponding symbols.

1 Gauge Theories in Noncommutative Clifford Spaces

Clifford algebras are deeply related and essential tools in many aspects in Physics. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory [3] whose generalized polyvector-valued coordinates are Clifford-valued quantities which incorporate lines, areas, volumes, hyper-volumes.... degrees of freedom associated with the collective particle, string, membrane, p-brane,... dynamics of p-loops (closed p-branes) in D-dimensional target spacetime backgrounds. C-space Relativity naturally incorporates the ideas of an invariant length (Planck scale), maximal acceleration, non-commuting coordinates, supersymmetry, holography, higher derivative gravity with torsion and variable dimensions/signatures. It permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing. It resolves the ordering ambiguities in QFT, the problem of time in Cosmology and admits superluminal propagation (tachyons) without violations of causality. The relativity of signatures of the underlying spacetime results from taking different slices of C-space.

The conformal group in spacetime emerges as a natural subgroup of the Clifford group and Relativity in C-spaces involves natural scale changes in the sizes of physical objects without the introduction of forces nor Weyl's gauge field of dilations. A generalization of Maxwell theory of Electrodynamics of point charges to a theory in C-spaces involves extended charges coupled to antisymmetric tensor fields of arbitrary rank and where the analog of photons are tensionless p-branes. The Extended Relativity Theory in Born-Clifford Phase Spaces with a Lower and Upper Length Scales and the program behind a Clifford Group Geometric Unification was advanced by [5]. Furthermore, there is no EPR paradox in Clifford spaces [6] and Clifford-space tensorial-gauge fields generalizations of Yang-Mills theories and the Standard Model allows to predict the existence of new particles (bosons, fermions) and tensor-gauge fields of higher-spins in the 10 TeV regime [2]. Clifford-spaces can also be extended to Clifford-Superspaces by including both orthogonal and symplectic Clifford algebras and generalizing the Clifford super-differential exterior calculus in ordinary superspace to the full fledged Clifford-Superspace outlined in [7]. Clifford-Superspace is far richer than ordinary superspace and Clifford Supergravity involving polyvector-valued extensions of Poincare and (Anti) de Sitter supergravity (antisymmetric tensorial charges of higher rank) is a very relevant generalization of ordinary supergravity with applications in M-theory.

It was recently shown [1] how an unification of Conformal Gravity and a $U(4) \times U(4)$ Yang-Mills theory in four dimensions could be attained from a Clifford Gauge Field Theory in C-spaces (Clifford spaces) based on the (complex) Clifford $Cl(4, C)$ algebra underlying a complexified four dimensional spacetime (8 real dimensions). Other approaches to unification based on Clifford algebras can be found in [8]. Tensorial Generalized Yang-Mills in C-spaces (Clifford spaces) based on poly-vector valued (anti-symmetric tensor fields) gauge fields $\mathcal{A}_M(\mathbf{X})$ and field strengths $\mathcal{F}_{MN}(\mathbf{X})$ have been studied in [2], [3] where $\mathbf{X} = X_M \Gamma^M$ is a C-space poly-vector valued coordinate

$$\mathbf{X} \ = \ s \ \mathbf{1} \ + \ x_\mu \ \gamma^\mu \ + \ x_{\mu_1 \mu_2} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \ + \ x_{\mu_1 \mu_2 \mu_3} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \ + \ \ldots\ldots \ + \\ x_{\mu_1 \mu_2 \mu_3 \ldots\ldots \mu_d} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \ \gamma^{\mu_3} \ldots\ldots\ \wedge \gamma^{\mu_d} \tag{1.1}$$

In order to match dimensions in each term of (1.1) a length scale parameter must be suitably introduced. In [3] we introduced the Planck scale as the expansion parameter in (1.1). The scalar component s of the C-space poly-vector valued coordinate $\mathbf{X}$ was interpreted by [4] as a Stuckelberg time-like parameter that solves the problem of time in Cosmology in a very elegant fashion.

A Clifford gauge field theory in the C-space associated with the ordinary $4D$ spacetime requires $\mathcal{A}_M(\mathbf{X}) = \mathcal{A}^A_M(\mathbf{X})\ \Gamma_A$ that is a poly-vector valued gauge field where M represents the poly-vector index associated with the C-space, and whose gauge group $\mathcal{G}$ is itself based on the Clifford algebra $Cl(3,1)$ of the tangent space spanned by 16 generators Γ_A. The expansion of the poly-vector Clifford-algebra-valued gauge field $\mathcal{A}^A_M$, for $fixed$ values of A, is of the form

$$\mathcal{A}^A_M\ \Gamma^M\ =\ \Phi^A\ +\ \mathcal{A}^A_\mu\ \gamma^\mu\ +\ \mathcal{A}^A_{\mu_1\mu_2}\ \gamma^{\mu_1}\wedge\gamma^{\mu_2}\ +\ \mathcal{A}^A_{\mu_1\mu_2\mu_3}\ \gamma^{\mu_1}\wedge\gamma^{\mu_2}\wedge\gamma^{\mu_3}\ +\ .... \tag{1.2}$$

The index A spans the 16-dim Clifford algebra $Cl(3,1)$ of the tangent space. In this letter we proceed with the construction of Polyvector-valued Gauge Field Theories in *noncommutative* Clifford Spaces (C-spaces) which are polyvector-valued *extensions* and *generalizations* of the ordinary *noncommutative* spacetimes. We begin firstly by writing the commutators $[\Gamma_A, \Gamma_B]$. For $pq = odd$ one has [9]

$$[\ \gamma_{b_1 b_2.....b_p},\ \gamma^{a_1 a_2......a_q}\ ]\ =\ 2\gamma^{a_1 a_2......a_q}_{b_1 b_2.....b_p}\ -$$
$$\frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3....a_q]}_{b_3.....b_p]}\ +\ \frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[a_1....a_4}_{[b_1....b_4}\ \gamma^{a_5....a_q]}_{b_5.....b_p]}\ -\ ...... \tag{1.3}$$

for $pq = even$ one has

$$[\ \gamma_{b_1 b_2.....b_p},\ \gamma^{a_1 a_2......a_q}\ ]\ =\ -\ \frac{(-1)^{p-1}2p!q!}{1!(p-1)!(q-1)!}\ \delta^{[a_1}_{[b_1}\ \gamma^{a_2 a_3....a_q]}_{b_2 b_3.....b_p]}\ -$$
$$\frac{(-1)^{p-1}2p!q!}{3!(p-3)!(q-3)!}\ \delta^{[a_1....a_3}_{[b_1....b_3}\ \gamma^{a_4....a_q]}_{b_4.....b_p]}\ +\ ...... \tag{1.4}$$

The anti-commutators for $pq = even$ are

$$\{\ \gamma_{b_1 b_2.....b_p}, \gamma^{a_1 a_2......a_q}\ \}\ =\ 2\gamma^{a_1 a_2......a_q}_{b_1 b_2.....b_p}\ -$$
$$\frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3....a_q]}_{b_3.....b_p]}\ +\ \frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[a_1....a_4}_{[b_1....b_4}\ \gamma^{a_5....a_q]}_{b_5.....b_p]}\ -\ ...... \tag{1.5}$$

and the anti-commutators for $pq = odd$ are

$$\{\ \gamma_{b_1 b_2.....b_p},\ \gamma^{a_1 a_2......a_q}\ \}\ =\ -\ \frac{(-1)^{p-1}2p!q!}{1!(p-1)!(q-1)!}\ \delta^{[a_1}_{[b_1}\ \gamma^{a_2 a_3....a_q]}_{b_2 b_3.....b_p]}\ -$$
$$\frac{(-1)^{p-1}2p!q!}{3!(p-3)!(q-3)!}\ \delta^{[a_1....a_3}_{[b_1....b_3}\ \gamma^{a_4....a_q]}_{b_4.....b_p]}\ +\ ...... \tag{1.6}$$

The second step is to write down the *noncommutative* algebra associated with the noncommuting poly-vector-valued coordinates in $D = 4$ and which can be obtained from the Clifford algebra (1.3-1.6) by performing the following replacements (and relabeling indices)

$$\gamma^{\mu} \leftrightarrow X^{\mu}, \quad \gamma^{\mu_1\mu_2} \leftrightarrow X^{\mu_1\mu_2}, \quad .... \gamma^{\mu_1\mu_2.....\mu_n} \leftrightarrow X^{\mu_1\mu_2....\mu_n}. \tag{1.7}$$

When the spacetime metric components $g_{\mu\nu}$ are *constant*, from the replacements (1.7) and the Clifford algebra (1.3-1.6) (after one relabels indices), one can then construct the following *noncommutative* algebra among the poly-vector-valued coordinates in $D = 4$, and *obeying* the Jacobi identities, given by the relations

$$[\ X^{\mu_1},\ X^{\mu_2}\ ]_* \ =\ X^{\mu_1}\ *\ X^{\mu_2}\ -\ X^{\mu_2}\ *\ X^{\mu_1}\ =\ 2\ X^{\mu_1\mu_2}. \tag{1.8}$$

In most of the remaining commutators a suitable length scale parameter must be introduced in order to match units. We shall set this length scale (let us say the Planck scale) to *unity*. Also, by choosing the C-space coordinates to behave like anti-Hermitian operators we avoid the need to introduce i factors in the right hand side. The (star) commutators are

$$[\ X^{\mu_1\mu_2},\ X^{\nu}\ ]_* \ =\ 4\ (\ g^{\mu_2\nu}\ X^{\mu_1}\ -\ g^{\mu_1\nu}\ X^{\mu_2}\ ). \tag{1.9}$$

$$[\ X^{\mu_1\mu_2\mu_3},\ X^{\nu}\ ]_* \ =\ 2\ X^{\mu_1\mu_2\mu_3\nu}, \quad [\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu}\ ]_* \ =\ -8\ g^{\mu_1\nu}\ X^{\mu_2\mu_3\mu_4} \pm ...... \tag{1.10}$$

$$[\ X^{\mu_1\mu_2},\ X^{\nu_1\nu_2}\ ]_* \ =\ -\ 8\ g^{\mu_1\nu_1}\ X^{\mu_2\nu_2}\ +\ 8\ g^{\mu_1\nu_2}\ X^{\mu_2\nu_1}\ +$$
$$8\ g^{\mu_2\nu_1}\ X^{\mu_1\nu_2}\ -\ 8\ g^{\mu_2\nu_2}\ X^{\mu_1\nu_1}. \tag{1.11}$$

$$[\ X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2}\ ]_* \ =\ 12\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\nu_2}\ \pm\ .... \tag{1.12}$$

$$[\ X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2\nu_3}\ ]_* \ =\ -\ 36\ G^{\mu_1\mu_2\ \nu_1\nu_2}\ X^{\mu_3\nu_3}\ \pm\ ...... \tag{1.13}$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}\ ]_* \ =\ -\ 16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ \pm ...... \tag{1.14}$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}\ ]_* \ =\ -\ 16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ +\ 16\ g^{\mu_1\nu_2}\ X^{\mu_2\mu_3\mu_4\nu_1}\ -\ .... \tag{1.15}$$

$$[\ X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3}\ ]_* \ =\ 48\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4} - 48\ G^{\mu_1\mu_2\mu_4\ \nu_1\nu_2\nu_3}\ X^{\mu_3} + ... \tag{1.16}$$

$$[\, X^{\mu_1\mu_2\mu_3\mu_4},\; X^{\nu_1\nu_2\nu_3\nu_4}\,]_* \;=\; 192\; G^{\mu_1\mu_2\mu_3\;\nu_1\nu_2\nu_3}\; X^{\mu_4\nu_4} \;-\; .... \tag{1.17}$$

etc...... where

$$G^{\mu_1\mu_2......\mu_n\;\nu_1\nu_2......\nu_n} \;=\; g^{\mu_1\nu_1}\; g^{\mu_2\nu_2}\; ....\; g^{\mu_n\nu_n} + signed\,permutations \tag{1.18}$$

One must emphasize that when the spacetime metric components $g_{\mu\nu}$ are *no* longer *constant*, the noncommutative algebra among the poly-vector-valued coordinates in $D = 4$, does *not* longer *obey* the Jacobi identities. For this reason we restrict our construction to a flat spacetime background $g_{\mu\nu} = \eta_{\mu\nu}$. The noncommutative conditions on the polyvector coordinates in condensed notation can be written as

$$[\, X^M,\; X^N\,]_* \;=\; X^M{*}X^N - X^N{*}X^M \;=\; \Theta^{MN}(X) \;=\; f^{MN}{}_L\, X^L \;=\; f^{MNL}\, X_L \tag{1.19}$$

the structure constants f^{MNL} are antisymmetric under the exchange of polyvector valued indices. The third step is to define the noncommutative star product of functions of X. The correct noncommutative and *associative* star product [16] corresponding to a Lie-algebraic structure for the noncommutative (C-space) coordinates requires the use of the Baker-Campbell-Hausdorff formula

$$exp(A)\, exp(B) \;=\; exp\left(\; A + B + \frac{1}{2}[A,B] + \frac{1}{12}(\; [A,[A,B]] - [B,[A,B]]\;) \;+ ....\right). \tag{1.20a}$$

and is given by

$$(\; A_1 \;*\; A_2\;)(X) \;=\; exp\left(\; \frac{i}{2}\; X^M \Lambda_M[\; i\;\partial_Y;\; i\;\partial_Z\;]\;\right)\; A_1(Y)\; A_2(Z)|_{X=Y=Z}. \tag{1.20b}$$

where the expression for the bilinear differential polynomial $\Lambda_M[i\partial_Y; i\partial_Z]$ in eq-(1.32b) can be *read* from the Baker-Campbell-Hausdorff formula

$$e^{i\;K_M \hat X^M}\; e^{i\;P_N \hat X^N} \;=\; e^{i\;\hat X^M(\;K_M \;+\; P_M \;+\; \frac{1}{2}\;\Lambda_M[K,P]\;)}. \tag{1.20c}$$

and is given in terms of the structure constants $[X^N, X^Q] = f^{NQ}_M X^M$, after setting $K_N = i\;\partial_{Y^N},\; P_Q = i\;\partial_{Z^Q}$, by the following expression

$$\Lambda_M[K,P] \;=\; i\;K_N\;P_Q\;f^{NQ}_M \;+\; \frac{i^2}{6}\;K_{N_1}\;P_{Q_1}\;(P_{N_2} - K_{N_2})\;f^{N_1Q_1}_S\;f^{SN_2}_M \;+$$

$$\frac{i^3}{24}\;(P_{N_2}\;K_{Q_2} \;+\; K_{N_2}\;P_{Q_2})\;K_{N_1}\;P_{Q_1}\;f^{N_1Q_1}_{S_1}\;f^{S_1N_2}_{S_2}\;f^{S_2Q_2}_M \;+\; .... \tag{1.20d}$$

In the very special case of canonical noncommutativity $[X^M, X^N]_* = \Theta^{MN} =$ *constants*, the star product is the standard Moyal one. If the fields and their derivatives vanishing fast enough at infinity (or there are no boundaries), one has the *cyclicity* property of the integral when $\Theta^{MN} =$ *constants*

$$\int A * B = \int A\ B + \textit{total derivative} = \int B\ A = \int B * A \quad (1.21a)$$

$$\int A * B * C = \int B * C * A = \int C * A * B . \quad (1.21b)$$

The relations (1.21) are essential in order to construct invariant actions under star gauge transformations. However, when one has a Lie-algebraic type of noncommutativity, the $\Theta's$ are X-dependent $[X^M, X^N]_* = \Theta^{MN}(X) = f^{MN}_K X^K$ and the cyclicity property (1.21a, 1.21b) *no longer holds* since the star product is X-dependent. Nevertheless, one can still construct a cyclic integral by introducing an auxiliary measure [13]

$$\int [DX]\ \mu(X)\ A * B = \int [DX]\ \mu(X)\ B * A = \int [DX]\ \mu(X)\ A\ B \quad (1.22)$$

In the case when the components $\Theta^{MN} =$ *constants*, the Clifford-algebra valued field strength $F^C_{MN}\Gamma_C$ is given by

$$F_{[MN]} = F^C_{[MN]}\ \Gamma_C = (\ \partial_M\ \mathcal{A}^C_N - \partial_N\ \mathcal{A}^C_M\ )\ \Gamma_C +$$

$$\frac{1}{2}\ (\ \mathcal{A}^A_M * \mathcal{A}^B_N - \mathcal{A}^B_N * \mathcal{A}^A_M\ )\ \{\ \Gamma_A,\ \Gamma_B\ \} + \frac{1}{2}\ (\ \mathcal{A}^A_M * \mathcal{A}^B_N + \mathcal{A}^B_N * \mathcal{A}^A_M\ )\ [\ \Gamma_A,\ \Gamma_B\ ]. \quad (1.23)$$

The commutators $[\ \Gamma_A,\ \Gamma_B\ ]$ and anti-commutators $\{\ \Gamma_A,\ \Gamma_B\ \}$, where A, B are polyvector-valued indices, can be read from the relations in eqs-(1.3-1.6) . The symmetrized star product when the components $\Theta^{MN} =$ *constants* is

$$\mathcal{A}^A_M\ *_s\ \mathcal{A}^B_N \equiv \frac{1}{2}\left(\mathcal{A}^A_M * \mathcal{A}^B_N + \mathcal{A}^B_N * \mathcal{A}^A_M\right) = \mathcal{A}^A_M\ \mathcal{A}^B_N +$$

$$\Theta^{\mu\nu}\ \Theta^{\kappa\lambda}\ (\partial_\mu\ \partial_\kappa\ \mathcal{A}^A_M)\ (\partial_\nu\ \partial_\lambda\ \mathcal{A}^B_N) + ... \quad (1.24a)$$

the antisymmetrized (Moyal bracket) star product is

$$\mathcal{A}^A_M *_a \mathcal{A}^B_N \equiv \frac{1}{2}\left(\mathcal{A}^A_M * \mathcal{A}^B_N - \mathcal{A}^B_N * \mathcal{A}^A_M\right) = \Theta^{\mu\nu}\ (\partial_\mu\ \mathcal{A}^A_M)\ (\partial_\nu\ \mathcal{A}^B_N) +$$

$$\Theta^{\mu\nu}\ \Theta^{\kappa\lambda}\ \Theta^{\alpha\beta}\ (\partial_\mu\ \partial_\kappa\ \partial_\alpha\ \mathcal{A}^A_M)\ (\partial_\nu\ \partial_\lambda\ \partial_\beta\ \mathcal{A}^B_N) + ... \quad (1.24b)$$

However, when one has a Lie-algebraic type of noncommutativity the $\Theta's$ are X-dependent $[X^M, X^N]_* = \Theta^{MN}(X) = f^{MN}_K X^K$, and consequently the components of the Clifford-algebra valued field strength $F^C_{MN}\Gamma_C$ in *noncommutative*

C-spaces are *no* longer given by eq-(1.23) because the *ordinary* derivative operators ∂_M obeying the standard Liebnitz rule are *no* longer *consistent* with the relations $[X^M, X^N]_* = \Theta^{MN}(X) = f_K^{MN} X^K$ due to the X-dependence of $\Theta^{MN}(X)$.

Modified derivative operators obeying a *modified* Liebnitz rule must be introduced. For instance, in the case of κ-Minkowski space, which is a very special case of a Lie-algebraic noncommutativity of coordinates, the *deformed* derivative operators consistent with the κ-Minkwoski space commutation relations were obtained by [13]. Secondly, an extension of the Seiberg – Witten (SW) map for x-dependent $\theta^{\mu\nu}(x)$ was provided by [13], [17] relating the non-Abelian noncommutative gauge fields based on *noncommutative* coordinates and non-Abelian gauge fields based on *commutative* coordinates. It is then when one may construct the proper expressions for *deformed* field strengths associated with the *noncommutative* coordinates. A C-space extension of the results by [17] based on the Seiberg-Witten map leads to an action, after introducing the auxiliary measure $\mu(X)$ to ensure cyclicity,

$$\int [DX]\ \mu(X)\ <\ \hat{F}^A_{MN}(X)\ \Gamma_A\ *\ \hat{F}_B^{MN}(X)\ \Gamma^B\ >_{scalar\ part}. \quad (1.25)$$

However, the above action *is not gauge invariant* despite the cyclicity property of the integral (1.25) because F_{MN} does *not* transform properly

$$\delta\ \hat{F}_{MN}\ =\ \delta\ \left(\ \Theta_{MK}\ \Theta_{NL}\ \hat{F}^{KL}\ \right)\ =\ \Theta_{MK}\ \Theta_{NL}\ \delta\ \hat{F}^{KL}\ =$$

$$\Theta_{MK}\ \Theta_{NL}\ [\ \hat{\Lambda},\ \hat{F}^{KL}\ ]_*\ \neq\ [\ \hat{\Lambda},\ \Theta_{MK}\ \Theta_{NL}\ \hat{F}^{KL}\ ]_*\ =\ [\ \hat{\Lambda},\ \hat{F}_{MN}\ ]_*. \quad (1.26)$$

Therefore, one learns that due to the X-dependence of $\Theta^{MN}(X)$ the gauge transformation for $F_{MN} : \delta\ \hat{F}_{MN}\ \neq\ [\ \hat{\Lambda},\ \hat{F}_{MN}\ ]_*$ does *not* transform properly despite that F^{MN} does have the correct transformation law : $\delta\ \hat{F}^{MN}\ =\ [\ \hat{\Lambda},\ \hat{F}^{MN}\ ]_*$

As far as we know, a fully satisfactory gauge-invariant physical action obeying the cyclicity property and also invariant under the *deformed* symmetries has not been constructed yet. Scalar field actions on κ-deformed spacetimes were constructed by [14] and similar conclusions were found : it was not possible to build actions obeying the cyclicity property while being also invariant under the deformed κ-Poincare symmetries. It remains to be seen if similar problems arise in building gauge-invariant actions based on *covariant* star products between two arbitrary Lie algebra valued differential forms on a *symplectic* manifold [18]. It will be interesting to see if the Seiberg-Witten map can be generalized to the case when the ordinary derivatives are replaced with the covariant derivatives and the Moyal star product is replaced by the covariant one [18].

Having provided the basic ideas and results behind polyvector gauge field theories in Noncommutative Clifford spaces, the construction of Noncommutative Clifford-space gravity as polyvector valued gauge theories of twisted diffeomorphisms in C-spaces will be the subject of future investigations. It would require quantum Hopf algebraic deformations of Clifford algebras. Such theory should be richer than gravity in Noncommutative spacetimes.

2 N-ary algebras, Clifford Spaces, p-branes and hyper-matrix QM.

Next we shall show the relation between n-ary Algebras and Clifford Spaces. Ternary algebras have recently resurfaced with great intensity in the study of $M2$-brane duality where M theory on $AdS_4 \times S^7$ is dual to a superconformal field theory in three dimensions, with the supergroup $OSp(8|4)$, after Bagger-Lambert-Gustavsson (BLG) [22] constructed a Chern-Simons gauge theory in three dimensions with maximal supersymmety $\mathcal{N} = 8$. The authors [23] later have shown that the dual gauge theory is actually an $\mathcal{N} = 6$ superconformal Chern-Simons theory in three-dimensions and is associated to M-theory on $AdS_4 \times S^7/Z_k$, with N units of flux. The $M5$-brane duality is based on M theory on $AdS_7 \times S^4$ being dual to a six dimensional superconformal field theory whose super group is $OSp(6,2|4)$. N-ary algebras have been known for some time [20] since Nambu introduced his bracket (a Jacobian) in the study of branes and the generalizations of Hamiltonian mechanics based on Poisson brackets. In this section we shall show how poly-vector valued coordinates admit a very natural interpretation in terms of $n-$ary commutators.

The ternary commutator for noncommuting coordinates is defined as

$$[X^1, X^2, X^3] \;=\; X^1\,[X^2, X^3] \;+\; X^2\,[X^3, X^1] \;+\; X^3\,[X^1, X^2] \;=$$

$$\frac{1}{2}\,\{\; X^1,\; [X^2, X^3]\;\} \;+\; \frac{1}{2}\,[\; X^1,\; [X^2, X^3]\;] \;+\; \textit{cyclic permutations} \qquad (2.1)$$

Due to the Jacobi identities, the terms

$$\frac{1}{2}\,[\; X^1,\; [X^2, X^3]\;] \;+\; \textit{cyclic permutations} \;=\; 0. \qquad (2.2)$$

so that the ternary commutators become

$$[X^1, X^2, X^3] \;=\; \frac{1}{2}\,\{\; X^1,\; [X^2, X^3]\;\} \;+\; \textit{cyclic permutations}. \qquad (2.3)$$

After using the relations, from eqs-(1.8-1.17),

$$[X^2, X^3] \;=\; 2\;X^{23}, \quad \{\; X^1,\; X^{23}\;\} \;=\; 2\;X^{123}. \qquad (2.4)$$

one gets finally

$$[X^1, X^2, X^3] \;=\; 2\;X^{123} \;+\; \textit{cyclic permutations} \;=\; 6\;X^{123}. \qquad (2.5)$$

since $X^{123} = X^{231} = X^{312} = -X^{132} = ...$ The 4-ary commutator is defined as

$$[X^1, X^2, X^3, X^4] \;=\; X^1\,[X^2, X^3, X^4] \;-\; X^2\,[X^3, X^4, X^1] \;+$$
$$X^3\,[X^4, X^1, X^2] \;-\; X^4\,[X^1, X^2, X^3] \;=$$

$$\frac{1}{2}\ \{\ X^1,\ [X^2,X^3,X^4]\ \}\ +\ \frac{1}{2}\ [\ X^1,\ [X^2,X^3,X^4]\ ]\ -\ ....\ =$$

$$3\ \{\ X^1,\ X^{234}\ \}\ +\ 3\ [\ X^1,\ X^{234}\ ]\ -\ ....\ =$$

$$6\ X^{1234}\ +\ 18\ (\ g^{12}\ X^{34}\ +\ g^{13}\ X^{42}\ +\ g^{14}\ X^{23}\ )\ -\ ....\ =\ 24\ X^{1234} \tag{2.6}$$

due to the cancellations

$$(\ g^{12}\ X^{34}\ +\ g^{13}\ X^{42}\ +\ g^{14}\ X^{23}\ )\ -\ (\ g^{23}\ X^{41}\ +\ g^{24}\ X^{13}\ +\ g^{21}\ X^{34}\ )\ +$$

$$(\ g^{34}\ X^{12} + g^{31}\ X^{24} + g^{32}\ X^{41}\ ) - (\ g^{41}\ X^{23} + g^{42}\ X^{31} + g^{43}\ X^{12}\ )\ =\ 0. \tag{2.7}$$

resulting from the conditions $X^{\mu\nu} = -X^{\nu\mu}$, $g^{\mu\nu} = g^{\nu\mu}$ after recurring to the (anti) commutators

$$[X^1, X^{234}]\ =\ 2\ X^{1234},\quad \{X^1, X^{234}\}\ =\ 6\ (g^{12}\ X^{34}\ +\ g^{13}\ X^{42}\ +\ g^{14}\ X^{23}\ ). \tag{2.8}$$

and the conditions $X^{1234} = -X^{2341} = X^{3412} = -X^{4123}$. For example, given a Noncommutative Clifford space in $D = 4$, one arrives at

$$[X^1, X^2]\ =\ 2\ X^{12},\ [X^1, X^2, X^3]\ =\ 6\ X^{123},\ [X^1, X^2, X^3, X^4]\ =\ 24\ X^{1234}. \tag{2.9}$$

where X^1, X^2, X^3, X^4 is a shorthand notation for $X^{\mu_1}, X^{\mu_2}, X^{\mu_3}, X^{\mu_4}$. Therefore, one finds that the poly-vector coordinates $X^{\mu_1\mu_2}, X^{\mu_1\mu_2\mu_3}, X^{\mu_1\mu_2\mu_3\mu_4}$ can be seen, respectively, as the binary, ternary and 4-ary commutators of the noncommuting vector coordinates X^μ. In the general case, using the noncommutative algebra of eqs-(1.8-1.17) in Clifford spaces one arrives by recursion at

$$[\ X^1,\ X^2,\ ....,\ X^n\ ]\ =\ n!\ X^{123.....n}. \tag{2.10}$$

This n-ary commutator interpretation of the poly-vector valued coordinates of a noncommutative Clifford space warrants further investigation.

N-ary algebras are relevant to the large N limit of covariant Matrix Models based on generalized n-th power matrices (hyper-matrices) [21] $\mathbf{X}_{i_1 i_2 ......i_n}$, that are extensions of square, cubic, quartic, ... matrices (hyper-matrices). These Matrix models bear a relationship to Eguchi-Schild p-brane actions for $p+1 = n$. The range of indices is $i_1, i_2, ..., i_n \subset I = 1, 2, ...N$. The n-ary commutator of n generalized n-th power matrices (hyper matrices) in the large $N \to \infty$ has a correspondence with the Nambu-brackets (NB) as follows

$$[\ \mathbf{X}^1,\ \mathbf{X}^2,\ ....,\ \mathbf{X}^n\ ]_{i_1 i_2 ......i_n}\ \to\ \{\ X^1,\ X^2,\ ....,\ X^n\ \}_{NB}. \tag{2.11}$$

by replacing the hyper matrix $\mathbf{X}_{i_1 i_2 ......i_n}$ in the large $N \to \infty$ limit for the c-function of n-variables $X(\sigma^1, \sigma^2, ..., \sigma^n)$. The trace operation in the large N limit has a correspondence with the integral $\int d^n\sigma$ so that

$$Trace\ \left(\ [\ \mathbf{X}^1,\ \mathbf{X}^2,\ ....,\ \mathbf{X}^n\ ]^2\ \right)\ \to\ \int\ d^n\sigma\ \{\ X^1,\ X^2,\ ....,\ X^n\ \}^2_{NB}. \tag{2.12}$$

recovering in this fashion the Eguchi-Schild p-brane actions for $p+1=n$. The fermionic version of (2.12) is

$$\int d^n\sigma \ \bar{\Psi} \ \Gamma_{12....n-1} \ \{ \ X^1, \ X^2, \ ...., \ X^{n-1} \ , \Psi\}. \tag{2.13}$$

The canonical Moyal noncommutative (but assocciative) star product is defined as

$$(f \ * \ g)(x,p) \ = \left(e^{\frac{i\hbar\omega_{ij}}{2!} \ \partial_{Z'_i} \ \wedge \ \partial_{Z''_j}} \right) \ f(Z') \ g(Z'')|_{Z=Z'=Z''}. \tag{2.14}$$

where the derivatives are evaluated at $Z=Z'=Z''$ and the phase coordinates are defined by $Z=(x,p)$; $Z'=(x',p'')$; $Z''=(x'',p'')$. By analogy one can define the ternary $\bullet$ product of three functions of x,y,z in terms of a deformation parameter κ as

$$(f \ \bullet \ g \ \bullet \ h)(x,y,z) \ = \left(e^{\frac{i\kappa\epsilon_{ijk}}{3!} \ \partial_{X'_i} \ \wedge \ \partial_{X''_j} \ \wedge \ \partial_{X'''_k}} \right) \ f(X') \ g(X'') \ h(X'''). \tag{2.15}$$

where the derivatives are evaluated at $X_i=X'_i=X''_i=X'''_i$; the range of indices is $i=1,2,3$. The coordinates are defined by

$$X_i=x,y,z; \ X'_i=x',y',z'; \ X''_i=x'',y'',z''; \ X'''_i=x''',y''',z'''. \tag{2.16}$$

The author [24] has also proposed such ternary product. The n-ary extension of (2.15) is straightforward. It remains to be seen whether or not the ternary $\bullet$ product obeys the ternary associativity condition

$$A\bullet B\bullet(C\bullet D\bullet E) \ = \ A\bullet(B\bullet C\bullet D)\bullet E \ = \ (A\bullet B\bullet C)\bullet D\bullet E. \tag{2.17}$$

The Moyal canonical star product (2.14) can also be recast in integral form as [20]

$$(f \ * \ g)(x,p) \ = \ (\frac{1}{\pi\hbar})^2 \int \ du_1 \ du_2 \ dv_1 \ dv_2 \ e^{\frac{2i}{\hbar} \ \Delta(u_i,v_i)} \ \times$$

$$f(x+u_1,p+v_1) \ g(x+u_2,p+v_2). \tag{2.18}$$

where the integral limits are $-\infty,+\infty$ and the kernel of the exponential is given by the determinant

$$\Delta(u_i,v_i)=det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}. \tag{2.19}$$

The analog of the integral (2.18) for the ternary case is

$$(f\bullet g\bullet h)(x,y,z) \ = \ (\frac{1}{\kappa})^3 \int \ du_1du_2du_3dv_1dv_2dv_3dw_1dw_2dw_3 \ e^{\frac{2\pi i}{\kappa} \ \Delta(u_i,v_i,w_i)} \ \times$$

$$f(x+u_1, y+v_1, z+w_1)\ g(x+u_2, y+v_2, z+w_2)\ h(x+u_3, y+v_3, z+w_3). \quad (2.20)$$

where the kernel of the exponential is given by the determinant

$$\Delta(u_i, v_i, w_i) = det \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}. \quad (2.21)$$

However, the latter integral expression (2.20) for the putative ternary $\bullet$ product does not appear to yield the same expression as the ternary $\bullet$ product provided by eq-(2.15). In the Weyl-Wigner-Groenewold-Moyal (WWGM) deformation quantization procedure, the operator/function in classical phase space correspondence $\hat{A}(\hat{q}, \hat{p}) \leftrightarrow A(q, p; \hbar)$ is given by [20]

$$A(q,p) \ = \ \mathbf{W}[\ \hat{A}(\hat{q},\hat{p})\ ] \ = \ \int dy\ e^{\frac{-2i\pi py}{\hbar}} \ < q+y|\ \hat{A}(\hat{q},\hat{p})\ |q-y> . \quad (2.22)$$

such that the WWGM map of the product of two Weyl-ordered operators $\hat{A}(\hat{q}, \hat{p})\ \hat{B}(\hat{q}, \hat{p})$ into the star product of their *symbols* $A(q, p; \hbar)\ * B(q, p; \hbar)$ obeys the relations

$$\mathbf{W}(\ \hat{A}(\hat{q},\hat{p})\ \hat{B}(\hat{q},\hat{p})\ ) \ = \ A(q,p,\hbar)\ *\ B(q,p,\hbar) \ \Rightarrow$$

$$\mathbf{W}(\ [\ \hat{A}(\hat{q},\hat{p}),\ \hat{B}(\hat{q},\hat{p}\,]\ ) \ = \ \{\ A(q,p,\hbar),\ B(q,p,\hbar)\ \}_* \ = \ A*B - B*A. \quad (2.23).$$

Given the noncommutative ternary $\bullet$ product of three functions of x, y, z as shown in eq-(2.15), and which is associated with the deformation quantization of ternary structures [24], the immediate question is how to generalize the WWGM map (2.22) in the binary star product case to the ternary $\bullet$ product case. In particular, how to map the Nambu-Heisenberg n-ary commutation relations of linear operators into the *deformed* Nambu-Poisson brackets of their corresponding symbols. For instance, to find the correspondence

$$\{\ A,\ B,\ C\ \}_\bullet \ = \ A \bullet B \bullet C \ \pm \ permutations \ \leftrightarrow \ [\ \hat{A},\ \hat{B},\ \hat{C}\ ] \quad (2.24)$$

such that the Nambu-Weyl-Heisenberg *ternary* commutation relations among a *triad* of canonical "conjugate" operators has a one-to-one correspondence to the *deformed* Nambu-Poisson brackets of their symbols as follows

$$[\ \hat{A},\ \hat{B},\ \hat{C}\ ] \ = \ i\ \kappa\ \mathbf{I} \ \leftrightarrow \ \{\ A,\ B,\ C\ \}_\bullet \ = \ i\ \kappa. \quad (2.25)$$

The deformation parameter κ appearing in (2.15) plays now the role of Planck's constant h in (2.25). To find the linear operator $\hat{A}(\hat{x}, \hat{y}, \hat{z}) \leftrightarrow A(x, y, z)$ correspondence such that the relations (2.24, 2.25) are obeyed in conjunction with the Nambu-Filippov fundamental identity [19], etc .... is a very *challenging* problem; i.e. to construct a Hypermatrix formulation of QM based on a deformation quantization of Nambu-Poisson classical mechanics. For example, the ternary product of three Hypermatrices which preserves the rank is

$$\delta_{i_3 j_1} \; \delta_{j_3 k_1} \; \delta_{k_3 i_1} \; A_{i_1 i_2 i_3} \; B_{j_1 j_2 j_3} \; C_{k_1 k_2 k_3} \; = \; (ABC)_{i_2 j_2 k_2}. \qquad (2.26)$$

Following Heisenberg's formulation of ordinary QM, the large $N = \infty$ limit of a Hypermatrix should correspond to an operator in a Hilbert space. It is warranted to pursue these ideas further to see whether or not one can construct a Hypermatrix formulation/extension of QM.

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