AN ELEMENTARY PROOF OF THE NON-EXISTENCE OF ANY NON-TRIVIAL SOLUTIONS TO THE ERDŐS-MOSER DIOPHANTINE EQUATION $1^N + 2^N + \cdots + (M-1)^N + M^N = (M+1)^N$

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Abstract

Erdős had conjectured that the equation of the title had no solutions in natural numbers except the trivial $1^1 + 2^1 = 3^1$. Moser (1953) had shown that there are no solutions for $M + 1 < 10^{10^6}$. Butske *et al* (1993) had further shown that there are no solutions for $M + 1 < 9.3\text{x}10^6$. In this paper I show that a solution to this equation cannot exist for any value of $M > 2$ hence proving Erdős' conjecture. This is achieved using elementary number theoretic methods employing congruences and well-known identities.

1 Introduction

It is first assumed that a solution to the Diophantine equation in question exists. Therefore, being a natural number it should be either odd or even. Considering residues of derivative expressions modulo 4, 8, and 16 I show that this "solution" can neither be odd nor even. Hence, I show that this equation has no solutions.

2 Notation

All notations used are as in standard mathematical texts.Specifically

$\mathbb{N}$ is the set of natural numbers.

$\mathbb{Z}$ is the set of integers.

If a non-zero integer a divides b i.e if there exists an integer c such that $ac = b$ then we write $a|b$.

Again, if a is congruent to b modulo m then we write $a \equiv b$ modulo m.

A solution to the Erdős-Moser equation such that $M = k$ and $N = n$ is written as (k, n).

3 Lemmata

Lemma 1

Let a, b, c, $m \neq 0$, $n \geqq 0$ be integers. Then,

1. *$a \equiv b$ modulo m, $b \equiv a$ modulo m and $a - b \equiv 0$ modulo m are equivalent statements.*
2. *If $a \equiv b$ modulo m and $b \equiv c$ modulo m then $a \equiv c$ modulo m.*
3. *If $a \equiv b$ modulo m then $(a + c) \equiv (b + c)$ modulo m.*
4. *If $a \equiv b$ modulo m then $ac \equiv bc$ modulo m.*
5. *If $a \equiv b$ modulo m and $m \equiv 0$ modulo c where $c \neq 0$ then $a \equiv b$ modulo c.*

Proof :

1. Suppose $a \equiv b$ modulo m . Then, by definition, $m|(a-b)$ i.e. $m| - (b-a)$ which implies $m|(b-a)$. Therefore,by definition, $b \equiv a$ modulo m. Again, $m|(a - b)$ implies $m| \{(a - b) - 0\}$which implies $a - b \equiv 0$ modulo m.
2. Since $a \equiv b$ modulo m therefore $m|(a - b)$. Thus, $a - b = mx$, for some $x\in \mathbb{Z}$. Similarly, $b - c = my$, for some $y\in \mathbb{Z}$.Therefore $a - c = (a - b) + (b - c) = mx + my = m(x + y)$ i.e. $m|(a - c)$. Hence, $a \equiv c$ modulo m.
3. Since $a \equiv b$ modulo m therefore $m|(a - b)$. Thus, $a - b = mx$, for some $x\in \mathbb{Z}$. Observe that, $(a+c)-(b+c) = a-b = mx$. Hence, $(a+c) \equiv (b+c)$ modulo m.
4. Since $a \equiv b$ modulo m therefore $m|(a-b)$ i.e. $a-b = mx$, for some $x\in \mathbb{Z}$. Then $(a - b)c = mxc$ i.e. $ac - bc = mxc$.Therefore $m|(ac - bc)$ which implies $ac \equiv bc$ modulo m.
5. Since $a \equiv b$ modulo m therefore $m|(a-b) \Rightarrow (a-b) = mx$ for some $x \in \mathbb{Z}$. Now, $m \equiv 0$ modulo $c \Rightarrow m = cy$ for some $y \in \mathbb{Z}$. Therefore, $(a-b) = cxy$. Hence, $a \equiv b$ modulo c.

Lemma 2

If a, b and n are integers such that $a \neq b$ then $a^n - b^n \equiv 0$ modulo $(a - b)$ or $a^n \equiv b^n$modulo $(a - b)$.

Proof :

We have the identity $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1})$. Since, a and b are integers therefore all terms of the form a^xb^y where x,$y \in \mathbb{Z}$ are integers. Thus, $a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1}$ is an integer. Therefore $(a - b)|(a^n - b^n)$. Hence, $a^n \equiv b^n$ modulo $(a - b)$ or $a^n - b^n \equiv 0$ modulo $(a - b)$.

Lemma 3

If a, b, $m \neq 0$ and $n \geqq 0$ are integers such that $a \equiv b$ modulo m then $a^n \equiv b^n$ modulo m.

Proof :

We have $a \equiv b$ modulo m
$\Rightarrow a - b \equiv 0$ modulo m.
$\Rightarrow (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1}) \equiv 0$ modulo m (using Lemma 1.4)
$\Rightarrow a^n - b^n \equiv 0$ modulo m
$\Rightarrow a^n \equiv b^n$ modulo m

Lemma 4

If a and n are integers then $(a-1)^n \equiv \pm 1$ modulo a.

Proof:

We have, $a \equiv 0$ modulo a
$\Rightarrow (a-1) \equiv (-1)$ modulo a
$\Rightarrow (a-1)^n \equiv (-1)^n$ modulo a
$\Rightarrow (a-1)^n \equiv \pm 1$ modulo a

Lemma 5

If there exists a set $S \subseteq \mathbb{N}$ such that $S = \{a_i | 1 \leqq i \leqq t,\ t \in \mathbb{N}\}$ and $\sum_{i=1}^{t} a_i^{2m} \equiv 0$ modulo 4,$m \in \mathbb{N}$ then the number of odd elements in the set S is a multiple of 4.

Proof :

Let us partition the set S into disjoint subsets $S_1 = \{$set of all even numbers$\in S\}$ and $S_2 = \{$set of all odd numbers$\in S\}$.

Now, any number $e \in S_1$ is of the form $4f$ or $4f + 2$, where $f \in \mathbb{N}$. Thus, $e \equiv 0, 2$ modulo $4 \Rightarrow e^2 \equiv 0$ modulo 4.

Likewise, for any number $g \in S_2$, $g \equiv 1, 3$ modulo $4 \Rightarrow g^2 \equiv 1, 9 \equiv 1$ modulo 4.

Let, the number of elements in S_1 be n_e and the number of S_2 be n_o.Then, $\sum_{i=1}^{t} a_i^{2m} \equiv \sum_{i=1}^{t} (a_i^2)^m \equiv n_e.(0^m) + n_o.(1^m)$ modulo 4

$\Rightarrow \sum_{i=1}^{t} a_i^{2m} \equiv n_o$ modulo 4. Now, $\sum_{i=1}^{t} a_i^{2m} \equiv 0$ modulo 4. Therefore, Lemma 1.2 $\Rightarrow n_o \equiv 0$ modulo 4.

Therefore, the number of elements in S_2 must be a multiple of 4.

Now, the number of elements in S_2 =the number of odd elements in S.

Hence, the number of elements in S must be a multiple of 4.

Lemma 6

If there exists a set $S \subseteq \mathbb{N}$ such that $S = \{a_i | 1 \leqq i \leqq t, t \in \mathbb{N}\}$ and $\sum_{i=1}^{t} a_i^{2m} \equiv 0$ modulo 8, then (number of odd elements in S)+(number of elements in S of the form $8f \pm 2$, $f \in \mathbb{N}$).(4^m)$\equiv 0$ modulo 8.

Proof :

Let us partition the set S into disjoint subsets $S_1 = \{$set of all numbers of the form $8f+2 \in S$, $f \in \mathbb{N}\}$, $S_2 = \{$set of all numbers of the form $8f-2 \in S$, $f \in \mathbb{N}\}$, $S_3 = \{$set of all odd numbers $\in S\}$ and $S_4 = \{$set of all numbers of the form $8f$ or $8f+4 \in S$, $f \in \mathbb{N}\}$.

Now, for any number $e \in S_1$, $e \equiv 2$ modulo $8 \Rightarrow e^2 \equiv 4$ modulo 8. Similarly, for any number $g \in S_2$, $g \equiv -2$ modulo 4.

Likewise, for any $h \in S_3$, $h \equiv \pm 1, \pm 3$ modulo $8 \Rightarrow h^2 \equiv 1, 9 \equiv 1$ modulo 8 while for any number $x \in S_4$, $x \equiv 0, \pm 4$ modulo $8 \Rightarrow x^2 \equiv 0, 16 \equiv 0$ modulo 8.

Let, the number of elements in S_1be n_{e1}, the number of elements in S_2 be n_{e2}, the number of elements in S_3 be n_o and the number of elements in S_4be n_r.

Then, $\sum_{i=1}^{t} a_i^{2m} \equiv \sum_{i=1}^{t} (a_i^2)^m \equiv (n_{e1} + n_{e2}).(4^m) + n_o.(1^m) + n_r.(0^m) \equiv n_o + (n_{e1} + n_{e2}).(4^m)$ modulo 8.

Now, $\sum_{i=1}^{t} a_i^{2m} \equiv 0$ modulo 8. Therefore, Lemma 1.2 $\Rightarrow n_o + (n_{e1} + n_{e2}).(4^m) \equiv 0$ modulo 8.

Now, $n_{e1}$is the number of elements of S of the form $8f-2$ and n_{e2} is the number of elements of S of the form $8f+2$, $f \in \mathbb{N}$.

Therefore, $n_{e1} + n_{e2}$ is the number of elements in S of the form $8f \pm 2$, $f \in \mathbb{N}$ while n_o is the number of odd elements in S.

Hence, the result.

Lemma 7

If there exists a set $S \subseteq \mathbb{N}$ such that $S = \{a_i | 1 \leqq i \leqq t, t \in \mathbb{N}\}$ and $\sum_{i=1}^{t} a_i^{2m} \equiv r$ modulo 16, $r \in \mathbb{N}$, then (number of elements in S of the forms$16f \pm 1$ and $16f \pm 7$, $f \in \mathbb{N}$)+(number of elements in S of the forms $16f \pm 2$ and $16f \pm 6$, $f \in \mathbb{N}$).(4^m)+(number of elements in S of the forms $16f \pm 3$ and $16f \pm 5$, $f \in \mathbb{N}$).(9^m)$\equiv r$ modulo 8.

Proof :

Let us partition the set S into disjoint subsets $S_1 = \{$set of all elements of S of the forms $16f$, $16f \pm 4$ and $16f \pm 8$, $f \in \mathbb{N}\}$, $S_2 = \{$set of all elements of S of the forms $16f \pm 1$ and $16f \pm 7$, $f \in \mathbb{N}\}$, $S_3 = \{$set of all elements of S of the forms $16f \pm 2$ and $16f \pm 6$, $f \in \mathbb{N}\}$ and $S_4 = \{$set of all elements of S of the forms $16f \pm 3$ and $16f \pm 5$, $f \in \mathbb{N}\}$.

Now, for any element e_1 of S_1, $e_1 \equiv 0, \pm 4, \pm 8$ modulo 16 $\Rightarrow e_1^2 \equiv 0, 16, 64 \equiv 0$ modulo 16, for any element $e_2$of S_2, $e_2 \equiv \pm 1, \pm 7$ modulo 16 $\Rightarrow e_2^2 \equiv 1, 49 \equiv 1$ modulo 16, for any element e_3 of S_3, $e_3 \equiv \pm 2, \pm 6$ modulo 16 $\Rightarrow e_3^2 \equiv 4, 36 \equiv 4$ modulo 16 and for any element e_4 of S_4, $e_4 \equiv \pm 3, \pm 5$ modulo 16 $\Rightarrow e_4^2 \equiv 9, 25 \equiv$ 9 modulo 16.

Let, the number of elements in S_1be n_{e1}, the number of elements in S_2 be n_{e2}, the number of elements in S_3 be n_{e3} and the number of elements in S_4be n_{e4}.

Then, $\sum_{i=1}^{t} a_i^{2m} \equiv \sum_{i=1}^{t} (a_i^2)^m \equiv n_{e1}.(0^m) + n_{e2}.(1^m) + n_{e3}.(4^m) + n_{e4}.(9^m) \equiv n_{e2} + n_{e3}.(4^m) + n_{e4}.(9^m)$ modulo 16.

Now, $\sum_{i=1}^{t} a_i^{2m} \equiv r$ modulo 16. Therefore, Lemma 1.2 $\Rightarrow n_{e2} + n_{e3}.(4^m) + n_{e4}.(9^m) \equiv r$ modulo 16.

Now, n_{e2} is the number of elements in S of the forms $16f \pm 1$ and $16f \pm 7$, n_{e3} is the number of elements in S of the forms $16f \pm 2$ and $16f \pm 6$ and n_{e4} is the number of elements in S of the forms $16f \pm 3$ and $16f \pm 5$, $f \in \mathbb{N}$.

Hence, the result.

4 THE PROOF PROPER

.Let us assume that there exists a non-trivial solution to the Erdős-Moser equation. We shall denote this solution by (k, n).Evidently $k \in \mathbb{N}$. $(\exists)$

Obviously, $3^n = (2+1)^n > 2^n + 1^n$ for any $n > 1$, $n \in \mathbb{N}$. Thus, k must be greater than 2.

$$\text{Now,} 1^n + 2^n + 3^n + \cdots + (k-1)^n + k^n = (k+1)^n \tag{1}$$

$\Rightarrow 2^n + 3^n + \cdots + (k-1)^n + k^n = (k+1)^n - 1^n$

$\Rightarrow k|(2^n + 3^n + \cdots + (k-1)^n + k^n)$ (using Lemma 1.4)

$$\text{Therefore, } 2^n + 3^n + \cdots + (k-2)^n + (k-1)^n \equiv 0 \text{ modulo } k \tag{2}$$

Again, $k - x \equiv -x$ modulo k for any $x \in \mathbb{N}$

$$\Rightarrow (k-x)^n \equiv (-x)^n \text{ modulo } k \tag{3}$$

$$\Rightarrow (k-x)^n + x^n \equiv x^n + (-x)^n \text{ modulo } k \tag{4}$$

Assume n is odd

If k is odd then $2^n + 3^n + \cdots + (k-2)^n + (k-1)^n \equiv (k-1)^n + \sum_{x=2}^{\frac{k-1}{2}} x^n + (-x)^n$ modulo k

but $\sum_{x=2}^{\frac{k-1}{2}} x^n + (-x)^n = \sum_{x=2}^{\frac{k-1}{2}} 0 = 0$

$\Rightarrow 2^n + 3^n + \cdots + (k-2)^n + (k-1)^n \equiv (k-1)^n$ modulo k

Therefore, (2)$\Rightarrow (k-1)^n \equiv 0$ modulo k which is absurd! (Using (3))

If k is even then $2^n + 3^n + \cdots + (k-2)^n + (k-1)^n \equiv (k-1)^n + (\frac{k}{2})^n + \sum_{x=2}^{(\frac{k}{2}-1)} x^n + (-x)^n$ modulo k

but $\sum_{x=2}^{(\frac{k}{2}-1)} x^n + (-x)^n = \sum_{x=2}^{(\frac{k}{2}-1)} 0 = 0$

$\Rightarrow 2^n + 3^n + \cdots + (k-2)^n + (k-1)^n \equiv (k-1)^n + (\frac{k}{2})^n$ modulo k

$$\text{Therefore, (2)} \Rightarrow (k-1)^n + (\tfrac{k}{2})^n \equiv 0 \text{ modulo } k \tag{5a}$$

$$\text{(5a) \& (3)} \Rightarrow (\tfrac{k}{2})^n + (-1)^n \equiv 0 \text{ modulo } k \qquad \Rightarrow (\tfrac{k}{2})^n \equiv 1 \text{ modulo } k \tag{5b}$$

Let $k = 2b$ then (5b)$\Rightarrow b^n \equiv 1$ modulo $2b \Rightarrow b^n \equiv 1$ modulo b which is absurd!

Therefore, n must be even.

Let $n = 2n_1$, $n_1 \in \mathbb{N}$

Therefore, $1^{2n_1} + 2^{2n_1} + \cdots + (k-1)^{2n_1} + k^{2n_1} = (k+1)^{2n_1}$

$\Rightarrow 1^{2n_1} + 2^{2n_1} + \cdots + (k-3)^{2n_1} + (k-2)^{2n_1} + k^{2n_1} = (k+1)^{2n_1} - (k-1)^{2n_1}$

$\Rightarrow 4k|(1^{2n_1} + 2^{2n_1} + \cdots + (k-3)^{2n_1} + (k-2)^{2n_1} + k^{2n_1})$

Therefore, $1^{2n_1} + 2^{2n_1} + \cdots + (k-3)^{2n_1} + (k-2)^{2n_1} + k^{2n_1} \equiv 0$ modulo $4k$ (6)

If $n_1 = 1$ then (1) $\Rightarrow 1^2 + 2^2 + \cdots + (k-1)^2 + k^2 = (k+1)^2$

$\Rightarrow \frac{k(k+1)(2k+1}{6}) = (k+1)^2$

$\Rightarrow 2k^2 - 5k - 6 = 0$

$\Rightarrow k = \frac{5+\sqrt{73}}{4} \notin \mathbb{N}$

Therefore, $n_1 > 1$

$\Rightarrow 4^{n_1} = 16$x(some power of 4) $\Rightarrow 4^{n_1} \equiv 0$ modulo 8 (7)

Case I : Assume k is odd.

Now, (6)$\Rightarrow 1^{2n_1} + 2^{2n_1} + \cdots + (k-3)^{2n_1} + (k-2)^{2n_1} + k^{2n_1} \equiv 0$ modulo 4

Therefore, Lemma 5 implies that the number of odd elements in the set $\{1, 2, 3, \ldots, (k-3), (k-2), k\}$ is a multiple of 4.

Now, number of odd elements in the set $\{1, 2, 3, \ldots, (k-3), (k-2), k\} = \frac{k-1}{2} + 1 = \frac{k+1}{2}$

Therefore, $\frac{k+1}{2} = 4m$, $m \in \mathbb{N} \Rightarrow k = 8m - 1$ (8)

Now, (8) $\Rightarrow k + 1 \equiv 0$ modulo 8

Therefore, (1) $\Rightarrow 1^{2n_1} + 2^{2n_1} + \cdots + (k-1)^{2n_1} + k^{2n_1} \equiv 0$ modulo 8

Consider the set $S = \{1,\ 2,\ \ldots,\ (8m-2),\ (8m-1)\}$.

Number of odd elements in $S = 4m$

Number of elements of the form $8f - 2$ in $S = m$

Number of elements of the form $8f + 2$ in $S = m - 1$ where $f \in \mathbb{N}$

Therefore, Lemma 6 $\Rightarrow 4^{n_1}(2m-1) + 4m \equiv 0$ modulo 8

Therefore, (7) $\Rightarrow 4m \equiv 0$ modulo 8 $\Rightarrow m$ is even.

Let $m = 2m_1$. Therefore, $k = 16m_1 - 1$ (9)

Now, (9) $\Rightarrow k + 1 \equiv 0$ modulo 16

Therefore, (1) $\Rightarrow 1^{2n_1} + 2^{2n_1} + \cdots + (k-1)^{2n_1} + k^{2n_1} \equiv 0$ modulo 16

Consider the set $S' = \{1,\ 2,\ \ldots,\ (16m_1-2),\ (16m_1-1)\}$.

Number of elements of the form $16f - 1$ in $S' = m_1$

Number of elements of the form $16f + 1$ in $S' = m_1 - 1$

Number of elements of the form $16f - 7$ in $S' = m_1$

Number of elements of the form $16f + 7$ in $S' = m_1 - 1$

Number of elements of the form $16f - 2$ in $S' = m_1$

Number of elements of the form $16f + 2$ in $S' = m_1 - 1$

Number of elements of the form $16f-6$ in $S' = m_1$

Number of elements of the form $16f+6$ in $S' = m_1 - 1$

Number of elements of the form $16f-3$ in $S' = m_1$

Number of elements of the form $16f+3$ in $S' = m_1 - 1$

Number of elements of the form $16f-5$ in $S' = m_1$

Number of elements of the form $16f+5$ in $S' = m_1 - 1$ where $f \in \mathbb{N}$

Therefore, Lemma 7 $\Rightarrow (4m_1 - 2)(1^{n_1} + 4^{n_1} + 9^{n_1}) \equiv 0$ modulo 16

$$(7) \Rightarrow (4m_1 - 2)(1 + 9^{n_1}) \equiv 0 \text{ modulo } 16 \qquad (10)$$

If n_1 is even then $n_1 = 2n_2$, $n_2 \in \mathbb{N}$

$\Rightarrow 9^{n_1} \equiv 81^{n_2} \equiv 1^{n_2} \equiv 1$ modulo 16

$\Rightarrow 1 + 9^{n_1} \equiv 2$ modulo 16

$\Rightarrow (1 + 9^{n_1})(4m_1 - 2) \equiv 2(4m_1 - 2)$ modulo 16

Therefore, (10) $\Rightarrow 2(4m_1 - 2) \equiv 0$ modulo 16

$\Rightarrow 2(4m_1 - 2) = 16t$, $t \in \mathbb{N}$

$\Rightarrow 2m_1 - 1 = 4t$ which is absurd!

Again, if n_1 is odd then $n_1 = 2n_2 + 1$, $n_2 \in \mathbb{N}$

$\Rightarrow 9^{n_1} \equiv 81^{n_2}.9 \equiv 1^{n_2}.9 \equiv 9$ modulo 16

$\Rightarrow 1 + 9^{n_1} \equiv 10$ modulo 16

$\Rightarrow (1 + 9^{n_1})(4m_1 - 2) \equiv 10(4m_1 - 2)$ modulo 16

Therefore, (10) $\Rightarrow 10(4m_1 - 2) \equiv 0$ modulo 16

$\Rightarrow 10(4m_1 - 2) = 16t$, $t \in \mathbb{N}$

$\Rightarrow 5(2m_1 - 1) = 4t$ which is absurd!

Hence, k cannot be odd.

Case II :

Assume k is even.

Now, (6) $\Rightarrow 1^{2n_1} + 2^{2n_1} + \cdots + (k-3)^{2n_1} + (k-2)^{2n_1} + k^{2n_1} \equiv 0$ modulo 4

Therefore, Lemma 5 implies that the number of odd elements in the set $\{1, 2, 3, \ldots, (k-3), (k-2), k\}$ is a multiple of 4.

Now, number of odd elements in the set $\{1, 2, 3, \ldots, (k-3), (k-2), k\} = \frac{k}{2} - 1$

Therefore, $\frac{k}{2} - 1 = 4m$, $m \in \mathbb{N} \Rightarrow k = 8m + 2$ (11)

Now, (11) $\Rightarrow k \equiv 2$ modulo $8 \Rightarrow 4k \equiv 8 \equiv 0$ modulo 8

Therefore, (6) and Lemma 1.5 $\Rightarrow 1^{2n_1} + 2^{2n_1} + \cdots + (k-3)^{2n_1} + (k-2)^{2n_1} + k^{2n_1} \equiv 0$ modulo 8

Consider the set $S = \{1,\ 2,\ \ldots,\ (8m-2),\ (8m-1),\ 8m,\ (8m+2)\}$.

Number of odd elements in $S = 4m$

Number of elements of the form $8f - 2$ in $S = m$

Number of elements of the form $8f + 2$ in $S = m$ where $f \in \mathbb{N}$

Therefore, Lemma 6 $\Rightarrow 4^{n_1}.2m + 4m \equiv 0$ modulo 8

Therefore, (7) $\Rightarrow 4m \equiv 0$ modulo $8 \Rightarrow m$ is even.

Let $m = 2m_1$. Therefore, $k = 16m_1 + 2$ (12)

$\Rightarrow k \equiv 2$ modulo 16 (13)

$\Rightarrow k + 1 \equiv 3$ modulo 16 $\Rightarrow (k+1)^{2n_1} \equiv 9^{n_1}$modulo 16 (14)

Again, (13) $\Rightarrow k - 1 \equiv 1$ modulo 16

$\Rightarrow (k-1)^{2n_1} \equiv 1^{2n_1} \equiv 1$ modulo 16

Therefore,(1) $\Rightarrow (k+1)^{2n_1} - \{1^{2n_1} + 2^{2n_2} + \cdots + (k-3)^{2n_1} + (k-2)^{2n_1} + k^{2n_1} \equiv 1$ modulo 16

$\Rightarrow 1^{2n_1} + 2^{2n_1} + \cdots + (k-3)^{2n_1} + (k-2)^{2n_1} + k^{2n_1} \equiv (k+1)^{2n_1} - 1$ modulo 16

Therefore,(14) $\Rightarrow 1^{2n_1} + 2^{2n_1} + \cdots + (k-3)^{2n_1} + (k-2)^{2n_1} + k^{2n_1} \equiv 9^{n_1} - 1$ modulo 16

Consider the set $S' = \{1,\ 2,\ \ldots,\ (16m_1 - 2),\ (16m_1 - 1),\ 16m_1,\ (16m_1 + 2)\}$.

Number of elements of the form $16f - 1$ in $S' = m_1$

Number of elements of the form $16f + 1$ in $S' = m_1 - 1$

Number of elements of the form $16f - 7$ in $S' = m_1$

Number of elements of the form $16f + 7$ in $S' = m_1 - 1$

Number of elements of the form $16f - 2$ in $S' = m_1$

Number of elements of the form $16f + 2$ in $S' = m_1$

Number of elements of the form $16f - 6$ in $S' = m_1$

Number of elements of the form $16f + 6$ in $S' = m_1 - 1$

Number of elements of the form $16f - 3$ in $S' = m_1$

Number of elements of the form $16f + 3$ in $S' = m_1 - 1$

Number of elements of the form $16f - 5$ in $S' = m_1$

Number of elements of the form $16f + 5$ in $S' = m_1 - 1$ where $f \in \mathbb{N}$

Therefore, Lemma 7 $\Rightarrow (4m_1 - 2)(1^{n_1} + 9^{n_1}) + 4^{n_1}.(4m_1 - 1) \equiv 9^{n_1} - 1$ modulo 16

Therefore, (7) $\Rightarrow (4m_1 - 2)(1 + 9^{n_1}) \equiv 9^{n_1} - 1$modulo 16 (15)

If n_1 is even then $n_1 = 2n_2$, $n_2 \in \mathbb{N}$

$\Rightarrow 9^{n_1} \equiv 81^{n_2} \equiv 1^{n_2} \equiv 1$ modulo 16 (16)

$\Rightarrow 1 + 9^{n_1} \equiv 2$ modulo 16

$\Rightarrow (1 + 9^{n_1})(4m_1 - 2) \equiv 2(4m_1 - 2)$ modulo 16 (17)

Now, (16) $\Rightarrow 9^{n_1} - 1 \equiv 0$ modulo 16 (18)

Therefore, (15),(17) & (18) $\Rightarrow 2(4m_1 - 2) \equiv 0$ modulo 16

$\Rightarrow 2(4m_1 - 2) = 16t$, $t \in \mathbb{N}$

$\Rightarrow 2m_1 - 1 = 4t$ which is absurd!

Again, if n_1 is odd then $n_1 = 2n_2 + 1$, $n_2 \in \mathbb{N}$

$\Rightarrow 9^{n_1} \equiv 81^{n_2}.9 \equiv 1^{n_2}.9 \equiv 9$ modulo 16 (19)

$\Rightarrow 1 + 9^{n_1} \equiv 10$ modulo 16

$\Rightarrow (1 + 9^{n_1})(4m_1 - 2) \equiv 10(4m_1 - 2)$ modulo 16 (20)

Now, (19) $\Rightarrow 9^{n_1} - 1 \equiv 0$ modulo 16 (21)

Therefore, (15),(20) & (21) $\Rightarrow 10(4m_1 - 2) \equiv 0$ modulo 16

$\Rightarrow 10(4m_1 - 2) = 16t$, $t \in \mathbb{N}$

$\Rightarrow 5(2m_1 - 1) = 4t$ which is absurd!

Hence, k cannot be even.

Since, k is neither even nor odd, therefore $k \notin \mathbb{N}$.This contradicts ($\exists$).

Therefore, there exists no $a \in \mathbb{N}$ with $a > 2$ such that it is a solution to the Erdős-Moser equation.

Hence, there exist no non-trivial solutions to the Erdős-Moser equation.

5 References

1. *The Erdős-Moser Equation* at Wolfram Mathworld : http://mathworld.wolfram.com/Erdos-MoserEquation.html

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