Getting path integrals physically and technically right

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Abstract

Feynman's Lagrangian path integral was an outgrowth of Dirac's vague surmise that Lagrangians have a role in quantum mechanics. Lagrangians implicitly incorporate Hamilton's first equation of motion, so their use contravenes the uncertainty principle, but they are relevant to semiclassical approximations and relatedly to the ubiquitous case that the Hamiltonian is quadratic in the canonical momenta, which accounts for the Lagrangian path integral's "success". Feynman also invented the Hamiltonian phase-space path integral, which is fully compatible with the uncertainty principle. We recast this as an *ordinary functional integral* by changing direct integration over subpaths constrained to all have the same two endpoints into an equivalent integration over those subpaths' unconstrained second derivatives. Function expansion with generalized Legendre polynomials of time then enables the functional integral to be unambiguously evaluated through first order in the elapsed time, yielding the Schrödinger equation with a unique quantization of the classical Hamiltonian. Widespread disbelief in that uniqueness stemmed from the mistaken notion that no subpath can have its two endpoints arbitrarily far separated when its nonzero elapsed time is made arbitrarily short. We also obtain the quantum amplitude for any specified configuration or momentum path, which turns out to be an ordinary functional integral over, respectively, all momentum or all configuration paths. The first of these results is directly compared with Feynman's mistaken Lagrangian-action hypothesis for such a configuration path amplitude, with special heed to the case that the Hamiltonian is quadratic in the canonical momenta.

Introduction

The incorporation of the correspondence principle into quantum mechanics has proceeded along two profound and elegant parallel tracks, namely Dirac's canonical commutation rules and Feynman's path integrals. It is, however, unfortunately the case that from their inceptions the *prescribed implementations* of both of these have had some physically unrefined aspects—albeit these conceivable stumbling blocks turn out to be of little or no *practical* consequence in light of the fact that the Hamiltonians which have been of interest are almost invariably quadratic forms in the canonical momenta and as well usually consist of sums of terms which themselves depend either on *only* the canonical coordinates or on *only* the canonical momenta, which makes their unique quantization unmistakably obvious. In this paper we nonetheless show that the *physically* *called-for* refinements of the prescribed implementations of both the canonical commutation rules and the path integrals result in the unique quantization of *all* classical Hamiltonians rather than *only* those which have heretofore been of practical interest. This endows quantum mechanics with a degree of coherence and consistency which is entirely comparable to that of classical mechanics, and also renders fully transparent its precise relationship to the latter.

Whereas the called-for refinement of Dirac's canonical commutation rule prescription is the straightforward strengthening of its classical correspondence to the maximum that is still self-consistent, the physical issue which besets Feynman's prescribed *Lagrangian* path integral is more drastic. Because Lagrangians *implicitly* incorporate Hamilton's first equation of motion, they likewise *implicitly* contravene the uncertainty principle, which makes their utilization in rigorous quantum theory impermissible—albeit they *do* play a role in semiclassical approximations and, relatedly, in the practically ubiquitous special circumstance that the Hamiltonian is a *quadratic form in the canonical momenta*.

In general, however, the *Lagrangian* path integral must be regarded as invalid, and should be replaced by the *Hamiltonian phase-space path integral*, also invented by Feynman, which is fully compatible with the uncertainty principle. We recast this as an *ordinary functional integral* by changing direct integration over subpaths constrained to all have the same two endpoints into an equivalent integration over those subpaths' unconstrained second derivatives. Function expansion with generalized Legendre polynomials of time then enables the functional integral to be unambiguously evaluated through first order in the elapsed time, yielding the Schrödinger equation with a unique quantization of the classical Hamiltonian. Widespread disbelief in that uniqueness stemmed from *misapprehension* of the fact that *arbitrary* endpoint stipulations *can always be fulfilled by an infinite number of subpaths no matter how short the nonzero time interval allotted for such a subpath may be.*

The unique quantization of the classical Hamiltonian produced by the Hamiltonian phase-space path integral turns out to be in complete accord with the unambiguous quantization of that classical Hamiltonian which emerges from a slightly strengthened, but still self-consistent, variant of Dirac's canonical commutation rule prescription that is alluded to above.

This paper also obtains the formal quantum amplitude for a *specified* configuration-space path or a *specified* momentum-space path as an ordinary functional integral over, respectively, *all* momentum-space paths or *all* configuration-space paths. The first of these two results is then instructively *directly* compared and contrasted with Feynman's mistaken *Lagrangian-action hypothesis* for such a specified configuration-space path amplitude, with special attention given to the case that the Hamiltonian is a quadratic form in the canonical momenta.

The Lagrangian path integral

In the preface to *Quantum Mechanics and Path Integrals* by R. P. Feynman and A. R. Hibbs [1], which treats *only* the *Lagrangian* path integral, the reader encounters the revelation that, "Over the succeeding years, ... Dr. Feynman's approach to teaching the subject of quantum mechanics evolved somewhat away from the initial path integral approach. At the present time, it appears that the operator technique is both deeper and more powerful for the solution of more general quantum-mechanical problems." Unfortunately, no recognizable elaboration of this cautionary note regarding the *Lagrangian* path integral is to be found in the book's main text. But in what might be construed as a muffled echo of this theme, we *do* learn in the second paragraph of page 33 of the book that to define the "normalizing factor" $1/A$ which is required to *convert* the Dirac-inspired very short-time Lagrangian-action phase factor [2] into the *actual* very short-time quantum mechanical propagator in configuration representation "seems to be a very difficult problem and we do not know how to do it in general terms" [1]. This makes it clear that the authors, *contrary* to a widely held impression, did *not succeed* in making *Lagrangian* path integration into a systematic alternate approach to quantum mechanics—which one could suppose may have been reason enough for Feynman to have *turned away* from teaching it.

On page 33 Feynman and Hibbs interpret this "normalizing factor" $1/A$ as *also* being the "path measure normalization factor", which, when paired with each of multiple integrations over configuration space (at successive, narrowly spaced points in time), converts the whole lot of those integrations into an actual integration over *all paths* in the limit that the spacing of the successive time points is taken to zero. For the *particular* class of one-degree-of-freedom Lagrangians which have the form, $L(\dot{q}, q, t) = \frac{1}{2}m\dot{q}^2 - V(q, t)$—to which corresponds the class of quantized Hamiltonians that have the form, $\widehat{H}(t) = \widehat{p}^2/(2m) + V(\widehat{q}, t)$—Feynman and Hibbs point out on page 33 that the factor $1/A$ comes out to equal $\sqrt{m/(2\pi i\hbar\delta t)}$, as that particular quantity properly converts the δt-time-interval Lagrangian-action phase factor into the *actual* δt-time-interval *quantum mechanical propagator in configuration representation.* Feynman and Hibbs fail, however, to scrutinize the issue of whether this object can pass muster as *also* being the "path measure normalization factor" which they have, on page 33, *explicitly* claimed it must be. One notes immediately that this particular $1/A$ depends on the particle mass m, whereas the *set* of *all* paths could not possibly depend on anything other than the time interval on which they are defined and the constraints on their endpoints. The "measure normalization factor" for such paths could also feature constants of mathematics and of nature, but that *set* of *all* paths clearly *does not change in the slightest* if a *different* value is selected for the particle's *mass*! The particle mass is a *parameter* of the Lagrangian, which is supposed to be at the heart of the path *integrand*—the *measure* aspect of any integral is always supposed to be *independent* of the choice of *integrand*! Furthermore, "measure normalization factors" are, by their nature, supposed to be *positive* numbers, whereas this particular $1/A$ is complex-valued! It can only be concluded that the "Lagrangian path integral" simply *cannot* make sense as a "path integral" at all! It is a great pity that Feynman failed to recognize these *surface* anomalies of the "Lagrangian path integral" immediately, as digging deeper only unearths ever worse ones.

Feynman does not seem to have reflected at all on the fact that mechanical systems that are described by *configuration* Lagrangians $L(\dot{q}, q, t)$ can in most instances *also* be described by *momentum* Lagrangians $L(\dot{p}, p, t)$. Indeed, if $L(\dot{q}, q, t) = \frac{1}{2}m\dot{q}^2 - V(q, t)$, then it turns out that $L(\dot{p}, p, t) = -\dot{p}F^{-1}(\dot{p}; t) - V(F^{-1}(\dot{p}; t)) - p^2/(2m)$, where $F(q; t) \stackrel{\text{def}}{=} -\partial V(q, t)/\partial q$. Unpleasant though this $L(\dot{p}, p, t)$ appears for *general* $V(q, t)$, it greatly simplifies when $V(q, t)$ is a *quadratic form* in q, e.g., for the harmonic oscillator $V(q, t) = \frac{1}{2}kq^2$, $L(\dot{p}, p, t) = \dot{p}^2/(2k) - p^2/(2m)$. Indeed it will pretty much be for *only* those $V(q, t)$ which *are* quadratic forms in q that the very short-time quantum mechanical propagator in *momentum representation*, which is simply a *Fourier transformation* of the one in configuration representation, will bear much resemblance to the desired very short-time *momentum* Lagrangian-action phase factor that arises from the quite ugly $L(\dot{p}, p, t)$ given above—the good correspondence in the quadratic form cases is an instance of the fact that the Fourier transformation of an exponentiated quadratic form generally comes out to itself be an exponentiated quadratic form times a simple factor (albeit that factor is by *no means* assured to make sense in the role of "path measure normalization factor", as we have seen above). When $V(q, t)$ is *not* a quadratic form in q, it will usually be quite impossible to transparently relate the Fourier transformation of the very short-time quantum propagator in configuration representation to the very short-time Lagrangian-action phase factor which arises from the fraught $L(\dot{p}, p, t)$ given above. The burden of reconciling the two will then have been loaded *entirely* onto the shoulders of the $1/A$ factor, whose role as a *fudge factor* will thus have been starkly exposed (its forlorn cause as a "path measure normalization factor" will certainly *not* have been furthered).

The *inability* of the *Lagrangian* approach to cope in all but very fortuitous circumstances with the *Fourier transformations* that take the quantum mechanics configuration representation to its momentum representation and conversely, suggest a *fundamental incompatability* of Lagrangians with the canonical *commutation rule*, $\widehat{q}\widehat{p} - \widehat{p}\widehat{q} = i\hbar\mathrm{I}$, as that *underlies* the Fourier relation between those representations. It *also*, of course, is the *heart* of the *uncertainty principle*. Now Feynman took pains to try to move well away from classical dynamics by attempting (albeit not so successfully!) to *integrate* quantum amplitudes over *all* paths, so it does not seem likely that conflicts with the above quantum canonical commutation rule could be rooted in that aspect of his approach. We have, however, just seen that, aside from Lagrangians of quadratic form, the relationships between $L(\dot{q}, q, t)$ and $L(\dot{p}, p, t)$ exhibit *no suggestion of compatibility* with that commutation rule. This seems to hint that there may be something *intrinsic to Lagrangians* that is generally incompatible with the quantum momentum-configuration commutation rule. So might $L(\dot{\mathbf{q}}, \mathbf{q}, t)$ *itself* have a property that *clashes* with the *uncertainty principle*? It turns out that one need *not* look very far to locate *that* culprit: Dirac (and later Feynman) *simply failed* to bear in mind the *basic fact* that to *any* configuration path $\mathbf{q}(t)$, $L(\dot{\mathbf{q}}, \mathbf{q}, t)$ *automatically associates a uniquely determined momentum path* $\mathbf{p}(t) = \nabla_{\dot{\mathbf{q}}(t)} L(\dot{\mathbf{q}}(t), \mathbf{q}(t), t)$, a relation that is *patently incompatible* with the uncertainty principle!

Dirac's vague 1933 surmise about the role of the *Lagrangian* in quantum mechanics [2] has clearly done a long-lived disservice to physics, but Feynman and also all those who sought to educate themselves in Feynman's Lagrangian path integral results were as well scientifically obliged to ponder and pursue any apparently dubious peculiarities which emanate from them. H. Bethe blurted out that there are no paths in quantum mechanics upon hearing Feynman's ideas for the first time at a Cornell University seminar. While this initial visceral reaction cannot be defended as stated, it seems clear that discomfort concerning the uncertainty principle was percolating in Bethe's mind. It is a very great pity that Bethe did not *persist* in pondering that discomfort, seeking to pin down and confront its source.

The Hamiltonian actions and the phase-space path integral concept

Feynman not only originated the Lagrangian path integral idea, he was also the first to publish the idea of the *Hamiltonian phase-space path integral*—which he deeply buried in Appendix B of his major 1951 paper [3]. Apparently he attached little importance to it, and it conceivably slipped from his mind by 1965, as there is *no mention* of it in the book by Feynman and Hibbs. Perhaps Feynman had a reflexive aversion to all Hamiltonian approaches because of the fact that the Hamiltonian *density* in field theories is *not* Lorentz-invariant, whereas the Lagrangian density is—that would have been a pity: the full *action density in Hamiltonian form* is *also* a Lorentz invariant; indeed the Lagrangian density is merely a *restricted version* of this. For quantum theory the Hamiltonian is far superior, as it does *not* harbor the uncertainty principle trap that is *implicit* in the Lagrangian. To be sure, *either one* of the *two* classical Hamiltonian equations of motion *does* implicitly contradict the uncertainty principle (indeed, the Lagrangian is a version of the Hamiltonian action integrand that has been *restricted* according to *one* of the classical Hamiltonian equations of motion). But if we firmly drop *both* classical Hamiltonian equations of motion, $\mathbf{q}(t)$ and $\mathbf{p}(t)$ become *independent* argument functions of the Hamiltonian action functional, and thus do *not* challenge the uncertainty principle.

The path integral concept in this context then becomes one of summing quantum amplitudes over *all phase-space paths.* This states what must be done a bit too expansively, however, as we know that in order to obtain a physically *useful* summed amplitude, we must *restrict* the $\mathbf{q}(t)$ paths to ones which all have the *same* value $\mathbf{q}_i$ at the initial time t_i and *also* all have the *same* value $\mathbf{q}_f$ at the final time t_f. An *alternate* useful restriction is, of course, to require the $\mathbf{p}(t)$ paths to all have the *same* value $\mathbf{p}_i$ at the initial time t_i and *also* to all have the *same* value $\mathbf{p}_f$ at the final time t_f. As is well known, when the configuration paths $\mathbf{q}(t)$ are endpoint-constrained as just described, the two classical Hamiltonian equations of motion result from setting to zero the first-order variation with respect to $[\mathbf{q}(t), \mathbf{p}(t)]$ of the Hamiltonian action functional,

$$S_H([\mathbf{q}(t), \mathbf{p}(t)]; t_f, t_i) \stackrel{\text{def}}{=} \int_{t_i}^{t_f} dt\ (\dot{\mathbf{q}}(t) \cdot \mathbf{p}(t) - H(\mathbf{q}(t), \mathbf{p}(t), t))\,, \tag{1a}$$

whereas when it is the *momentum* paths $\mathbf{p}(t)$ that are endpoint-constrained as described above, the *same* two classical Hamiltonian equations of motion result from setting to zero the first-order variation with respect to $[\mathbf{q}(t), \mathbf{p}(t)]$ of the very slightly *different* Hamiltonian action functional,

$$S'_H([\mathbf{q}(t), \mathbf{p}(t)]; t_f, t_i) \stackrel{\text{def}}{=} \int_{t_i}^{t_f} dt\ (-\mathbf{q}(t) \cdot \dot{\mathbf{p}}(t) - H(\mathbf{q}(t), \mathbf{p}(t), t))\,. \tag{1b}$$

We are, to be sure, interested in *summing* the quantum amplitudes for *all* the appropriately endpoint-constrained phase-space paths rather than in finding *which* of those paths is the *classical* one by the variational approach. Nevertheless, in order to honor the *correspondence principle*, we must make it a path summand *requirement* that the *dominant* path as $\hbar \to 0$, i.e., the path of *stationary phase*, *matches* the classical path. For that reason, we must be careful to *also* match the very slightly *different* actions, S_H or S'_H, respectively, to their appropriate *corresponding* configuration or momentum endpoint constraints, respectively, *even* in the summands of our *path sums* over *quantum amplitudes*—which, in standard fashion, are taken to be proportional to the exponential of $(i/\hbar)$ times the *action* of the path in question.

We also note that that the *values* which the two endpoint-constraining vectors $\mathbf{q}_i$ and $\mathbf{q}_f$ (or, alternately, $\mathbf{p}_i$ and $\mathbf{p}_f$) are *permitted* to assume are *completely arbitrary* and *mutually independent.* We shall, in fact, in quantum mechanical practice frequently be *integrating* over the *full range* of either or both of $\mathbf{q}_i$ and $\mathbf{q}_f$ (or, alternately, of either or both of $\mathbf{p}_i$ and $\mathbf{p}_f$), so this utter freedom of choice is, in fact, a *necessity*—in the language of quantum mechanics the *range* of both $\mathbf{q}_i$ and $\mathbf{q}_f$ (or, alternately, of both $\mathbf{p}_i$ and $\mathbf{p}_f$) must, for *each*, describe a *complete set* of quantum states. The statements just made are neither modified nor qualified *in the slightest* when the *positive* quantity $|t_f - t_i|$ is made *increasingly small.* In other words, $|\mathbf{q}_f - \mathbf{q}_i|$ (or, alternately, $|\mathbf{p}_f - \mathbf{p}_i|$) remains *unbounded* no matter *how small* the positive value of $|t_f - t_i|$ may be. There *always exist* an *infinite* number of paths which *adhere* to the endpoint constraints no matter *how large* $|\mathbf{q}_f - \mathbf{q}_i|$ is or *how small* a positive value $|t_f - t_i|$ assumes. Indeed, given *any velocity* $\mathbf{v}(t)$ that is defined for $t \in [t_i, t_f]$ and *which satisfies* $\int_{t_i}^{t_f} dt\, \mathbf{v}(t) = \mathbf{q}_f - \mathbf{q}_i$, the path,

$$\mathbf{q}(t) = \mathbf{q}_i + \int_{t_i}^{t} dt'\, \mathbf{v}(t'),$$

obviously qualifies. *One* such velocity $\mathbf{v}(t)$ is, of course, the *constant* one, $(\mathbf{q}_f - \mathbf{q}_i)/(t_f - t_i)$, and to it may be added an *arbitrary* number of terms of the form, $\mathbf{v}^{(n)}(t_i)((t - t_i)^n/n! - (t_f - t_i)^n/(n+1)!)$, $n = 1, 2, \ldots$. These *utterly elementary* observations have, in fact, *completely eluded the grasp* of an astonishing number of "experts" in the field of path integrals. Time and again it is implicitly or explicitly *insisted* that,

$$\lim_{|t_f - t_i| \to 0} |\mathbf{q}_f - \mathbf{q}_i| = 0,$$

which is then taken to justify the resort to *completely unsound approximations*, in some instances even a *vast class* of these [5, 6]. This last approach can produce variegated results that are not merely wrong, but even mutually incompatible!

The endpoint-constraining *configuration* vectors $\mathbf{q}_i$ and $\mathbf{q}_f$ are, of course, as well part and parcel of the *Lagrangian* path integral, and on their page 38, Feynman and Hibbs make a variation of the blunder just described. Their Equation (2-33) on that page shows a very clear instance of $\mathbf{q}_i$ and $\mathbf{q}_f$ being *independently* integrated, *each* over its *full range.* That *notwithstanding*, just below their *very next* Equation (2-34), they effectively claim that for sufficiently small $|t_f - t_i|$, the error expression $|\mathbf{q}(t) - \frac{1}{2}(\mathbf{q}_f + \mathbf{q}_i)|$ is *first-order* in $|t_f - t_i|$ for all t in the interval $[t_i, t_f]$. Of course $\mathbf{q}(t)$ obeys the usual two fundamental endpoint constraints $\mathbf{q}(t_i) = \mathbf{q}_i$ and $\mathbf{q}(t_f) = \mathbf{q}_f$. These constraints immediately imply that the above error expression is equal to $\frac{1}{2}|\mathbf{q}_f - \mathbf{q}_i|$ *both* at $t = t_i$ *and* at $t = t_f$. But their *independent* integrations over the *full ranges* of $\mathbf{q}_i$ and $\mathbf{q}_f$ in their *adjacent* Equation (2-33) make it extremely obvious that $\frac{1}{2}|\mathbf{q}_f - \mathbf{q}_i|$ *has no upper bound*! Moreover, this conclusion is clearly *utterly independent* of *how small* a positive value $|t_f - t_i|$ may have!

Having no upper bound is *a very long way indeed* from being first-order in $|t_f - t_i|$ as $|t_f - t_i| \to 0$! This massive blunder by the ostensible ultimate experts in the field drives home the lesson that all scientists bear the obligation to ponder and pursue apparently dubious peculiarities *irrespective* of their pedigree. Science has nothing to gain from the perpetuation of unrecognized mistakes *whatever* their source. The Lagrangian path integral is, of course, deficient because that approach violates the uncertainty principle, i.e., it is *physically* wrong. So adding a gross *mathematical* mistake *on top* of that doesn't really much matter. The critical issue with this particular category of mathematical blunder is that it has *also* infiltrated the Hamiltonian phase-space path integral, which has *no* known deficiencies of physical principle, and the *manner* of the blunder's intrusion has *completely obfuscated* the unique, straightforward result which the Hamiltonian path integral in fact yields.

The key consequences of the Hamiltonian phase-space path integral were first *correctly* worked out in a groundbreaking paper by Kerner and Sutcliffe [4]. That paper was quickly taken to task by L. Cohen [5] *because it failed to take into account the full consequences of the "fact" that* $\lim_{|t_f - t_i| \to 0} |\mathbf{q}_f - \mathbf{q}_i| = 0$! Cohen's "fact" is, of course, as we have gone to great pains above to demonstrate, a *baneful fiction*! A consequence of the toxic assumption that $\lim_{|t_f - t_i| \to 0} |\mathbf{q}_f - \mathbf{q}_i| = 0$ is, according to Cohen and his followers Tirapegui et al. [6], that for all sufficiently small positive values of $|t_f - t_i|$, the term $H(\mathbf{q}(t), \mathbf{p}(t), t)$ in the integrand of the Hamiltonian action in Eq. (1a) may, for *all* t in the interval $[t_i, t_f]$, always be replaced by *any constant-in-time* "discretization" entity of the form $h(\mathbf{q}_f, \mathbf{q}_i, \bar{\mathbf{p}}, \bar{t})$, where $\bar{\mathbf{p}}$ can be regarded as a type of average value of $\mathbf{p}(t)$ for t in the interval $[t_i, t_f]$, $\bar{t}$ is some *fixed* element of that interval, and h is *any* smooth function that satisfies $h(\mathbf{q}, \mathbf{q}, \mathbf{p}, t) = H(\mathbf{q}, \mathbf{p}, t)$. Thus, $H(\frac{1}{2}(\mathbf{q}_f + \mathbf{q}_i), \bar{\mathbf{p}}, \bar{t})$—which is effectively the same as the *bad approximation* to $\mathbf{q}(t)$ by $\frac{1}{2}(\mathbf{q}_f + \mathbf{q}_i)$ advocated by Feynman and Hibbs—is one such "discretization". The quasi-*optimized* "discretization" $\frac{1}{2}(H(\mathbf{q}_f, \bar{\mathbf{p}}, \bar{t}) + H(\mathbf{q}_i, \bar{\mathbf{p}}, \bar{t}))$ is nonetheless *also* a *bad approximation*, as can be verified by examining its differences from $H(\mathbf{q}(t), \mathbf{p}(t), t)$ at the two endpoints $t = t_i$ and $t = t_f$ when $\mathbf{q}_f$ is assumed to be *arbitrarily different* from $\mathbf{q}_i$. The remaining members of this *vast* class of "discretizations" are *bad approximations* as well, as similar arguments about how badly they can miss at one or the other or both of those two time endpoints shows. One upshot of the *misguided imposition* of this vast "discretization" class of *unsound approximations* on the Hamiltonian phase-space path integral is to foster the *false impression* that the Hamiltonian path integral does *not* yield a *unique* result—indeed that it even paradoxically simultaneously yields quite a few *mutually incompatible* results! The *correct* treatment of this path integral *does* in fact yield a unique result; it is merely the fact that *different* members of this *vast class* of *unsound* "discretization" *approximations* can *differ substantially from each other* that lies behind the pedestrian phenomenon that two *different* unsound "discretization" approximations can produce two *sufficiently different* wrong results such that they are in fact *mutually incompatible.* Tirapegui et al. [6] actually set to work *categorizing* this vast class of *unsound* "discretization" approximations and their frequently mutually incompatible results—*all* of which are, in fact, nothing more than the *counterproductive fruit* of Cohen's *completely erroneous assertion* that $\lim_{|t_f - t_i| \to 0} |\mathbf{q}_f - \mathbf{q}_i| = 0$!

Expressing the Hamiltonian phase-space path integral in efficacious form

With the burden of Cohen's counterproductive mathematical lapse—which has been permitted to block understanding for far too many decades—lifted, we turn our attention to trying to express the Hamiltonian phase-space path integral in a form that is as understandable and efficacious technically as it is physically. We wish to make the concept of summing quantum amplitudes over all phase-space paths *completely central* ab initio, *not* to *stumble* on it in conseqence of first having written down a great many repeated integrations over configuration (or momentum) space that arise from some *other* approach to quantum mechanics. This strategy automatically orients our thinking toward the concept of *functional integration*. However, a stumbling block to the immediate identification of the phase-space path integral as simply a functional integral arises from the physically key but technically awkward set-of-measure-zero *endpoint constraints* on the configuration (or momentum) paths. Because of this obstacle we *begin* by writing the configuration version of the phase-space path integral in the standard merely *schematic* form, with those problematic endpoint constraints *only* expressed in words,

$$K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{q}(t),\mathbf{p}(t)]} \exp(iS_H([\mathbf{q}(t), \mathbf{p}(t)]; t_f, t_i)/\hbar), \tag{2}$$

where it is *understood* that all the $\mathbf{q}(t)$ configuration paths that enter into the "functional integral" on the right-hand side of Eq. (2) are *restricted* by the endpoint constraints $\mathbf{q}(t_i) = \mathbf{q}_i$ and $\mathbf{q}(t_f) = \mathbf{q}_f$. It now behooves us, of course, to discover mathematical machinery which gives *actual effect* to that understanding! Before getting to grips with this issue, however, we note that the configuration path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ is to be given its usual quantum-mechanical interpretation as the time-evolution operator for the wave function in configuration representation, i.e.,

$$\psi(\mathbf{q}_f, t_f) = \int K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)\psi(\mathbf{q}_i, t_i)d^n\mathbf{q}_i, \tag{3a}$$

which requires, inter alia, that,

$$\lim_{t_f\to t_i} K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \delta^{(n)}(\mathbf{q}_f - \mathbf{q}_i). \tag{3b}$$

Eq. (3a) certainly underlines the critical quantum-mechanical role which the two endpoint constraints $\mathbf{q}(t_i) = \mathbf{q}_i$ and $\mathbf{q}(t_f) = \mathbf{q}_f$ play in the schematic Eq. (2) path integral. But these endpoint constraints would have little practical effect if the $\mathbf{q}(t)$ paths were not *simultaneously* required to be *sufficiently smooth* for all $t \in [t_i, t_f]$. If the $\mathbf{q}(t)$ paths were permitted to have *jump discontinuities*, for example, it is obvious that the two endpoint requirements would insignificantly constrain them. Now in *classical* mechanics one *varies* rather than *sums* over the phase-space $(\mathbf{q}(t), \mathbf{p}(t))$ paths, and since the classical path obeys a differential equation that is second-order in time, it suffices to vary *only* over phase-space paths which are continuously twice differentiable. We shall therefore impose exactly this smoothness requirement on the phase-space paths that we *sum* over, i.e., we *only* sum over phase-space paths which are *smooth enough* to be classical path *candidates*. In this regard, we take note of the fact that given *any* continuous acceleration $\mathbf{a}(t)$ defined for $t \in [t_i, t_f]$, there exists a continuously twice-differentiable $\mathbf{q}(t)$ defined for $t \in [t_i, t_f]$ which satisfies the three conditions $\mathbf{q}(t_i) = \mathbf{q}_i$, $\mathbf{q}(t_f) = \mathbf{q}_f$ and $\ddot{\mathbf{q}}(t) = \mathbf{a}(t)$. Such a $\mathbf{q}(t)$ is explicitly given in particular by the object $\mathbf{q}\,(t; [\mathbf{a}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i)$, a function-valued functional whose definition is,

$$\mathbf{q}\,(t; [\mathbf{a}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i) \stackrel{\text{def}}{=} \mathbf{q}_i + (t - t_i)[(\mathbf{q}_f - \mathbf{q}_i)/(t_f - t_i) + \int_{t_f}^{t} dt'\,(t' - t_i)^{-2} \int_{t_i}^{t'} dt''\,(t'' - t_i)\mathbf{a}(t'')]. \tag{4a}$$

This particular $\mathbf{q}(t)$ clearly satisfies $\mathbf{q}(t_i) = \mathbf{q}_i$ and $\mathbf{q}(t_f) = \mathbf{q}_f$, and its first derivative with respect to t is given by the function-valued functional,

$$\begin{gathered}\dot{\mathbf{q}}\,(t; [\mathbf{a}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i) = \\ (\mathbf{q}_f - \mathbf{q}_i)/(t_f - t_i) + \int_{t_f}^{t} dt'\,(t' - t_i)^{-2} \int_{t_i}^{t'} dt''\,(t'' - t_i)\mathbf{a}(t'') + (t - t_i)^{-1} \int_{t_i}^{t} dt'\,(t' - t_i)\mathbf{a}(t'),\end{gathered} \tag{4b}$$

from which we readily calculate that its second derivative with respect to t comes out to equal $\mathbf{a}(t)$. Conversely, *any* continuously twice-differentiable path $\mathbf{q}(t)$ that is defined for $t \in [t_i, t_f]$ and which satisfies the two endpoint constraints $\mathbf{q}(t_i) = \mathbf{q}_i$ and $\mathbf{q}(t_f) = \mathbf{q}_f$ is *equal* to $\mathbf{q}\,(t; [\ddot{\mathbf{q}}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i)$. This follows from the fact that for any continuously twice-differentiable path $\mathbf{q}(t)$ that is defined for $t \in [t_i, t_f]$ the relation,

$$\mathbf{q}(t) = \mathbf{q}\left(t; [\ddot{\mathbf{q}}(t')], \mathbf{q}(t_f), t_f, \mathbf{q}(t_i), t_i\right), \tag{4c}$$

is an *identity*, as can be straightforwardly, albeit tediously, verified by the use of the definition in Eq. (4a) to expand out its right-hand side, followed by repeated integrations by parts and applications of the fundamental theorem of the calculus.

Since the *identity* given by Eq. (4c) is not widely known, we briefly digress to indicate how it can be derived. For the continuously twice-differentiable path $\mathbf{q}(t)$ that is defined for $t \in [t_i, t_f]$, we form the *error remainder* $\mathbf{R}\left(t; [\mathbf{q}(t')], t_i, t_f\right)$ with respect to its *linear interpolation* from t_i to t_f,

$$\mathbf{R}\left(t; [\mathbf{q}(t')], t_i, t_f\right) \stackrel{\text{def}}{=} \mathbf{q}(t) - \mathbf{q}(t_i) - (t - t_i)(\mathbf{q}(t_f) - \mathbf{q}(t_i))/(t_f - t_i). \tag{5a}$$

Because this error remainder *vanishes* at both $t = t_i$ and $t = t_f$, we make the *Ansatz* that,

$$\mathbf{R}\left(t; [\mathbf{q}(t')], t_i, t_f\right) = (t - t_i)\textstyle\int_{t_f}^{t} dt'\, \boldsymbol{\Upsilon}\left(t'; [\mathbf{q}(t'')], t_i, t_f\right). \tag{5b}$$

Now Eq. (5a) implies that,

$$\ddot{\mathbf{R}}\left(t; [\mathbf{q}(t')], t_i, t_f\right) = \ddot{\mathbf{q}}(t), \tag{5c}$$

which, given the *Ansatz* of Eq. (5b), implies that,

$$(t - t_i)\dot{\boldsymbol{\Upsilon}}\left(t; [\mathbf{q}(t')], t_i, t_f\right) + 2\boldsymbol{\Upsilon}\left(t; [\mathbf{q}(t')], t_i, t_f\right) = \ddot{\mathbf{q}}(t). \tag{5d}$$

This is a first-order inhomogeneous linear differential equation for $\boldsymbol{\Upsilon}\left(t; [\mathbf{q}(t')], t_i, t_f\right)$ whose general solution is,

$$\boldsymbol{\Upsilon}\left(t; [\mathbf{q}(t')], t_i, t_f\right) = (t - t_i)^{-2}\textstyle\int_{t_0}^{t} dt'\, (t' - t_i)\ddot{\mathbf{q}}(t'). \tag{5e}$$

Since $\mathbf{R}\left(t; [\mathbf{q}(t')], t_i, t_f\right)$ vanishes at $t = t_i$, it is seen from the *Ansatz* of Eq. (5b) that the only correct choice for the unknown constant t_0 in Eq. (5e) is t_i. With that, Eq. (4c) is obtained from Eqs. (5e), (5b) and (5a), together with the definition given in Eq. (4a). The relation of the Eq. (4c) identity to the error remainder of *linear interpolation* is quite analogous to the relation of the identity,

$$\mathbf{q}(t) = \mathbf{q}(t_0) + (t - t_0)\dot{\mathbf{q}}(t_0) + \textstyle\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, \ddot{\mathbf{q}}(t''), \tag{6}$$

to the error remainder of *first-order Taylor expansion*. (Note that the Eq. (6) identity follows from straightforward iteration of the fundamental theorem of the calculus.)

With the above theorems regarding the function-valued functional $\mathbf{q}\left(t; [\mathbf{a}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i\right)$ that is defined by Eq. (4a) in hand, we can now efficaciously integrate over all continuously twice-differentiable configuration paths $\mathbf{q}(t)$ that are defined for $t \in [t_i, t_f]$ and which adhere to the two endpoint constraints $\mathbf{q}(t_i) = \mathbf{q}_i$ and $\mathbf{q}(t_f) = \mathbf{q}_f$. It is clear that this is done by replacing all occurrences of $\mathbf{q}(t)$ in the path integrand of Eq. (2) by $\mathbf{q}\left(t; [\mathbf{a}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i\right)$, followed by functionally integrating *without restriction* over *all* continuous accelerations $\mathbf{a}(t)$ that are defined for $t \in [t_i, t_f]$. Therefore the merely schematic phase-space path integral of Eq. (2) is mathematically joined to the understanding given below it concerning the $\mathbf{q}(t)$ configuration-path endpoint constraints by rewriting it as the following *unconstrained functional integral*,

$$K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i) = \textstyle\int \mathcal{D}_{[\mathbf{a}(t), \mathbf{p}(t)]}^{(t \in [t_i, t_f])} \exp\left(iS_H\left(\left[\mathbf{q}\left(t; [\mathbf{a}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i\right), \mathbf{p}(t)\right]; t_f, t_i\right)/\hbar\right), \tag{7}$$

where the functional integration embraces *all* continuous $\mathbf{a}(t)$ and *all* continuously twice-differentiable $\mathbf{p}(t)$ that are defined for $t \in [t_i, t_f]$. To make futher progress with the path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ described by Eq. (7), we now need to address the question of how to *actually carry out* its indicated unconstrained integration over all the above-described functions $(\mathbf{a}(t), \mathbf{p}(t))$ that are defined for $t \in [t_i, t_f]$.

Normalized multiple integration over the orthogonal components of functions

The set of functions $(\mathbf{a}(t), \mathbf{p}(t))$ defined for $t \in [t_i, t_f]$ that are described in connection with Eq. (7) above comprises an *infinite*-dimensional vector space. Integration over any *finite*-dimensional vector space is, of course, routinely carried out as normalized *multiple ordinary integration* over all values of the *components* of the vectors of that space which arise from *any of its complete orthogonal decompositions.* Thus to integrate over the space of N-dimensional vectors $\mathbf{X}$, we simply perform an appropriately normalized multiple integration over any of its complete sets of N mutually orthogonal components,

$$\int d^N\mathbf{X} = M_N \int dX_1 \int dX_2 \ldots \int dX_N.$$

Here $\mathbf{X} = \Sigma_{k=1}^N X_k \mathbf{b}_k$, where the $\mathbf{b}_k$ comprise *any complete set of N mutually orthogonal basis vectors*, i.e., that satisfy $\mathbf{b}_k \cdot \mathbf{b}_{k'} = 0$ if $k \neq k'$. Thus the N multiple *integration variables X_k* are the N *orthogonal expansion coefficients*, i.e. $X_k = \mathbf{b}_k \cdot \mathbf{X}/\mathbf{b}_k \cdot \mathbf{b}_k$.

Now the time interval $[t_i, t_f]$ *also* has *complete sets* of real-valued, discrete *mutually orthogonal basis functions $B_k(t)$*, $k = 0, 1, \ldots, K, \ldots,$ that satisfy,

$$\int_{t_i}^{t_f} B_k(t)B_{k'}(t)dt = 0 \text{ if } k \neq k'.$$

We can expand any of our functions $(\mathbf{a}(t), \mathbf{p}(t))$ in such a complete, real-valued, discrete orthogonal basis set,

$$(\mathbf{a}(t), \mathbf{p}(t)) = \Sigma_{k=0}^{\infty}(\mathbf{a}_k, \mathbf{p}_k)B_k(t),$$

where the $(\mathbf{a}_k, \mathbf{p}_k)$ are that function's *orthogonal expansion coefficients*, i.e.,

$$(\mathbf{a}_k, \mathbf{p}_k) = \int_{t_i}^{t_f} B_k(t)(\mathbf{a}(t), \mathbf{p}(t))dt / \int_{t_i}^{t_f} (B_k(t))^2\, dt.$$

The integration in Eq. (7) over *all* the functions $(\mathbf{a}(t), \mathbf{p}(t))$ is therefore an appropriately normalized multiple integration over all the orthogonal expansion coefficients $(\mathbf{a}_k, \mathbf{p}_k)$,

$$\int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{a}(t),\mathbf{p}(t)]} = \lim_{K\to\infty} M_K \int d^n\mathbf{a}_0\, d^n\mathbf{p}_0 \int d^n\mathbf{a}_1\, d^n\mathbf{p}_1 \ldots \int d^n\mathbf{a}_K\, d^n\mathbf{p}_K, \tag{8}$$

where in the particular case of the Eq. (7) functional integration the *measure normalization factor M_K* is determined by the requirement of Eq. (3b).

A very commonly invoked *slight variation* of the above *complete* discrete orthogonal basis set approach to functional integration involves a *sequence* of *incomplete* discrete *approximation* orthogonal basis sets to the intuitively appealing complete *continuum* orthogonal basis set of delta functions in time, $B_{t_c}(t) \stackrel{\text{def}}{=} \delta(t - t_c)$, where $t_c \in [t_i, t_f]$. Given a partition of the time interval $[t_i, t_f]$ into $K + 1$ disjoint time subintervals, where $K = 0, 1, 2, \ldots$, we can *approximate* $B_{t_c}(t)$ by $B_{t_c}^K(t)$, which, for t in any of the $K+1$ disjoint time subintervals of $[t_i, t_f]$ equals the *inverse* of the *duration of that time subinterval* when t_c is *also* in that subinterval, but equals zero otherwise. Obviously there are *only* $K + 1$ *distinct* such approximating functions $B_{t_c}^K(t)$, so we may define $B_k^K(t) \stackrel{\text{def}}{=} B_{t_c}^K(t)$, where t_c is *any* time element of time subinterval number k, $k = 0, 1, \ldots, K$. It is clear that $B_k^K(t)$ is *orthogonal* to $B_{k'}^K(t)$ for $k \neq k'$. One develops in this way a *sequence* in K of *incomplete* orthogonal basis sets that each have *only* $K+1$ members. When $K \to \infty$, the intuitively appealing *continuum* orthogonal basis set of delta functions $B_{t_c}(t) = \delta(t - t_c)$, which is, of course, *complete*, will (very nonuniformly) be recovered *provided* that care is taken to ensure that the *durations* of *all* of the *individual time subintervals* of partition number K tend toward *zero* in that limit. Notwithstanding that this is the intuitively appealing "standard" method of functional integration, its highly *nonuniform* approach to the continuum delta function orthogonal basis set via *sequentially disjoint* incomplete discrete orthogonal basis sets could conceivably spawn convergence issues in cases of unfavorably disposed functionals.

The momentum path integral

In addition to the configuration path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ of Eq. (7), which is based on the classical action $S_H([\mathbf{q}(t), \mathbf{p}(t)]; t_f, t_i)$ of Eq. (1a) that is classically appropriate to endpoint constraints on the *configuration* paths, there *also* exists a *momentum* path integral which is based on the classical action $S'_H([\mathbf{q}(t), \mathbf{p}(t)]; t_f, t_i)$ of Eq. (1b) that is classically appropriate to endpoint constraints on the *momentum* paths. In close analogy with Eq. (2), the *schematic* form of this *momentum* path integral is given by,

$$K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i) = \int\mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{q}(t),\mathbf{p}(t)]} \exp(iS'_H([\mathbf{q}(t), \mathbf{p}(t)]; t_f, t_i)/\hbar), \tag{9}$$

where the $(\mathbf{q}(t), \mathbf{p}(t))$ *phase-space paths* are continuously twice differentiable, but *here* it is supposed to be *understood* that it is the *momentum paths* $\mathbf{p}(t)$ which are all restricted by two endpoint constraints, namely that $\mathbf{p}(t_i) = \mathbf{p}_i$ and $\mathbf{p}(t_f) = \mathbf{p}_f$. Just as the configuration path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ is the time-evolution operator for the wave function in configuration representation, the momentum path integral $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ is the time-evolution operator for the wave function in momentum representation, so that we have, in analogy to Eq. (3a),

$$\phi(\mathbf{p}_f, t_f) = \int K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)\phi(\mathbf{p}_i, t_i)d^n\mathbf{p}_i, \tag{10a}$$

which requires that,

$$\lim_{t_f\to t_i} K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i) = \delta^{(n)}(\mathbf{p}_f - \mathbf{p}_i). \tag{10b}$$

In line with the discussion which follows Eq. (4a), any continuously twice-differentiable $\mathbf{p}(t)$ defined for $t \in [t_i, t_f]$ that satisfies the two endpoint constraints $\mathbf{p}(t_i) = \mathbf{p}_i$ and $\mathbf{p}(t_f) = \mathbf{p}_f$ has the explicit representation $\mathbf{p}\,(t; [\mathbf{w}(t')], \mathbf{p}_f, t_f, \mathbf{p}_i, t_i)$ whose definition is,

$$\mathbf{p}\,(t; [\mathbf{w}(t')], \mathbf{p}_f, t_f, \mathbf{p}_i, t_i) \stackrel{\text{def}}{=} \mathbf{p}_i + (t - t_i)[(\mathbf{p}_f - \mathbf{p}_i)/(t_f - t_i) + \int_{t_f}^{t} dt'\,(t' - t_i)^{-2} \int_{t_i}^{t'} dt''\,(t'' - t_i)\mathbf{w}(t'')], \tag{11}$$

where the "force-rate" $\mathbf{w}(t)$ is an unconstrained continuous function that satisfies,

$$\mathbf{w}(t) = \ddot{\mathbf{p}}\,(t; [\mathbf{w}(t')], \mathbf{p}_f, t_f, \mathbf{p}_i, t_i)\,.$$

Therefore we now follow the example set by the replacement of the schematic Eq. (2) by the explicit Eq. (7) by replacing the schematic presentation of $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ given in Eq. (9) by the explicit,

$$K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i) = \int\mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{q}(t),\mathbf{w}(t)]} \exp\left(iS'_H\left([\mathbf{q}(t), \mathbf{p}\,(t; [\mathbf{w}(t')], \mathbf{p}_f, t_f, \mathbf{p}_i, t_i)]\,; t_f, t_i\right)/\hbar\right), \tag{12}$$

which is an unconstrained *functional integral* over $(\mathbf{q}(t), \mathbf{w}(t))$ that uses the function-valued functional defined by Eq. (11) *to explicitly incorporate the two endpoint constraints* which are required of all the *permitted* momentum paths $\mathbf{p}(t)$ by the *supplementary words* that are given *below* the schematic Eq. (9). The measure normalization factor for the functional integration of Eq. (12) will, of course, be determined by the requirement of Eq. (10b).

We note that the structure of the momentum path integral $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ which is given by Eq. (12) is highly analogous to that of the configuration path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ as given by Eq. (7). Therefore the steps of any derivation concerning $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ which flows from Eq. (12) will obviously proceed in close parallel with the steps of a *corresponding* derivation concerning $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ which flows from Eq. (7). For that reason we shall in the remainder of this paper be pointing out only steps for derivations concerning $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ which flow from Eq. (7), and simply leave the analogous steps for the corresponding derivations concerning the momentum path integral $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ which flow from Eq. (12) as straightforward exercises for the interested reader.

Path integral evaluation through first order in the elapsed time

From Eq. (3b) it is apparent that we already know the value of the configuration path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ through *zeroth order* in the elapsed time $\delta t_{fi} \stackrel{\text{def}}{=} (t_f - t_i)$. If we can *extend* the evaluation of $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ through *first order* in δt_{fi}, that result, together with Eq. (3a), will yield a *first-order differential equation in time* for the quantum-mechanical wave function $\psi(\mathbf{q}, t)$ in configuration representation. Obtaining the *solution* of that differential equation in time when the wave function has the *initial* value $\psi(\mathbf{q}, t_i)$ at time t_i is *equivalent* to Eq. (3a) itself—i.e., that solution of the differential equation in time *reproduces* the effect of applying the *full* path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ to $\psi(\mathbf{q}, t_i)$, the initial value of the wave function at time t_i. In other words, the evaluation of the path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ through *just first order* in the elapsed time δt_{fi} yields a first-order in time differential equation for the wave function whose solution *duplicates* evaluation of the effect of the *full* path integral on an initial wave-function value. This permits evaluation of the *full* path integral

$K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ to, in principle, be sidestepped, which is a considerable incentive to work it out through *just first order* in the elapsed time δt_{fi}.

To carry out the evaluation of $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ through just first order in δt_{fi}, we will obviously first need to expand out its *integrand* functional, namely the integrand of the functional integral on the right-hand side of Eq. (7), through first order in δt_{fi}. That Eq. (7) integrand functional is, of course, given by,

$$I_H([\mathbf{a}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_f, \mathbf{q}_i, t_i) \stackrel{\text{def}}{=} \exp\left(iS_H\left([\mathbf{q}\left(t; [\mathbf{a}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i\right), \mathbf{p}(t)]\,; t_f, t_i\right)/\hbar\right). \tag{13a}$$

We can express the above Eq. (7) integrand functional in greater detail by applying Eq. (1a) to it,

$$I_H([\mathbf{a}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_f, \mathbf{q}_i, t_i) = e^{i\int_{t_i}^{t_f}\left(\dot{\mathbf{q}}\left(t;[\mathbf{a}(t')],\mathbf{q}_f,t_f,\mathbf{q}_i,t_i\right)\cdot\mathbf{p}(t) - H\left(\mathbf{q}\left(t;[\mathbf{a}(t')],\mathbf{q}_f,t_f,\mathbf{q}_i,t_i\right),\, \mathbf{p}(t),\, t\right)\right)dt/\hbar}. \tag{13b}$$

The expansion of $I_H([\mathbf{a}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_f, \mathbf{q}_i, t_i)$ through first order in δt_{fi} would obviously be facilitated if we could expand out $(\mathbf{a}(t), \mathbf{p}(t))$ for $t \in [t_i, t_f]$ in the familiar but *nonorthogonal* Taylor-expansion monomial basis,

$$T_k(t) = (t - \tfrac{1}{2}(t_f + t_i))^k/k!,\ k = 0, 1, 2 \ldots\ .$$

Use of this Taylor-expansion basis isn't very feasible here, however, because the straightforward approach to functional integration requires that an *orthogonal* basis be utilized. But an *orthogonal-polynomial* basis $\{B_k(t),\ k = 0, 1, 2, \ldots\}$, whose members have the *same leading behavior as those of the Taylor-expansion monomial basis*, and which as well *share the crucial property of being of order* $O\big((\delta t_{fi})^k\big)$ *for* $t \in [t_i, t_f]$, is readily constructed by systematic successive orthogonalization of the monomial $T_k(t)$ in the interval $[t_i, t_f]$. One thus obtains $B_0(t) = T_0(t) = 1$ and,

$$B_k(t) = (t - \tfrac{1}{2}(t_f + t_i))^k/k! + \Sigma_{j=1}^k c_k^{(j)}(t - \tfrac{1}{2}(t_f + t_i))^{k-j}(\tfrac{1}{2}(t_f - t_i))^j \text{ for } k = 1, 2, \ldots,$$

where the k dimensionless constants $c_k^{(1)}, \ldots, c_k^{(k)}$ are determined by the k *orthogonality* requirements that,

$$\textstyle\int_{t_i}^{t_f}(t - \tfrac{1}{2}(t_f + t_i))^{k'}B_k(t)dt = 0 \text{ for } k' = 0, 1, \ldots, k - 1.$$

With *dimensionless* $c_k^{(j)}$, $j = 1, 2, \ldots, k$, it is clear that $B_k(t)$ is of order $O\big((\delta t_{fi})^k\big)$ for $t \in [t_i, t_f]$, as is foreshadowed above, and the above scheme for $B_k(t)$ *does indeed* produce dimensionless $c_k^{(j)}$ because,

$$\textstyle\int_{t_i}^{t_f}(t - \tfrac{1}{2}(t_f + t_i))^N dt = (\tfrac{1}{2}(t_f - t_i))^{N+1}(1 + (-1)^N)/(N + 1).$$

It is convenient to note here that the first three $B_k(t)$ are, explicitly,

$$B_0(t) = 1,\ \ B_1(t) = (t - \tfrac{1}{2}(t_f + t_i)),\ \ B_2(t) = (t - \tfrac{1}{2}(t_f + t_i))^2/2 - (\tfrac{1}{2}(t_f - t_i))^2/6,$$

which again illustrates the key fact that $B_k(t)$ is of order $O\big((\delta t_{fi})^k\big)$ for $t \in [t_i, t_f]$. In view of their properties, we can recognize these orthogonal basis polynomials $B_k(t)$ as *scaled, translated Legendre polynomials which have Taylor-like normalizations.*

The expansion in this orthogonal-polynomial basis of a continuously twice-differentiable momentum path $\mathbf{p}(t)$ that is defined for $t \in [t_i, t_f]$ is,

$$\mathbf{p}(t) = \Sigma_{k=0}^{\infty}\mathbf{p}_k B_k(t),$$

where the momentum path's orthogonal expansion coefficients $\mathbf{p}_k$, $k = 0, 1, 2, \ldots$, with respect to this basis are given by,

$$\mathbf{p}_k = \textstyle\int_{t_i}^{t_f} B_k(t)\mathbf{p}(t)dt/\int_{t_i}^{t_f}(B_k(t))^2 dt.$$

If this expansion is replaced by just its leading $k = 0$ term, the *error* made is obviously $(\mathbf{p}(t) - \mathbf{p}_0)$, which, in detail, is $\Big(\mathbf{p}(t) - \int_{t_i}^{t_f}\mathbf{p}(t')dt'/(t_f - t_i)\Big)$. If we now adopt the following $[t_i, t_f]$-interval *mean-value notation* for functions $g(t')$ of t' in $[t_i, t_f]$,

$$\langle g(t')\rangle_{t'\in[t_i,t_f]} \stackrel{\text{def}}{=} \textstyle\int_{t_i}^{t_f} g(t')dt'/(t_f - t_i),$$

then this error can be rewritten as $\langle(\mathbf{p}(t) - \mathbf{p}(t'))\rangle_{t'\in[t_i,t_f]}$. Since $\mathbf{p}(t)$ is *continuously differentiable* for $t \in [t_i, t_f]$, the fundamental theorem of the calculus tells us that for $t, t' \in [t_i, t_f]$, we may replace $(\mathbf{p}(t) - \mathbf{p}(t'))$

by $\int_{t'}^{t} \dot{\mathbf{p}}(t'')dt''$. With that we are able to conclude that the error made by substituting for a continuously differentiable momentum path $\mathbf{p}(t)$ just its leading $k = 0$ orthogonal-polynomial expansion term $\mathbf{p}_0$ can, for $t \in [t_i, t_f]$, be written,

$$(\mathbf{p}(t) - \mathbf{p}_0) = \left\langle \int_{t'}^{t} \dot{\mathbf{p}}(t'')dt'' \right\rangle_{t' \in [t_i, t_f]},$$

whose right-hand side is clearly of order $O(\delta t_{fi})$ when $t \in [t_i, t_f]$—note the *key role* which the fact that $\mathbf{p}(t)$ is *continuously differentiable* plays in this conclusion.

Now such a momentum path $\mathbf{p}(t)$ is, in fact, continuously *twice* differentiable, which leads us to suspect that an *even stronger* theorem holds for the error which is made by substituting for $\mathbf{p}(t)$ the *sum* of its leading *two* orthogonal-polynomial expansion terms, which is $[\mathbf{p}_0 + \mathbf{p}_1(t - \frac{1}{2}(t_f + t_i))]$. Indeed, the fact that $\mathbf{p}(t)$ is continuously *twice* differentiable implies that this considerably more complicated error can, via *two successive integrations by parts*, followed by repeated regroupings and applications of the fundamental theorem of the calculus, be written for $t \in [t_i, t_f]$ in the form,

$$(\mathbf{p}(t) - [\mathbf{p}_0 + \mathbf{p}_1(t - \tfrac{1}{2}(t_f + t_i))]) = (3/2) \left\langle \int_{t'}^{t} dt'' \left\langle \int_{t^{(3)}}^{t''} dt^{(4)} \ddot{\mathbf{p}}(t^{(4)}) \right\rangle_{t^{(3)} \in [t_i, t_f]} \right\rangle_{t' \in [t_i, t_f]}$$

$$-(1/4) \left\langle \int_{t'}^{t} dt'' \left(\int_{t_f}^{t''} dt^{(3)} \ddot{\mathbf{p}}(t^{(3)}) + \int_{t_i}^{t''} dt^{(3)} \ddot{\mathbf{p}}(t^{(3)}) \right) \right\rangle_{t' \in [t_i, t_f]}$$

$$-2(t - \tfrac{1}{2}(t_i + t_f))(t_f - t_i)^{-2} \left\langle (t' - \tfrac{1}{2}(t_i + t_f))^3 \ddot{\mathbf{p}}(t') \right\rangle_{t' \in [t_i, t_f]},$$

whose right-hand side consists of *three* terms, *each* of which is clearly of order $O\big((\delta t_{fi})^2\big)$ when $t \in [t_i, t_f]$.

Another entity which enters into the integrand functional $I_H([\mathbf{a}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_f, \mathbf{q}_i, t_i)$ on the right-hand side of Eq. (13b) is the *time derivative of the configuration path functional,* $\dot{\mathbf{q}}\,(t; [\mathbf{a}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i)$, which is explicitly given by Eq. (4b). Because the acceleration function $\mathbf{a}(t)$ is *continuous* for $t \in [t_i, t_f]$, Eq. (4b) tells us that the error made by substituting for $\dot{\mathbf{q}}\,(t; [\mathbf{a}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i)$ the *constant velocity* $(\mathbf{q}_f - \mathbf{q}_i)/(t_f - t_i)$ is of order $O(\delta t_{fi})$. It is important, however, to be aware of the fact that this constant velocity *itself* is of order $O\big((\delta t_{fi})^{-1}\big)$! Therefore the error made by *substituting for the composite object*,

$$\int_{t_i}^{t_f} \dot{\mathbf{q}}\,(t; [\mathbf{a}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i) \cdot \mathbf{p}(t)dt,$$

that appears on the right-hand side of Eq. (13b) the *approximation*,

$$\langle (\mathbf{q}_f - \mathbf{q}_i) \cdot [\mathbf{p}_0 + \mathbf{p}_1(t - \tfrac{1}{2}(t_f + t_i))] \rangle_{t \in [t_i, t_f]},$$

which fortuitously evaluates to simply $(\mathbf{q}_f - \mathbf{q}_i) \cdot \mathbf{p}_0$, is clearly of order $O\big((\delta t_{fi})^2\big)$. With that, a significant contribution to the integrand functional $I_H([\mathbf{a}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_f, \mathbf{q}_i, t_i)$, as it appears on the right-hand side of Eq. (13b), has been evaluated and found to have the simple form $(\mathbf{q}_f - \mathbf{q}_i) \cdot \mathbf{p}_0$ through first order in δt_{fi}. This particular contribution *itself* is obviously of order $O\big((\delta t_{fi})^0\big)$.

By way of making further progress toward completing the task of evaluating through first order in δt_{fi} the integrand functional $I_H([\mathbf{a}(t), \mathbf{p}(t)]; \mathbf{q}_f, t_f, \mathbf{q}_i, t_i)$ as it appears on the right-hand side of Eq. (13b), we also note from the fact that the the acceleration $\mathbf{a}(t)$ is continuous for $t \in [t_i, t_f]$, together with the representation of configuration path functional $\mathbf{q}\,(t; [\mathbf{a}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i)$ which is given by Eq. (4a), that the error made by substituting for this entity the *straight-line path* $\mathbf{q}_i + (t - t_i)(\mathbf{q}_f - \mathbf{q}_i)/(t_f - t_i)$ when $t \in [t_i, t_f]$, is of order $O\big((\delta t_{fi})^2\big)$. Of course this straight-line path *itself* is of order $O\big((\delta t_{fi})^0\big)$. Therefore, assuming that all the *gradients* of the *classical Hamiltonian* $H(\mathbf{q}, \mathbf{p}, t)$ are *continuous*, the error made by *substituting for the composite object*,

$$\int_{t_i}^{t_f} H\left(\mathbf{q}\,(t; [\mathbf{a}(t')], \mathbf{q}_f, t_f, \mathbf{q}_i, t_i)\,,\, \mathbf{p}(t),\, t\right) dt,$$

that occurs on the right-hand side of Eq. (13b) the *approximation*,

$$\int_{t_i}^{t_f} H(\mathbf{q}_i + (t - t_i)(\mathbf{q}_f - \mathbf{q}_i)/(t_f - t_i),\, \mathbf{p}_0,\, t_i)dt,$$

which, on changing the integration variable from t to the dimensionless $\lambda \stackrel{\text{def}}{=} (t - t_i)/(t_f - t_i)$, assumes the form,

$$(\delta t_{fi}) \int_0^1 H(\mathbf{q}_i + \lambda(\mathbf{q}_f - \mathbf{q}_i),\, \mathbf{p}_0,\, t_i)d\lambda,$$

is clearly of order $O\big((\delta t_{fi})^2\big)$. With that, the key *remaining* contribution to $I_H([\mathbf{a}(t),\mathbf{p}(t)];\mathbf{q}_f,t_f,\mathbf{q}_i,t_i)$, as it appears on the right-hand side of Eq. (13b), has been evaluated through first order in δt_{fi}. As we see from its displayed form given just above, this particular contribution *itself* is of order $O(\delta t_{fi})$.

With *both key contributions* through first order in δt_{fi} to $I_H([\mathbf{a}(t),\mathbf{p}(t)];\mathbf{q}_f,t_f,\mathbf{q}_i,t_i)$ as it appears on the right-hand side of Eq. (13b) *now in hand*, we are finally in a position to write down the result for $I_H([\mathbf{a}(t),\mathbf{p}(t)];\mathbf{q}_f,t_f,\mathbf{q}_i,t_i)$ *itself* through first order in δt_{fi},

$$I_H([\mathbf{a}(t),\mathbf{p}(t)];\mathbf{q}_f,t_i+\delta t_{fi},\mathbf{q}_i,t_i)=$$

$$e^{i(\mathbf{q}_f-\mathbf{q}_i)\cdot\mathbf{p}_0/\hbar-i(\delta t_{fi}/\hbar)\int_0^1 H(\mathbf{q}_i+\lambda(\mathbf{q}_f-\mathbf{q}_i),\,\mathbf{p}_0,\,t_i))d\lambda+O\left((\delta t_{fi})^2\right)}= \tag{14}$$

$$e^{i(\mathbf{q}_f-\mathbf{q}_i)\cdot\mathbf{p}_0/\hbar}\left(1-i(\delta t_{fi}/\hbar)\int_0^1 H(\mathbf{q}_i+\lambda(\mathbf{q}_f-\mathbf{q}_i),\,\mathbf{p}_0,\,t_i)d\lambda\right)+O\big((\delta t_{fi})^2\big)\,.$$

We see from Eqs. (14) and (13a) that the integrand functional of the functional integral on the right-hand side of Eq. (7) has, through first order in δt_{fi}, *no* dependence on the acceleration function $\mathbf{a}(t)$ and *only* depends on the momentum path $\mathbf{p}(t)$ through its $k=0$ orthogonal-polynomial expansion coefficient $\mathbf{p}_0$. This *independence through first order in δt_{fi}* of the *integrand* of the functional integral on the right-hand side of Eq. (7) of *all but one* of the orthogonal-polynomial expansion coefficients $\mathbf{a}_0,\mathbf{p}_0,\mathbf{a}_1,\mathbf{p}_1,\ldots,\mathbf{a}_K,\mathbf{p}_K,\ldots$ of the function $(\mathbf{a}(t),\mathbf{p}(t))$ implies, via the multiple-integration prescription given by Eq. (8) for that functional integration, that we have a *formally divergent*, undefined result for $K_H(\mathbf{q}_f,t_i+\delta t_{fi};\mathbf{q}_i,t_i)$ through first order in δt_{fi}! The *cause* of this predicament is that the Eq. (14) expansion through first order in δt_{fi} of the integrand functional of the functional integral on the right-hand side of Eq. (7), while true for *any given* function $(\mathbf{a}(t),\mathbf{p}(t))$, *loses its validity* when considered over the *entire set* of such functions. For example, over the *entire set* of continuous acceleration functions $\mathbf{a}(t)$, the *error* made in substituting for the time derivative of the configuration path functional $\dot{\mathbf{q}}\,(t;[\mathbf{a}(t')],\mathbf{q}_f,t_f,\mathbf{q}_i,t_i)$ of Eq. (4b) the constant velocity $(\mathbf{q}_f-\mathbf{q}_i)/(t_f-t_i)$ can obviously be made *arbitrarily large*, notwithstanding that it is clearly of order $O(\delta t_{fi})$ for *any particular* continuous acceleration function $\mathbf{a}(t)$.

Therefore the *only way* to deal with this quandary concerning the functional integration of integrand functionals that are given *through just a certain order of δt_{fi}* is to *cut off* the integration which *normally* runs over the *entire set* of applicable functions. From a physics point of view the imposition of such a function integration cutoff does not seem very concerning because path integrals are typically dominated by a *quite narrow range of functions* whose actions differ by no more than several times $\hbar$ from the action of the *classical* solution. Nevertheless, the *possibility* that the value of the thus cut-off functional integral *might* irrevocably depend on the *details* of the cutoff which is imposed is a dismaying one. Fortunately, such cutoffs seem to typically affect only the value of an *overall factor* which multiplies the *rest* of the functional integration result, and the path-integral requirement of Eq. (3b) ensures that the measure normalization factor M_K present in the Eq. (8) multiple-integration prescription for the Eq. (7) functional integration *cancels out* such factors.

Returning now to the Eq. (14) result for the *integrand functional* through first order in δt_{fi} of the functional integral which appears on the right-hand side of Eq. (7), we note that this integrand's *lack of dependence* on the orthogonal-polynomial expansion coefficients $\mathbf{a}_0,\mathbf{a}_1,\mathbf{p}_1,\mathbf{a}_2,\mathbf{p}_2,\ldots,\mathbf{a}_K,\mathbf{p}_K,\ldots$ *leaves us no choice but to cut off* the Eq. (8) multiple integrations *over these particular coefficients*. From this *unavoidably cutoff-modified* Eq. (8) prescription for the functional integral of Eq. (7) that applies to its integrand through just first order in δt_{fi} as given by Eq. (14), we obtain for $K_H(\mathbf{q}_f,t_i+\delta t_{fi};\mathbf{q}_i,t_i)$ through first order in δt_{fi},

$$K_H(\mathbf{q}_f,t_i+\delta t_{fi};\mathbf{q}_i,t_i)=$$

$$\lim_{K\to\infty} M_K\,F_K(A_0,A_1,P_1,A_2,P_2,\ldots,A_K,P_K)\times \tag{15}$$

$$\left(\int d^n\mathbf{p}_0\,e^{i(\mathbf{q}_f-\mathbf{q}_i)\cdot\mathbf{p}_0/\hbar}\left(1-i(\delta t_{fi}/\hbar)\int_0^1 H(\mathbf{q}_i+\lambda(\mathbf{q}_f-\mathbf{q}_i),\,\mathbf{p}_0,\,t_i)d\lambda\right)\right)+O\big((\delta t_{fi})^2\big)\,,$$

where the overall multiplicative factor $F_K(A_0,A_1,P_1,A_2,P_2,\ldots,A_K,P_K)$ is the product of all the *unavoidably cut-off integrals* over Eq. (8) orthogonal-polynomial expansion coefficients, namely,

$$F_K(A_0,A_1,P_1,A_2,P_2,\ldots,A_K,P_K)\stackrel{\text{def}}{=}$$

$$\int_{\{|\mathbf{a}_0|\le A_0\}}d^n\mathbf{a}_0\int_{\{|\mathbf{a}_1|\le A_1,|\mathbf{p}_1|\le P_1\}}d^n\mathbf{a}_1d^n\mathbf{p}_1\int_{\{|\mathbf{a}_2|\le A_2,|\mathbf{p}_2|\le P_2\}}d^n\mathbf{a}_2d^n\mathbf{p}_2\ldots\int_{\{|\mathbf{a}_K|\le A_K,|\mathbf{p}_K|\le P_K\}}d^n\mathbf{a}_Kd^n\mathbf{p}_K\,.$$

Now since,

$$\int d^n\mathbf{p}_0\, e^{i(\mathbf{q}_f-\mathbf{q}_i)\cdot\mathbf{p}_0/\hbar} = (2\pi\hbar)^n\delta^{(n)}(\mathbf{q}_f-\mathbf{q}_i),$$

we must choose the Eq. (8) measure normalization factor M_K on the right-hand side of Eq. (15) to be equal to $((2\pi\hbar)^n F_K(A_0, A_1, P_1, A_2, P_2, \ldots, A_K, P_K))^{-1}$ in order to satisfy the $\delta t_{fi} \to 0$ path-integral limit requirement of Eq. (3b). With that, Eq. (15) becomes,

$$K_H(\mathbf{q}_f, t_i+\delta t_{fi}; \mathbf{q}_i, t_i) = \delta^{(n)}(\mathbf{q}_f-\mathbf{q}_i) - i(\delta t_{fi}/\hbar)Q_H(t_i; \mathbf{q}_f; \mathbf{q}_i) + O\big((\delta t_{fi})^2\big)\,, \tag{16a}$$

where,

$$Q_H(t_i; \mathbf{q}_f; \mathbf{q}_i) \stackrel{\text{def}}{=} \int_0^1 d\lambda\,(2\pi\hbar)^{-n}\int d^n\mathbf{p}\, H(\mathbf{q}_i+\lambda(\mathbf{q}_f-\mathbf{q}_i), \mathbf{p}, t_i)e^{i(\mathbf{q}_f-\mathbf{q}_i)\cdot\mathbf{p}/\hbar}. \tag{16b}$$

It is easily demonstrated that $Q_H(t_i; \mathbf{q}_f; \mathbf{q}_i)$ is *Hermitian*, i.e. that,

$$Q_H(t_i; \mathbf{q}_f; \mathbf{q}_i) = (Q_H(t_i; \mathbf{q}_i; \mathbf{q}_f))^*. \tag{16c}$$

When Eq. (16a) is combined with Eq. (3a), we at long last obtain the first-order differential equation in time that we have been seeking for the wave function in configuration representation,

$$i\hbar\partial\psi(\mathbf{q}_f, t_i)/\partial t_i = \int Q_H(t_i; \mathbf{q}_f; \mathbf{q}_i)\psi(\mathbf{q}_i, t_i)d^n\mathbf{q}_i, \tag{17}$$

The quantized Hamiltonian operator and the Schrödinger equation

At this point we wish to mention the results for the momentum path integral $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ that parallel those which we have just demonstrated for the configuration path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$. Through first order in $\delta t_{fi} = (t_f - t_i)$, $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ satisfies relations which are highly analogous to those given in Eqs. (16) for $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$, namely,

$$K'_H(\mathbf{p}_f, t_i+\delta t_{fi}; \mathbf{p}_i, t_i) = \delta^{(n)}(\mathbf{p}_f-\mathbf{p}_i) - i(\delta t_{fi}/\hbar)Q'_H(t_i; \mathbf{p}_f; \mathbf{p}_i) + O\big((\delta t_{fi})^2\big)\,, \tag{18a}$$

where,

$$Q'_H(t_i; \mathbf{p}_f; \mathbf{p}_i) \stackrel{\text{def}}{=} \int_0^1 d\lambda\,(2\pi\hbar)^{-n}\int d^n\mathbf{q}\, H(\mathbf{q}, \mathbf{p}_i+\lambda(\mathbf{p}_f-\mathbf{p}_i), t_i)e^{-i(\mathbf{p}_f-\mathbf{p}_i)\cdot\mathbf{q}/\hbar}. \tag{18b}$$

It is easily demonstrated that $Q'_H(t_i; \mathbf{p}_f; \mathbf{p}_i)$ is *Hermitian*, i.e. that,

$$Q'_H(t_i; \mathbf{p}_f; \mathbf{p}_i) = (Q'_H(t_i; \mathbf{p}_i; \mathbf{p}_f))^*. \tag{18c}$$

When Eq. (18a) is combined with Eq. (10a), we obtain in analogy with Eq. (17) a first-order differential equation in time for the wave function in momentum representation,

$$i\hbar\partial\phi(\mathbf{p}_f, t_i)/\partial t_i = \int Q'_H(t_i; \mathbf{p}_f; \mathbf{p}_i)\phi(\mathbf{p}_i, t_i)d^n\mathbf{p}_i, \tag{19}$$

A *crucial* relationship which holds between $Q'_H(t_i; \mathbf{p}_f; \mathbf{p}_i)$ and $Q_H(t_i; \mathbf{q}_f; \mathbf{q}_i)$ is that,

$$\int d^n\mathbf{p}_f d^n\mathbf{p}_i\,\langle\mathbf{q}_f|\mathbf{p}_f\rangle Q'_H(t_i; \mathbf{p}_f; \mathbf{p}_i)\langle\mathbf{p}_i|\mathbf{q}_i\rangle = Q_H(t_i; \mathbf{q}_f; \mathbf{q}_i), \tag{20}$$

where we have used the standard Dirac quantum mechanics notation for the overlap amplitude between a configuration state and a momentum state, i.e., $\langle\mathbf{q}|\mathbf{p}\rangle = e^{i\mathbf{p}\cdot\mathbf{q}/\hbar}/(2\pi\hbar)^{n/2}$ and $\langle\mathbf{p}|\mathbf{q}\rangle = (\langle\mathbf{q}|\mathbf{p}\rangle)^*$. To carry out the verification of Eq. (20), it is useful to make the $d\lambda$-integration that arises from $Q'_H(t_i; \mathbf{p}_f; \mathbf{p}_i)$ via Eq. (18b) the *outermost* integration, and then change integration variables from the $(\mathbf{p}_f, \mathbf{p}_i)$ pair to the $\mathbf{p} = \mathbf{p}_i + \lambda(\mathbf{p}_f - \mathbf{p}_i)$ and $\mathbf{p}_- = (\mathbf{p}_f - \mathbf{p}_i)$ pair. This variable transformation has unit Jacobian, and the $d^n\mathbf{p}_-$-integration will give rise to a delta function which, in turn, permits the $d^n\mathbf{q}$-integration that arises from $Q'_H(t_i; \mathbf{p}_f; \mathbf{p}_i)$ via Eq. (18b) to be carried out. The upshot is to leave only the $d^n\mathbf{p}$-integration and the $d\lambda$-integration, both of which indeed occur in $Q_H(t_i; \mathbf{q}_f; \mathbf{q}_i)$, which is itself, of course, the result being sought. With this outline of the procedure, we leave the remaining straightforward details of verifying Eq. (20) to the reader.

Eq. (20) shows that $Q_H(t_i;\mathbf{q}_f;\mathbf{q}_i)$ and $Q'_H(t_i;\mathbf{p}_f;\mathbf{p}_i)$ are, respectively, the configuration and momentum representations of the *very same quantum mechanical operator*, which we shall now denote as $\widehat{H}(t_i)$. Thus, in the standard Dirac quantum mechanics notation,

$$Q_H(t_i;\mathbf{q}_f;\mathbf{q}_i) = \langle\mathbf{q}_f|\widehat{H}(t_i)|\mathbf{q}_i\rangle, \tag{21a}$$

and,

$$Q'_H(t_i;\mathbf{p}_f;\mathbf{p}_i) = \langle\mathbf{p}_f|\widehat{H}(t_i)|\mathbf{p}_i\rangle. \tag{21b}$$

Eqs. (16c) and (18c) of course show that this quantum mechanical operator $\widehat{H}(t_i)$ is a *Hermitian* one. Furthermore, if we transcribe the first-order in time differential equation that appears in Eq. (17) into standard Dirac quantum mechanics notation by rewriting $\psi(\mathbf{q},t)$ as $\langle\mathbf{q}|\psi(t)\rangle$ and switching to the notation on the *right-hand side* of Eq. (21a), we can readily demonstrate that Eq. (17) is equivalent to,

$$i\hbar\partial|\psi(t)\rangle/\partial t = \widehat{H}(t)|\psi(t)\rangle, \tag{22}$$

which is the familiar Schrödinger equation for the time evolution of the quantum state vector $|\psi(t)\rangle$ under the influence of the Hermitian Hamiltonian operator $\widehat{H}(t)$. Here, however, this Hamiltonian operator $\widehat{H}(t)$ is *uniquely determined*, via Eqs. (21a) and (16b) (or alternatively via Eqs. (21b) and (18b)) by the *classical Hamiltonian function* $H(\mathbf{q},\mathbf{p},t)$ for the physical system. Inter alia, this *crystallizes* the correspondence principle in a *very strong form* indeed.

Of course one can tread a highly analogous route with the first-order in time differential equation that appears in Eq. (19) by rewriting $\phi(\mathbf{p},t)$ as $\langle\mathbf{p}|\phi(t)\rangle$ and switching to the notation on the right-hand side of Eq. (21b), following which it is readily demonstrated that Eq. (19) is equivalent to,

$$i\hbar\partial|\phi(t)\rangle/\partial t = \widehat{H}(t)|\phi(t)\rangle, \tag{23}$$

which is exactly the same quantum state vector Schrödinger equation as that of Eq. (22). Again, its controlling Hamiltonian operator $\widehat{H}(t)$ is *uniquely determined*, via Eqs. (21b) and (18b) (or alternatively via Eqs. (21a) and (16b)) by the *classical Hamiltonian function* $H(\mathbf{q},\mathbf{p},t)$ for the physical system.

The *unique* classically underpinned Hamiltonian operator $\widehat{H}(t_i)$ of Eqs. (21), (16b) and (18b) was first obtained from the Hamiltonian phase-space path integral by Kerner and Sutcliffe [4], but it had been mooted by Born and Jordan [8] in their pre-Dirac version of quantum mechanics. Born and Jordan's theory featured commutation rules which were more elaborate than those of Dirac, but those rules were nevertheless still *not* sufficiently strong to *uniquely* pin down the particular $\widehat{H}(t_i)$ of Eqs. (21). Therefore Born and Jordan's discovery of $\widehat{H}(t_i)$ must be regarded as fascinatingly fortuitous rather than wholly systematic. Dirac, with his Poisson bracket insight into quantum commutators, had an excellent chance to *uniquely* pin down *exactly* this $\widehat{H}(t_i)$, but ironically he ended up choosing commutation rules that were even much *weaker* [7] than those of his predecessors Born and Jordan! Kerner [9] was apparently the first to work out the slightly strengthened self-consistent canonical commutation rule that Dirac *ought*, by rights, to have lit upon, but very unfortunately Kerner failed to publish that work. We shall briefly develop the highly satisfactory canonical commutation rule that Dirac *missed* in the last section of this paper.

Quantum amplitudes for individual configuration or momentum paths

As an extension of the interpretation that we have given to the configuration path integrals of Eq. (2) and Eq. (7), it seems reasonable to interpret the corresponding unconstrained functional integral over *only momentum paths* $\mathbf{p}(t)$, namely,

$$A_H([\mathbf{q}(t)];t_f,t_i) \stackrel{\text{def}}{=} \textstyle\int\mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{p}(t)]}\exp(iS_H([\mathbf{q}(t),\mathbf{p}(t)];t_f,t_i)/\hbar), \tag{24a}$$

as the quantum amplitude that the dynamical system traverses a specified *configuration path* $\mathbf{q}(t)$ for $t\in[t_i,t_f]$. If we now also ponder the interpretation that we have given to the momentum path integrals of Eq. (9) and Eq. (12), it as well seems reasonable that the quantum amplitude that the dynamical system traverses a specified *momentum path* $\mathbf{p}(t)$ for $t \in [t_i, t_f]$ ought to similarly be given by the corresponding unconstrained functional integral over *only configuration paths* $\mathbf{q}(t)$,

$$A'_H([\mathbf{p}(t)]; t_f, t_i) \stackrel{\text{def}}{=} \int \mathcal{D}^{(t\in[t_i,t_f])}_{[\mathbf{q}(t)]} \exp(iS'_H([\mathbf{q}(t), \mathbf{p}(t)]; t_f, t_i)/\hbar). \tag{24b}$$

Now we note from Eq. (1a) that the *unconstrained* variation of the classical action $S_H([\mathbf{q}(t), \mathbf{p}(t)]; t_f, t_i)$ with respect to the *momentum path* $\mathbf{p}(t)$ yields the *first* classical Hamiltonian equation, and from Eq. (1b) that the *unconstrained* variation of the classical action $S'_H([\mathbf{q}(t), \mathbf{p}(t)]; t_f, t_i)$ with respect to the *configuration path* $\mathbf{q}(t)$ yields the *second* classical Hamiltonian equation. We therefore see that our above unconstrained functional integrals in Eq. (24a) for $A_H([\mathbf{q}(t)]; t_f, t_i)$ and in Eq. (24b) for $A'_H([\mathbf{p}(t)]; t_f, t_i)$ are the *precise embodiments* of the principle that the *quantization of classical dynamics* is achieved by substituting *superposition* of the exponential of $(i/\hbar)$ times the classical action for *variation* of that action. (Additionally, of course, that classical action must *not* be one that *implicitly contravenes* the uncertainty principle!) This *validates* the interpretation of $A_H([\mathbf{q}(t)]; t_f, t_i)$ as the *quantum amplitude* that the dynamical system traverses the specified *configuration path* $\mathbf{q}(t)$ for $t \in [t_i, t_f]$ and of $A'_H([\mathbf{p}(t)]; t_f, t_i)$ as the *quantum amplitude* that the dynamical system traverses the specified *momentum path* $\mathbf{p}(t)$ for $t \in [t_i, t_f]$. The *dominant* stationary phase $\mathbf{p}(t)$ *momentum* path that *contributes* to $A_H([\mathbf{q}(t)]; t_f, t_i)$ is readily seen to be the one that comes from *algebraically* solving the *first* classical Hamiltonian equation, i.e.,

$$\dot{\mathbf{q}}(t) = \nabla_{\mathbf{p}(t)} H(\mathbf{q}(t), \mathbf{p}(t), t), \tag{25a}$$

whereas the *dominant* stationary phase $\mathbf{q}(t)$ *configuration* path that *contributes* to $A'_H([\mathbf{p}(t)]; t_f, t_i)$ is seen to be the one that comes from *algebraically* solving the *second* classical Hamiltonian equation, i.e.,

$$\dot{\mathbf{p}}(t) = -\nabla_{\mathbf{q}(t)} H(\mathbf{q}(t), \mathbf{p}(t), t). \tag{25b}$$

In order to obtain the configuration path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$ described below Eq. (2), we clearly must superpose the amplitudes for all configuration paths $\mathbf{q}(t)$ that satisfy the two *endpoint constraints* $\mathbf{q}(t_i) = \mathbf{q}_i$ and $\mathbf{q}(t_f) = \mathbf{q}_f$, i.e., we must superpose $A_H([\mathbf{q}(t)]; t_f, t_i)$ over all the $\mathbf{q}(t)$ which satisfy these two endpoint constraints. A mathematically efficacious method for superposing over only those $\mathbf{q}(t)$ which conform to these two endpoint constraints has already been developed with the aid of the configuration path functional of Eq. (4a), and application of this method to the Eq. (24a) representation of $A_H([\mathbf{q}(t)]; t_f, t_i)$ will clearly produce Eq. (7). Obtaining $K'_H(\mathbf{p}_f, t_f; \mathbf{p}_i, t_i)$ from the Eq. (24b) representation of $A'_H([\mathbf{p}(t)]; t_f, t_i)$ with the aid of the momentum path functional of Eq. (11) proceeds along closely parallel lines, and analogously produces Eq. (12).

We note that in the mistaken Feynman-Dirac *Lagrangian*-action version of the configuration path integral $K_H(\mathbf{q}_f, t_f; \mathbf{q}_i, t_i)$, the amplitude for the path $\mathbf{q}(t)$, namely $A_H([\mathbf{q}(t)]; t_f, t_i)$, is *not* given by Eq. (24a) but *instead* by the exponential of $(i/\hbar)$ times the *Lagrangian* action for that configuration path $\mathbf{q}(t)$. This *phase factor* is *only the integrand corresponding to one particular momentum path* of the Eq. (24a) *functional integral* for $A_H([\mathbf{q}(t)]; t_f, t_i)$ that runs over *all* momentum paths. The *particular* momentum path $\mathbf{p}(t)$ which the Feynman-Dirac Lagrangian-action hypothesis *inadvertently singles out* is the one which the Lagrangian *implicitly determines* (in *contravention* of the uncertainty principle) from the configuration path $\mathbf{q}(t)$, namely,

$$\mathbf{p}(t) = \nabla_{\dot{\mathbf{q}}(t)} L(\dot{\mathbf{q}}(t), \mathbf{q}(t), t).$$

From classical mechanics one readily verifies that this *particular* momentum path is in fact the *dominant* contributor to the *actually required sum* over momentum paths in Eq. (24a) because it is *precisely* the one which algebraically satisfies the Eq. (25a) first classical Hamiltonian equation. That explains why the Lagrangian path integral "works" under certain favorable conditions, and it *also* explains why, even under the *most favorable* of those conditions, namely Hamiltonians which are quadratic forms in $\mathbf{p}(t)$ whose Gaussian-phase functional integrals over the $\mathbf{p}(t)$ *automatically* produce the *dominant* phase factor, the Lagrangian path integral *still* requires a mysterious *additional* factor—this "mystery factor" arises because *integration* over even *Gaussian* phases yields *not only the dominant phase factor*, but a *non-phase* factor as well. In the *subsequent* integration over endpoint-constrained *configuration paths*, this factor is *not*, as Feynman's mistaken Lagrangian approach drove him to erroneously conclude, a *totally ad hoc* measure "normalizing factor" which needs to be puzzled out and *inserted by hand*, but rather *part of the correct integrand*. The Lagrangian path integral is thus seen to be a not-quite-adequate *relative* of *systematic* semiclassical asymptotic approximations to the correct Hamiltonian phase-space path integral.

The slightly stronger self-consistent canonical commutation rule Dirac missed

The *unique* Hamiltonian quantization given by Eqs. (21) in conjunction with Eq. (16b) or Eq. (18b) could *very well* have been discovered by Dirac when he was formulating his canonical commutation rule in 1925 [7], or at any time thereafter that he should have chosen to *revisit* that work. We now briefly explore just what it was that Dirac *failed to light on* during an entire lifetime (see reference [10] for greater detail). We note that the canonical commutation rules which Dirac ended up postulating in 1925 (after some struggling) can be gathered into the single formula,

$$[c_1 \mathrm{I} + \mathbf{k}_1 \cdot \widehat{\mathbf{q}} + \mathbf{l}_1 \cdot \widehat{\mathbf{p}}, c_2 \mathrm{I} + \mathbf{k}_2 \cdot \widehat{\mathbf{q}} + \mathbf{l}_2 \cdot \widehat{\mathbf{p}}] = i\hbar(\mathbf{k}_1 \cdot \mathbf{l}_2 - \mathbf{l}_1 \cdot \mathbf{k}_2)\mathrm{I}, \tag{26a}$$

where c_1 and c_2 are constant scalars, and $\mathbf{k}_1$, $\mathbf{l}_1$, $\mathbf{k}_2$, $\mathbf{l}_2$ are constant vectors. The above equation can be reexpressed in the much more suggestive form,

$$[\overbrace{c_1 + \mathbf{k}_1 \cdot \mathbf{q} + \mathbf{l}_1 \cdot \mathbf{p}}, \overbrace{c_2 + \mathbf{k}_2 \cdot \mathbf{q} + \mathbf{l}_2 \cdot \mathbf{p}}] = i\hbar \overbrace{\{c_1 + \mathbf{k}_1 \cdot \mathbf{q} + \mathbf{l}_1 \cdot \mathbf{p}, c_2 + \mathbf{k}_2 \cdot \mathbf{q} + \mathbf{l}_2 \cdot \mathbf{p}\}}, \tag{26b}$$

where the overbrace denotes the *quantization* of the classical dynamical variable beneath it, and the vertical curly brackets of course denote the *classical* Poisson bracket. (We use overbraces to denote quantization *only* where the orthodox "hat" accent $\hat{}$, which is the standard way to denote quantization, *fails* to be sufficiently *wide.*) Eq. (26b) is compellingly elegant in light of Dirac's amazing groundbreaking demonstration that the quantum mechanical analog of the classical Poisson bracket *must* be $(-i/\hbar)$ times the commutator bracket [7]. Indeed, this form of Dirac's postulate rather strongly *suggests* the possibility that it might perhaps be *strengthened* to simply read,

$$[\overbrace{F_1(\mathbf{q}, \mathbf{p})}, \overbrace{F_2(\mathbf{q}, \mathbf{p})}] = i\hbar \overbrace{\{F_1(\mathbf{q}, \mathbf{p}), F_2(\mathbf{q}, \mathbf{p})\}}. \tag{27}$$

We note that the Eq. (26b) form of Dirac's postulate is the *restriction* of Eq. (27) to $F_i(\mathbf{q}, \mathbf{p})$, $i = 1, 2$, that are both *inhomogeneous linear functions* of phase space. Another, equivalent way to make this restriction is to require that *all* the second-order partial derivatives of the $F_i(\mathbf{q}, \mathbf{p})$, $i = 1, 2$, *must vanish.* Dirac was very tempted by Eq. (27), but upon playing with it he found to his consternation that it *overdetermined* the quantization of classical dynamical variables, and thus would be *self-inconsistent* as a postulate [7]. Dismayed, he retreated to the *restriction* on the $F_i(\mathbf{q}, \mathbf{p})$ that results in Eq. (26b), which, however, *cannot* determine the *order* of noncommuting factors *at all*! Far better that abject *underdetermination* of the quantization of classical dynamical variables than the outright *self-inconsistency* of their *overdetermination* was undoubtedly the thought that ran through Dirac's mind.

But could there be a "middle way" that skirts overdetermination *without* having to settle for *not* determining the order of noncommuting factors *at all*? Very unfortunately, Dirac apparently *never* revisited this issue after 1925. If one plays with polynomial forms of the $F_i(\mathbf{q}, \mathbf{p})$, one realizes that the *overdetermination* does *not* occur if *no* monomials that are dependent on *both* $\mathbf{q}$ *and* $\mathbf{p}$ are present. This tells us that Dirac's restriction on the $F_i(\mathbf{q}, \mathbf{p})$, which requires that *all* their second-order partial derivatives must vanish, is *excessive*: to prevent the self-inconsistent overdetermination of quantization it is *quite enough* to require that *only* the *mixed* $\mathbf{q}, \mathbf{p}$ second-order partial derivatives of the $F_i(\mathbf{q}, \mathbf{p})$ must vanish, i.e., that,

$$\nabla_{\mathbf{p}} \nabla_{\mathbf{q}} F_i(\mathbf{q}, \mathbf{p}) = 0,\ i = 1, 2, \tag{28a}$$

which has the general solution, $F_i(\mathbf{q}, \mathbf{p}) = f_i(\mathbf{q}) + g_i(\mathbf{p})$, $i = 1, 2$. Therefore, if we merely *replace* the Eq. (26b) form of Dirac's postulate by,

$$[\overbrace{f_1(\mathbf{q}) + g_1(\mathbf{p})}, \overbrace{f_2(\mathbf{q}) + g_2(\mathbf{p})}] = i\hbar \overbrace{\{f_1(\mathbf{q}) + g_1(\mathbf{p}), f_2(\mathbf{q}) + g_2(\mathbf{p})\}}, \tag{28b}$$

we will *still* have a canonical commutation rule that does *not* provoke the self-inconsistent *overdetermination* of classical dynamical variables. But does it make any dent in the gross *nondetermination* of the ordering of noncommuting factors that characterizes Dirac's Eq. (26b)? The question of whether a proposed approach fully determines the quantization of *all* classical dynamical variables can be boiled down to the issue of whether it fully determines the quantization of the class of exponentials $\exp(i(\mathbf{k} \cdot \mathbf{q} + \mathbf{l} \cdot \mathbf{p}))$, because if it *does*, the *linearity* of quantization, combined with Fourier expansion, then determines the quantization of *all* dynamical variables. It is apparent that the only truly *new* consequence of Eq. (28b) versus Dirac's Eq. (26b) is that,

$$[f(\widehat{\mathbf{q}}), g(\widehat{\mathbf{p}})] = i\hbar\, \overbrace{\nabla_{\mathbf{q}} f(\mathbf{q}) \cdot \nabla_{\mathbf{p}} g(\mathbf{p})}\,. \tag{28c}$$

Putting now $f(\mathbf{q}) = e^{i\mathbf{k}\cdot\mathbf{q}}$ and $g(\mathbf{p}) = e^{i\mathbf{l}\cdot\mathbf{p}}$, we see that Eq. (28c) yields,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = (i/(\hbar\mathbf{k}\cdot\mathbf{l}))[e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}}, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}], \tag{29}$$

which clearly answers the question concerning *full* determination of quantization in the affirmative! It now remains to be worked out how the unique, self-consistent quantization that results from slightly *strengthening* Dirac's *excessively restricted* canonical commutation rule of Eq. (26b) to the marginally *less* restricted canonical quantization rule of Eq. (28b) in fact *compares* with the unique quantization rule of Eq. (16b), which is a *key consequence* of the Hamiltonian phase-space path integral. To carry out the comparison, it is very helpful to use the identity,

$$[e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}}, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}] = \textstyle\int_0^1 d\lambda\, \left(d(e^{i\lambda\mathbf{k}\cdot\widehat{\mathbf{q}}} e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}} e^{i(1-\lambda)\mathbf{k}\cdot\widehat{\mathbf{q}}})/d\lambda\right), \tag{30a}$$

which is simply a consequence of the fundamental theorem of the calculus. Now if we carry out the differentiation under the integral sign, there results,

$$\begin{gathered}[e^{i\mathbf{k}\cdot\widehat{\mathbf{q}}}, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}] = \\ \textstyle\int_0^1 d\lambda\, e^{i\lambda\mathbf{k}\cdot\widehat{\mathbf{q}}}[i\mathbf{k}\cdot\widehat{\mathbf{q}}, e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}] e^{i(1-\lambda)\mathbf{k}\cdot\widehat{\mathbf{q}}} = -i\hbar\mathbf{k}\cdot\mathbf{l}\int_0^1 d\lambda\, e^{i\lambda\mathbf{k}\cdot\widehat{\mathbf{q}}} e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}} e^{i(1-\lambda)\mathbf{k}\cdot\widehat{\mathbf{q}}},\end{gathered} \tag{30b}$$

Combining this identity with the quantization result of Eq. (29) yields,

$$\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} = \textstyle\int_0^1 d\lambda\, e^{i\lambda\mathbf{k}\cdot\widehat{\mathbf{q}}} e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}} e^{i(1-\lambda)\mathbf{k}\cdot\widehat{\mathbf{q}}}. \tag{31}$$

We note here that the *form* of Eq. (31) is that of a rule for the *ordering* of noncommuting factors—and that rule has a characteristically Born-Jordan [8] *appearance*, i.e., *all* of the orderings of the class that it embraces appear with *equal weight*. H. Weyl, a mathematician who liked to dabble in the new quantum mechanics, thought it highly plausible that Nature would *select* the *most symmetric* of that class of orderings [11], i.e., the one for which $\lambda = \frac{1}{2}$, but Eq. (31) has it that Nature does *not select* amongst orderings *at all*, that it *instead* achieves an *alternate* kind of symmetry through *utter nondiscrimination* amongst orderings (an echo, perhaps, of the need to sum over *all* paths). Now in order to compare the quantization given by Eq. (31) to the result of the integration which is called for by Eq. (16b), we must first obtain the *configuration representation* of the former, which is facilitated by the well-known result that,

$$\langle\mathbf{q}_f|e^{i\mathbf{l}\cdot\widehat{\mathbf{p}}}|\mathbf{q}_i\rangle = \delta^{(n)}(\mathbf{q}_f + \hbar\mathbf{l} - \mathbf{q}_i).$$

Using this, we obtain from Eq. (31) that,

$$\langle\mathbf{q}_f|\,\overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}}\,|\mathbf{q}_i\rangle = \textstyle\int_0^1 d\lambda\, e^{i\mathbf{k}\cdot(\mathbf{q}_i+\lambda(\mathbf{q}_f-\mathbf{q}_i))}\, \delta^{(n)}(\mathbf{q}_f + \hbar\mathbf{l} - \mathbf{q}_i), \tag{32}$$

which result, it is readily verified, is *also* produced by the path integral quantization formula of Eq. (16b) when $e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}$ is substituted for the classical Hamiltonian.

We do not really need to go further than this to have demonstrated that the quantization produced by the path integral is the *same* as that produced by the slightly strengthened canonical commutation rule of Eq. (28b). The reader may find it interesting, however, to follow out the full consequences of combining the *linearity* of quantization with the *Fourier expansion* of an *arbitrary* classical dynamical variable $F(\mathbf{q}, \mathbf{p})$, which together formally imply that,

$$\langle \mathbf{q}_f | \overbrace{F(\mathbf{q},\mathbf{p})} | \mathbf{q}_i \rangle =$$

$$(2\pi)^{-2n} \int d^n\mathbf{q}' d^n\mathbf{p}' \, F(\mathbf{q}',\mathbf{p}') \int d^n\mathbf{k} d^n\mathbf{l} \, e^{-i(\mathbf{k}\cdot\mathbf{q}'+\mathbf{l}\cdot\mathbf{p}')} \langle \mathbf{q}_f | \overbrace{e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}} | \mathbf{q}_i \rangle. \tag{33a}$$

The next step is, of course, to substitute the unambiguous result for the quantization of the exponential $e^{i(\mathbf{k}\cdot\mathbf{q}+\mathbf{l}\cdot\mathbf{p})}$, which was obtained in Eq. (32) from the slightly strengthened canonical commutation rule of Eq. (28b), for the last factor of the integrand on the right hand side of Eq. (33a). We leave it to the reader to then plow through all the integrations that can be carried out in closed form to obtain,

$$\langle \mathbf{q}_f | \overbrace{F(\mathbf{q},\mathbf{p})} | \mathbf{q}_i \rangle = \int_0^1 d\lambda \, (2\pi\hbar)^{-n} \int d^n\mathbf{p} \, F(\mathbf{q}_i + \lambda(\mathbf{q}_f - \mathbf{q}_i), \mathbf{p}) e^{i(\mathbf{q}_f - \mathbf{q}_i)\cdot\mathbf{p}/\hbar}, \tag{33b}$$

which is precisely the *same* quantization result as is obtained from the Hamiltonian phase-space path integral, namely that given by Eq. (16b), when $F(\mathbf{q},\mathbf{p})$ is subtituted for the classical Hamiltonian. Dirac's 1925 postulation of Eqs. (26) as *the* canonical commutation rule is thus seen to be a *purely historical aberration.* One can only suppose that if Dirac had *kept working* over the years on trying to obtain a more satisfactory canonical commutation rule than the abjectly deficient Eqs. (26), he would *surely* have eventually lit upon their slight strengthening to Eq. (28b), which *removes* their vexing ordering ambiguity *without* imperiling their self-consistency. The Hamiltonian phase-space path integral's utterly straightforward unique quantization *ought* to have been the needed wake-up call to the physics community on this issue, but by then the result of Dirac's inadequate work had become so *ingrained* that it was mentioned by Cohen [5] in his last paragraph as another reason to call into question the correct path integral results of Kerner and Sutcliffe [4]. Cohen's mention of the "usual" ambiguity of quantization may have been one of Kerner's motivations to revisit Dirac's canonical commutation rule. He soon came up with its slight strengthening to Eq. (28b) and showed this to produce the very same Born-Jordan [8] quantization as does the Hamiltonian phase-space path integral [9]. Stunningly, however, Kerner *never published* those results! Neither did he *ever reply* in print nor at any scholarly forum to the meritless $\lim_{|t_f - t_i| \to 0} |\mathbf{q}_f - \mathbf{q}_i| = 0$ objection that Cohen raised regarding his groundbreaking paper with Sutcliffe on the consequences of the Hamiltonian phase-space path integral. Pressed on why, he said that he "did not want to pick a fight with Leon Cohen" [9]. Kerner's apparently shy, retiring nature came within a hair of *denying* physics the gifts that his mind had produced. To read page after page of solemn classification by Tirapegui et al. [6] of wrong "discretization" results that flow from Cohen's lapse is to utterly despair of Kerner's choice of silence.

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