RESOLVING RUSSELL'S PARADOX WITHIN CANTOR'S INTUITIVE SET THEORY

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Abstract

The set of all the subsets of a set is its power set, and the cardinality of the power set is always larger than the set and its subsets. Based on the definition and the inequality in cardinality, a set cannot include its power set as element, and a power set cannot include itself as element. "Russell's set" is a putative set of all the sets that don't include themselves as element. It can be shown, however, that "Russell's set" can never take in *all* such sets. This is because its own power set, which (like any power set) is a set that doesn't include itself (thus qualifies as an element for "Russell's set"), cannot (although should) be taken in due to the cardinality inequality. Thus "Russell's set" can never be formed. Without it, Russell's paradox, which forced the modification of Cantor's intuitive set theory into a more restricted axiomatic theory, can never be formulated. The reported approach to resolve Russell's paradox is fundamentally different from the conventional approaches. It may restore the self-consistency of Cantor's original set theory, make the Axiom of Regularity unnecessary, and expand the coverage of set to assemblies that include themselves as element.

Keyword: Russell's paradox, Cantor's set theory

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1. RELATIONSHIP BETWEEN A SET AND ITS POWER SET

Set theory, originated by Cantor, is of fundamental importance to mathematics (see *e.g*. Potter 2004). To form a (nonempty) set X, a certain number of distinct objects are *identified*, as its elements x, and *assembled* by a proposition (or unary predicate) P(x):

$$X = \{x: P(x)\} \tag{1}$$

The proposition specifies the *property* (to identify "what kind") and the *inclusion* (to assemble "how many") of the elements.

A set, X', that includes only elements of X is a subset of X, denoted here as either $X' \subset X$, if X' includes only part of elements of X, or $X' \subseteq X$, if X' includes all the elements of X. The set of all the subsets is the power set of X, denoted here as $\wp X$.

The size of set X is measured by its cardinality, |X|. From his original, now often called "intuitive", set theory, Cantor proved that in general, the cardinalities of X, X', and $\wp X$ obey the following order (Cantor's theorem):

$$|X'| \leq |X| < |\wp X| \tag{2}$$

Let S be a set of sets. Then its subsets S' and power set $\wp S$ are also sets of sets. Depending on the formation proposition, S may include its S' (*i.e*. $S' \in S$), or even itself (*i.e*. $S \in S$), but never its $\wp S$ (*i.e*. $\wp S \notin S$) because of the cardinality inequality (2). Since the elements of $\wp S$ are the S' of S (including S), and $\wp S \neq S$ or any S' due to (2), $\wp S$ can never include itself (*i.e*. $\wp S \notin \wp S$). In summary, one has the following relationships:

$$\wp S \notin S \tag{3}$$
$$\wp S \notin \wp S \tag{4}$$

2. RUSSELL'S PARADOX

The paradox can be formulated as follows. Let's consider sets, denoted as Q, that don't include themselves, *i.e*. $Q \notin Q$. Let R ("Russell's set") be the set of *all* such sets:

$$R = \{Q: Q \notin Q\} \tag{5}$$

The proposition in (5) specifies "$Q \notin Q$" as the *property* and "all" as the *inclusion*. If one asks "Does $R \in R$?" then one gets two contradictions, resulting in the Russell's paradox:

If $R \in R$, then R is itself an element. Based on the *property* part of the proposition in (5), R shall have the property of $R \notin R$, same as Q. This leads to the 1st contradiction;

If $R \notin R$, then R has the same property as Q. Based on the *inclusion* part of the proposition in (5), R shall be included as an element, or $R \in R$. This leads to the 2nd contradiction.

The apparent inability to resolve Russell's paradox within Cantor's set theory forced various modifications on the theory, developed originally as the logic foundation of mathematics. All the attempts to remove this paradox have so far relied on the formulation restrictions of either set (*e.g*. type theory, class-set theory or axiomatic set theory) or proposition (*e.g*. 2nd order theory and New Foundation theory) (see *e.g*. Potter 2004). In the widely accepted Zermelo-Fraenkel axiomatic set theory, Russell's paradox is avoided by the Axiom of Regularity, which considers any assembly that includes itself as element (including "Russell's set") as "proper class" only but disqualifies it from being a set. The axiomitization allowed the further development of set theory, although it sacrificed the generality of Cantor's intuitive set principles.

Historically, Canton's original set theory is presumed unable to avoid Russell's paradox, because the *property* part of the Russell proposition (5) is allowable by Canton's principles (Unrestricted Comprehension axioms), and the "Russell's set" R is presumed constructible from Cantor's principles as directed by (5). The paradoxical property of R is presumed to reflect a fundamental inconsistency in Cantor's set theory, and presumably it can be resolved only after a specific axiom, the Axiom of Regularity, is applied to disallow any elements of the Russell *property* from forming a set.

Instead of the *property* part, could the Russell paradox arise from the *inclusion* part of (5)? *Is it possible that R cannot be formed within Cantor's theory?* If so, then there would be no Russell's paradox.

3. CAUSATION OF RUSSELL'S PARADOX

The formation of "Russell's set" R in (5) involves proposition formulation, element identification, and set assemblage. Which of these causes Russell's paradox?

Suppose that in the universe there are in total n sets, denoted as Q, that don't contain themselves as element (*i.e*. $Q \notin Q$): Q_1, Q_2, …Q_m, …Q_n. Let's consider a set R' which includes *some*, but not *all* these sets:

$$R' = \{Q_1, Q_2, \ldots Q_m\} \qquad (6)$$

The proposition in (6) specifies "$Q \notin Q$" as the *property* and "some" as the *inclusion*. It differs from the one in (5) only by the *inclusion* part. Does $R' \in R'$?

If $R' \in R'$, then R' is an element of itself. Based on the *property* part of the proposition in (6), R' shall have the property of $R' \notin R'$, same as Q. Obviously this leads to a contradiction (like the 1st contradiction of Russell's paradox);

If $R' \notin R'$, then R' has the same property as Q. Based on the *inclusion* part of the proposition in (6), R does not have to be included as an element in (6). Thus $R' \notin R'$ can stand, and no contradiction (like the 2nd contradiction of Russell's paradox) arises.

R' is a proper subset of "Russell's set" R (*i.e*. $R' \subset R$). In contrast to R, R' is not paradoxical. Comparing the analyses following (5) and (6), one notices that Russell's paradox is trigged when the *inclusion* part of the proposition in (6) is extended in (5) from "some" to "all". Then one may ask: Is such extension to include/assemble *all* Q into a set as R in (5) *allowed* by Cantor's set theory? If not, then "Russell's set" cannot be formed, and consequently Russell's paradox can never be formulated.

4. RESOLVING RUSSELL'S PARADOX WITHIN CANTOR'S SET THEORY

Suppose that "Russell's set" R could be assembled as in (5). Then the assemblage would ensure the existence of another set of sets, $\wp R$, the power set of R.

According to (4), $\wp R \notin \wp R$. Thus $\wp R$ is a set that has the same property as the elements of R. The proposition in (5) would then include $\wp R$ into R, making $\wp R \in R$. However, this is strictly prohibited by (3)!

Thus the assemblage of *all* Q in (5) would ensure the existence of a set that should be, but cannot be, included into the assembly as its element. Therefore the inclusion part, "all", of the proposition in (5) can never be executed. Consequently, *"Russell's set" R can never be formed*.

The fundamental relationships of a set and its power set, (2), (3), and (4), of Cantor's set theory prevents the formation of "Russell's set". Without "Russell's set", there is simply no Russell's paradox. *Thus, even without the Axiom of Regularity, Cantor's set theory can resolve Russell's paradox.*

It is emphasized that the approach to resolve Russell's paradox described above is fundamentally different from the conventional ones, which all act on the *property* part of the proposition (5) by modifying Cantor's general set-forming principles (Unrestricted Comprehension axioms). In contrast, the new approach acts on the *inclusion* part of the proposition (5) while preserving Cantor's principles. The fact that Russell's paradox can be resolved within Cantor's intuitive set theory makes the Axiom of Regularity, which is used in the axiomatic set theory to disallow Russell's paradox, unnecessary. Consequently any assembly that includes itself as element may again qualify being a set, and Cantor's original set theory may regain its self-consistency.

5. SETS OF ALL THE SETS THAT SHARE A COMMON PROPERTY

In Section 4, it is shown, by resolving Russell's paradox, that one can never assemble all the sets that don't include themselves as element into a set ("Russell's set"), due to the relationships of (2), (3), and (4). Other similar cases exist as well:

One is the "set of all power sets", P. The assemblage of P would ensure the existence of another power set, $\wp$P. $\wp$P should be an element of P, *i.e.* $\wp P \in P$, but cannot because of (3), *i.e.* $\wp P \notin P$. Thus P can never be formed.

Another is the "set of all sets" or "universal set". Historically, Cantor disproved its existence, by resolving Cantor's paradox with the help of (2) and (3).

The resolve of Russell's paradox as demonstrated above may loosen the axiomatic restriction on, and expand the horizon of, set theory.

6. INFINITUDE OF THE SETS THAT DO NOT INCLUDE THEMSELVES

Using paradox to prove theorem is widely practiced in mathematics. One can use Russell's paradox to prove the infinitude of the sets that don't include themselves:

Assume that there are only finite numbers of distinct sets, Q, that don't include themselves as elements ($Q \notin Q$): $Q_1, Q_2, \ldots Q_n$. Let's assemble all the Q into a set T:

$$T = \{Q_1, Q_2, \ldots Q_n\} \tag{7}$$

and investigate whether $T \in T$.

If $T \in T$, then T is an element of itself. Based on the *property* part of the proposition in (7), T shall have the property of $T \notin T$, same as Q. Obviously this leads to a contradiction (like the 1st contradiction of Russell's paradox);

If $T \notin T$, then T has the same property as Q. Based on the *inclusion* part of the proposition in (7), T must be one of the Q, making $T \in T$. Obviously this leads to another contradiction (like the 2nd contradiction of Russell's paradox).

Thus, T must be a set that does not include itself, like Q, and does not belong to the $Q_1, Q_2, \ldots Q_n$. Therefore any finite $Q_1, Q_2, \ldots Q_n$ cannot include all the sets that don't include themselves. In other words, the number of the sets that don't include themselves as element is infinite.

This proof is similar to Euclid's proof of the infinitude of primes. The same approach may be applied to prove the infinitude of power sets or even sets in general.

REFERENCE

Potter, M. D. (2004) Set theory and its philosophy, Oxford University Press, Oxford.