Mathematical Model of Information

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Dedicated to Marie-Louise Nykamp

Abstract

A simple and rather general mathematical model of the phenomenon of information is presented, followed by several examples and comments.

1. The Relative Novelty of the Concept of Information

Concepts such as mass, motion and velocity, for instance, have from times immemorial been in a form or another in human awareness. Contrary to that, however, the concepts of acceleration, energy, entropy, and even more so information, are quite recent. In this regard, it is amusing to recall, for instance, that Aristotle believed velocity to be proportional to the force exerted upon an object. And it took until Newton's Second Law to realize that, in fact, it is acceleration, that is, the velocity of velocity, which has that proportionality property, at least within the framework of Classical Physics.
No wonder, therefore, that - contrary to what many may tacitly believe, thus they do so mostly by default - information is still a less than sufficiently clarified concept when it comes to its deeper nature and not merely to measuring it quantitatively, and this lack of clarity

is with us even if we happen to live in an era of information overload.

A consequence is that the concept of information suffers from being subjected to various insufficiently supported, let alone well founded claims, such as for instance the one insistently stated by not a few physicists, namely that, "information is physical", [1,2], ...

Remaining for a moment with Physics, the insufficient clarity surrounding the concept of information can lead to apparently important conflicts of view which in fact may altogether have a questionable basis, and in which the various positions may equally be unfounded. One such case is the three decades old so called "black hole war", in which the main protagonists have been Stephen Hawking, and on the opposite side, Gerard t' Hooft and Leonard Susskind, [3,31,32].

Of much wider interest of course, yet of no less confusing and influential effect is that, meanwhile, the claim that "information is physical", as found in the mentioned and often cited [1,2] papers, has turned into nothing short of a rather noisy slogan . Such papers, and other similar ones, are quite typical in their rather vague and long winding repetitive discoursive, than in their well founded presentation. And they end up being merely about an acquisitive intent of certain physicists to incorporate yet another modern branch of science into their specific discipline, an intent tantamount in this case to imposing the considerably reductionist view according to which "information is physical". Indeed, the considerable reductionism involved in such a claim is by now becoming quite obvious, some of the remarkable developments in this regard being advanced in [4,11], mentioned in a few details below.

As for the failure to achieve a satisfactory clarity in this reductionist venture, let alone, persuasiveness in argumentation, suffice it to note the following.
Certainly, it should be obvious that, whenever the statement "A is B" is made, such a statement has no clear and well founded meaning, unless the entity "B" is well defined, and defined so *a priori*. Thus, in the case of the above reductionist slogan related to information, what is meant by "physical" must be clearly defined in advance, in order for that slogan to have a chance to avoid being a mere trivially unfounded

nonsense.

Here however, one faces a manifestly serious problem. Indeed, the term "physical" has even during recent times proved to have a significantly changing and expanding meaning. Just consider how since Newton it got enlarged by incorporating electro-magnetism, relativity, atoms, quanta, particle physics, and so on.

And then, the question arises : is the reductionist slogan "information is physical" a latest definition of Physics, one that chooses to further expand Physics by incorporating phenomena related to information, or on the contrary, that slogan is a mere claim in which the concepts of "physical" and "information" are only assumed to be defined in some vague and tacitly accepted ways ?

If that reductionist slogan is a new expansion in the definition of Physics, then everything is all right, provided of course that the concept of "information" is well defined, and defined before that slogan is launched upon the world.
Otherwise, as seen above, that slogan is quite nonsensical ...

A possible source or cause for such an insufficient clarity is pointed out quite clearly in [6], namely :

"The main problem with thermodynamical arguments is that the laws of thermodynamics are usually formulated in a natural language and have a common sense character. To apply them to some subtle problems one needs more rigorous formulations, than those found in most textbooks. This is particularly important in quantum theory, which often seems to be far from a 'common sense'..."

In this regard, in view of [11] for instance, it may be noted the need for a considerable care which should be exhibited whenever the concept of information is used in Physics. Indeed, as it turns out, information is in fact so *fundamental* that the whole of Quantum Mechanics can be reconstructed from no more than three axioms with clear empirical motivation, the first of which is called

- *Information Capacity* : All systems with information carrying capacity of one bit are equivalent.

As for how *fundamental* is the concept of information for the very formulation of all of presently known Physics, among others, a surprising major discovery has been presented in [4]. According to it, if one is indeed to fall for any kind of reductionist sloganeering, then a far more appropriate one would be "Physics is but a mere sub-realm of Information".
Indeed, as B. Roy Frieden shows it in convincing and rigorous detail in [4], major theories of Physics, both Classical and Quantum, can rather directly be obtained from an optimization of suitable applications of the well known statistical concept of *Fisher Information*.

Regarding the nuanced, varied and deeper role of information and entropy in Physics, recent literature, such as in [4-32], can be relevant.

In this paper, a mathematical model is suggested for the concept of information. This model incorporates what appear to be two novel ideals related to information, [31,32], namely, *total involvement* and *simultaneous presence*.

As can be note from the model, its interest is not limited to Physics.

2. A Mathematical Model of Information

Step 1.

Claude Shannon, in the 1940s, defined a *measure* for quantities of information, and based on it, has established several basic properties of channels that transmit information.
His studies, however, did not consider the *nature* or *structure* of the possible relationships between information and its physical support. In this regard, as noted in [31,32], two concepts, namely, *total involvement* and *simultaneous presence* can be useful.

In this paper, the focus will be on modelling the possible nature or structure of the more usual relationships between information and its physical support. And in doing so, it will emerge that, unlike earlier well entrenched basic physical concepts, such as for instance, mass, motion, velocity, acceleration, force, energy, electric charge, magnetism, spin, etc., which are indeed assumed to be inseparable from corresponding physical supports, in the case of information there can often be a far more loose connection or relationship between information and the supporting physical structures. And the freedom in this regard of which information may benefit when it comes to its involvement with physical structures may possibly go so far that the lack of a usual physical support need not necessarily disable the presence of information.

Such a state of affairs, however, need not seem so strange. Indeed, thoughts and ideas, for instance, may as well be seen as having a being all of their own in the sense of being not necessarily conditioned by the presence of some customarily known physical support. And such a view need not necessarily be based on the adoption of any Cartesian type duality with its division between "res extensa" and "res cogitans".
Indeed, as mentioned in [33], for instance, there are everyday and rather typical phenomena related to thinking and ideas in which the presence of a supporting usual kind of physical structure does not seem to be so obvious. As an immediate example, let us recall that in Einsteinian Mechanics a basic assumption is that there cannot be any propagation of physical action faster than light. Yet just like in the case we happen to think in terms of Newtonian Mechanics, our thinking in terms of Einsteinian Mechanics can again instantly and simultaneously be about phenomena no matter how far apart from one another in space and/or time. Consequently, the question arises :

- Given the mentioned relativistic limitation, how and where does such a thinking happen ?

And certainly, information can be seen in a somewhat similar way with entities like thoughts and ideas, rather than with mass, motion, velocity, acceleration, force, energy, electric charge, magnetism, or say, spin, etc.

In this regard, it is important to note that *intent* at its production, as well as *interpretation* at its reception, may play crucial roles in information. And such an intent or interpretation need clearly not necessarily be there when mass, motion, velocity, acceleration, force, energy, electric charge, magnetism, spin, etc., are manifested.
Indeed, here, namely with the possible presence of intent and interpretation in the case of information, can we see an important *similarity* between information and entropy, as mentioned in the sequel, see also [31,32].

Let us further note that the presence of *time* seems to be a sine-qua-non for the presence of information. Thus a model for information should include the presence of time.
Here, in order not to preclude generality, we shall take for time any *linearly ordered* set $(T, \leq)$, being thus able to model discrete, as well as continuous time.

We can also note that, as mentioned, information, in order to function as such, assumes a *production* process, followed by a *reception* one. Thus the whole process can be seen as having *three* successive stages

(2.1) $\quad A \longrightarrow B \longrightarrow C$

where A is the stage of production, C is the stage of reception, and in between, stage B represents the specific information involved in the process. For further clarification, we note that B is not seen as any sort of "channel" supposed to convey information from A to C, but instead, it is itself the information *intended* when its production in stage A is made. Consequently, if one wants to talk about any channel at all related to (2.1), then it is rather the two arrows "$\longrightarrow$" in (2.1) which may represent it.

Here, it is important to note that one may as well have an *incomplete* process, namely

(2.2) $\quad A \longrightarrow B \longrightarrow$

in which stage C, that is, of reception, is missing. Indeed, such a situation need not always render the remaining incomplete process meaningless, since as long as (2.2) keeps existing, it is always an open possibility that some C may join in, and thus complete the process to its form in (2.1).
On the contrary, the incomplete process

$$(2.3) \quad \longrightarrow B \longrightarrow C$$

appears of to be of no interest, if not even, as having no sense, since the information represented by stage B is of course supposed to be produced somewhere, thus the stage A of production of information cannot be absent.

Clearly, (2.1), (2.2) can be seen both in a static, and alternatively, dynamic setup.

A first remark about (2.1), and implicitly (2.2), is that stages A and C assume a certain physical existence, the first producing information, and doing so with a possible *intent*, as well as a possible *encoding* type interpretation, while the second receiving it based upon a certain possible *decoding* type interpretation.

On the other hand, and as a second remark about (2.1), (2.2), stage B, which represents information as such, need *not* actually be of any physical nature, since it can consist alone of the event of there being a certain intended information which has been produced by stage A.

The third remark on (2.1), (2.2) is about the fact that, *unlike* in stage A, where the presence of intent or of encoding type interpretation is optional, stage C is meaningless without assuming that it involves a decoding type interpretation.

Step 2.

In view of the above, the scheme (2.1) obtains the following more structured form

$$(2.4) \qquad A = (P_A, P, Intent, E) \longrightarrow B \longrightarrow C = (P_C, R, D)$$

where

(2.5)

P_A is the physical system which is involved in the production, presentation or support of the information

P is the information produced as a physical entity

$Intent$ is a parameter with values 0 and 1, according to the absence or presence of intent in the production of information

E is a possible encoding type interpretation

P_C is the physical system which is involved in the reception of the information

R is the information received as a physical entity

D is a necessary decoding type interpretation of the information received

Clearly, processes (2.4), (2.5) can have their respective incomplete versions corresponding to (2.2).

Step 3.

Let us now look more carefully to the structure in (2.4), (2.5).

A first fact to note is that P_A and P_C in such a situations may usually be subjected both to the phenomenon of *total involvement* and *simultaneous presence*, [31,32]. Let us, therefore, recall briefly these two phenomena. For that, it is useful to separate in two classes various *concepts* in physics, namely

$$
(2.6) \quad \begin{array}{l} \text{Class I : mass, motion, velocity, acceleration, force, energy,} \\ \qquad\qquad \text{electric charge, magnetism, spin, etc.} \\ \\ \text{Class II : information, entropy, etc.} \end{array}
$$

An important property of effective physical entities which embody manifestations of concepts of Class I is that, when they are in physical interactions with similar entities, they participate in one and only one way, namely, with their *total involvement*. For instance, a mass of m kg, when interacting according to Newton's Law of Universal Attraction with another mass of M kg, will always lead to a force of attraction $F = GmM/r^2$, where G is the gravitational constant and r is the distance between the two masses. Therefore, that mass of m kg will always interact with no less and no more than its given m kg.

We can note that the very possibility of measurements of effective physical entities which embody manifestations of concepts of Class I depends on that property of total involvement, since measurements are instances of interactions between effective physical entities, and each such interaction performed under the same conditions is supposed to deliver the same unique measurement result.

On the other hand, the same clearly need not happen with effective physical entities which embody manifestations of concepts of Class II. Indeed, the smallest possible amount of information, namely, one bit, can be produced or received by a large variety of physical systems. And such physical systems can differ rather arbitrarily with respect to their embodiment of concepts of Class I. For instance, their mass can vary from very tiny to considerable amounts.

Thus such physical systems need *not* be totally involved when they are parts of processes in (2.4), (2.5). And in case of a lack of total involvement, such physical systems exhibit a respective *redundancy*.

Furthermore, the possibility of a lack of total involvement of physical systems P_A and P_C may allow the *simultaneous* presence of the production or reception of *different* pieces of information.

Such a fact is contrary to what happens to effective physical entities which embody manifestations of concepts of Class I. Indeed, in such cases two different amounts of any concept of Class I *cannot* - due to total involvement - simultaneously be embodied by the very same effective physical entity, as long as such a concept is considered in a given fixed frame of reference.

On the other hand, an effective physical entity may simultaneously embody different concepts of both Classes I and II.

Consequently, in processes (2.4), (2.5), we *cannot* always assume

- the total involvement of the physical systems P_A or P_C,
- the existence of a unique piece of information produced and received.

3. The Surprisingly Deep Role of Information in Physics, and a Possible Postulate ...

As mentioned, in [11], it was recently shown that the whole of Quantum Mechanics can be reconstructed from no more than three axioms with clear empirical motivation, the first of which is called

- *Information Capacity* : An elementary system has the information carrying capacity of at most one bit. All systems of the same information carrying capacity are equivalent.

The question arises how is this axiom supposed to be formulated in terms of processes (2.4), (2.5) which try to present the concept of information in a more precise, yet analytically and syntactically minimal manner ?

First let us, therefore, see what limitations on processes (2.4), (2.5) are imposed by the requirement that they should only be able to carry one single bit of information.

Clearly, such limitations may affect either of P_A, P, P_C and R, that is, the manifestly physical parts involved. Also, at least in principle, the inability of processes (2.4), (2.5) to carry more than one single bit of information may be due to certain specific features of E or D.

It follows that the corresponding class

(3.1) $\quad \mathcal{PS}_{1\ bit}$

of physical systems performing processes (2.4), (2.5) may be considerably wide and diverse. Consequently, the above axiom in [11] could in fact amount to a rather strong assumption.

Of course, in this regard, one has first to clarify the meaning of "equivalent" used in the mentioned axiom.

4. Examples

Let us illustrate the above with a simple, yet relevant example.

We consider $A = (P_A, P, Intent, E)$ in (2.4) constituted as follows :

1) P_A is a finite tape τ together with a device, or in general, process δ which successively can record on the tape the sign "0" or "1", until a number $\nu \geq 1$ of such signs are recorded

2) P is the resulting physically existing record on the tape

3) $Intent$ and E are for the moment undetermined.

Regarding $C = (P_C, R, D)$ in (2.4), we assume that :

4) P_C contains the same tape τ, together with a device, or in general, process η which can read the signs "0" or "1" on the tape

5) R is the same with P

6) D is undetermined for the time being.

It follows that for B in (2.4) to have meaning it is necessary for E and D to be specified.

Further, it is obvious that a same given P is compatible with a considerable variety of P_A, $Intent$ and E. Similarly, a given specific R is compatible with a considerable variety of P_C and D.

In particular, P_A can be far from a situation of *total involvement* in (2.4), in order to be able to produce, present or support P. In the same way, P_C need not at all exhibit a *total involvement* in (2.4), in order to be able to receive the information R.

As for B itself, it is to a considerable extent independent of P_A, P, P_C and R.

These features of (2.4) in the above example are, of course, essentially different from the physical phenomena and processes in which effective physical entities that embody manifestations of concepts of Class I are alone involved.

So much, therefore, for ... "information being physical" ...

As for the presence of *time* in the above example of (2.4) one can consider it as happening in one single instant, or alternatively, in a suitable finite number of successive instances.

As a second example, let us consider a vessel which contains a certain amount of water, say $N \geq 1$ molecules. In this case we can take $A = (P_A, P, Intent, E)$ in (2.4) constituted as follows :

7) $P_A = P$ is given by the N molecules of water

8) $Intent = 0$

9) E is the identical encoding, that is, no encoding is involved.
As for $C = (P_C, R, D)$ in (2.4), we can assume that :

10) $P_C = R = P_A = P$

11) $D = E$.

In this case, state of the matter in the vessel has *maximum* entropy when that matter is considered as a collection of *molecules*, and the corresponding information involved can easily be computed.

However, as a third example, one can consider the matter in the vessel in an alternative way, namely, as a collection of *atoms*. In this case, instead of 7) above, we shall have

7*) $P_A = P$ is given by the $3N$ atoms of Hydrogen and Oxygen, respectively

with the corresponding modification in 10) above.

And now obviously, the state of the matter in the vessel does *no* longer have maximum entropy, while again, the corresponding information involved can easily be computed.

We can note that the "physics" of the situation in the last two examples above is the same, namely, N molecules of water. What is different is the manner in which P is encoded by E and R is decoded by D.

Thus again, so much for ... "information being physical" ...

Two further related examples can be considered. Namely, one can assume E so that the vessel is seen as containing atoms, while D sees the vessel as containing molecules. Alternatively, one can have the opposite situation, when E sees molecules, while R sees atoms.

In the last four examples we do happen to have the it total involvement of the respective P_A and P_C. Therefore, in this regard, there is no place left for *redundancy*. On the other hand, the same four examples illustrate the possibility of *simultaneous presence* of different kind of information.

5. A Remark

The phenomenon of *total involvement*, or for that matter, lack of it, and the phenomenon of *simultaneous presence*, both closely related to information, as they were mentioned above, can lead to a rather *loose* relationship between information and it physical support.

In this regard, it may be instructive to recall Bekenstein's argument in the estimation of the entropy of a black hole, [3], an argument which is basic for the so called "black hole war", [3]. Namely, one is assumed to throw a vessel full of a given amount of matter constituted, say, of the molecules of a specific chemical compound, and do so beyond the horizon of a black hole, following which one assumes that the whole amount of entropy in that amount of matter will simply disappear completely from the universe observable outside of that horizon.
What one can further assume here, based on a widespread enough agreement in General Relativity, is that the respective amount of matter will indeed disappear from the observable universe.

From here, however, to jump to the conclusion that the same complete disappearance will happen with the entropy of that amount of matter means to disregard the fact that, as seen in the last four examples above, we cannot automatically assume a unique meaning for the entropy of the amount of matter under consideration. Furthermore, we cannot either assume the total involvement of that amount of matter in all forms of entropy which it may support, [31].

Obviously, the same goes for the conclusion that the information corresponding to that entropy will also disappear completely.

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