Towards A Moyal Quantization Program of the Membrane

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, GA. 30314; perelmanc@hotmail.com

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Abstract

A Moyal deformation quantization approach to a spherical membrane (moving in flat target backgrounds) in the light cone gauge is presented. The physical picture behind this construction relies in viewing the two spatial membrane coordinates σ_1, σ_2 as the two phase space variables q, p, and the temporal membrane coordinate τ as *time*. Solutions to the Moyal-deformed equations of motion are explicitly constructed in terms of elliptic functions. A knowledge of the Moyal-deformed light-cone membrane's Hamiltonian density $H(q,p,\tau)$ allows to construct a time-dependent Wigner function $\rho(q,p,\tau)$ as solutions of the Moyal-Liouville equation, and from which one can obtain the expectation values of the operator $< \hat{H} >= Trace\ (\rho H)$ that define the quantum average values of the energy density configurations of the membrane at any instant of time. It is shown how a time-dependent quartic oscillator with q^4, p^4, q^2p^2 terms plays a fundamental role in the quantum treatment of membranes and displays an important $p \leftrightarrow q$ *duality* symmetry.

The complete and fully satisfactory quantization program of the membrane is a notoriously difficult, unsolved problem (to our knowledge) due to the intrinsic nonlinearities and the influence of the membrane's topology on its dynamics. A closed membrane can have the topology of an arbitrary Riemann surface of any genus. Since all orientable Riemann surfaces are cobordant, it is possible for the topology of the membrane to evolve [1] and the three-dim world volume of the membrane could have two boundaries : one a sphere and the other a torus. A thorough analysis of the conventional and tentative approaches to the quantization of membranes and extendons (p-branes) can be found in the monograph [1] and [2]. The purpose of this letter is to propose a Moyal deformation quantization approach [3], [5] to a spherical membrane (moving in flat target backgrounds) in the light cone gauge. In [7] we have shown how p-brane actions can be obtained from the large $N \rightarrow \infty$ limit of $SU(N)$ (Generalized) Yang-Mills theories, and which is related to the classical limit $\hbar \rightarrow 0 (\hbar \sim 1/N)$ of the

Moyal deformation of Yang-Mills theories. Consequently the Moyal deformation quantization program is very relevant to the physics of extended objects.

On a (flat) two-dim phase space $z^m = x, p$, the noncommutative and associative star product of two functions $A(x,p)$ and $B(x,p)$ is defined in terms of the inverse $\Omega^{mn} = -\Omega^{nm}$ ($m, n = 1, 2$) of the symplectic form as

$$A \ * \ B = \ A \ exp \left(\ \frac{i\hbar}{2} \ \overleftarrow{\partial}_m \ \Omega^{mn} \ \overrightarrow{\partial}_n \ \right) \ B \ =$$

$$A \ B + \frac{i\hbar}{2} \ \Omega^{mn} \ (\partial_m A \ \partial_n B) + \ \frac{(i\hbar/2)^2}{2!} \ \Omega^{m_1 n_1} \ \Omega^{m_2 n_2} \ (\partial^2_{m_1 m_2} \ A) \ (\partial^2_{n_1 n_2} \ B) + ... \tag{1}$$

when Ω^{mn} depends on x, p the star product on Poisson manifolds (that can be odd-dimensional) was provided by [4]. The Baker integral expression for the star product in phase space captures the noncommutativity explicitly and is given by

$$(f * g)(x,p) \ = \ (\frac{1}{\pi\hbar})^2 \ \int \ dx' \ dp' \ dx'' \ dp'' \ e^{\frac{2i}{\hbar} \ \Delta(x,x',x'';p,p',p'')} \ f(x',p') \ g(x'',p''). \tag{2}$$

where the kernel in the exponential is the phase-space area (determinant)

$$\Delta(x, x', x''; p, p', p'') = det \ \begin{pmatrix} 1 & 1 & 1 \\ x & x' & x'' \\ p & p' & p'' \end{pmatrix} \tag{3}$$

The product can also be recast as

$$(f \ * \ g)(x,p) \ = \ (\frac{1}{\pi\hbar})^2 \ \int \ du_1 \ du_2 \ dv_1 \ dv_2 \ e^{\frac{2i}{\hbar} \ \Delta(u_i,v_i)} \ \times$$

$$f(x + u_1, p + v_1) \ g(x + u_2, p + v_2). \tag{4}$$

$$\Delta(u_i, v_i) = det \ \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}. \tag{5}$$

After expanding in a Taylor series

$$f(x + u_1, p + v_1) \ = \ f(x,p) \ + \ u_1 \partial_x f \ + \ v_1 \partial_p f \ + \ ......$$

$$g(x + u_2, p + v_2) \ = \ g(x,p) + \ u_2 \partial_x g \ + \ v_2 \partial_p g \ + \ .... \tag{6}$$

and inserting the Taylor series expansion into the integral (4) one arrives at terms involving delta functions leading finally to the standard expression for the star product (1).

The deformed light-cone membrane action of spherical topology moving in a flat D-dim target background (excluding the zero modes) is

$$S \ = \ \frac{T}{2} \int d^3\sigma \ \left(\ (\mathcal{D}_\tau X^i)_* \ * \ (\mathcal{D}_\tau X^i)_* \ - \frac{1}{2(i\hbar)^2} \ \{X^i, X^j\}_* \ * \ \{X_i, X_j\}_* \ \right) \ =$$

$$\frac{T}{2}\int d^3\sigma\ \left((\mathcal{D}_\tau X^i)_* \ (\mathcal{D}_\tau X^i)_* \ - \frac{1}{2(i\hbar)^2}\ \{X^i, X^j\}_* \ \{X_i, X_j\}_* \ + \ .... \right) \tag{7}$$

The ellipsis ..... in (7) denote total derivative terms whose contribution to the integral is zero if the fields and their derivatives vanish fast enough at infinity and/or there are no spatial boundaries. The membrane's tension T can be set to unity and $\sigma_a = q, p, \tau$ denote the membrane's three-dimensional world-volume local coordinates. The $D-2$ coordinates X^i, X^j correspond to the *transverse* $D-2$ directions to the light-cone $X^0 \pm X^D$ coordinates in D-dimensions. The one-dimensional gauge covariant derivative along the temporal direction is defined by

$$(\mathcal{D}_\tau X^i)_* \ = \ \frac{\partial X^i}{\partial \tau} \ - \ \frac{1}{i\hbar}\ \{\mathcal{A}_\tau,\ X^i\}_*. \tag{8}$$

The deformed light-cone membrane action can be re-written as a deformed $SU(\infty)$ Yang-Mills action in $D-1$ dimensions dimensionally reduced to one temporal dimension. The $D-2$ gauge fields A^i have a one-to-one correspondence to the $D-2$ transverse coordinates X^i. The contribution of the temporal component $\mathcal{A}_\tau$ of the gauge field increases by one the total number of gauge fields : $D-2+1 = D-1$. A Yang-Mills theory in $D-1$ has $D-1-2 = D-3$ physical degrees of freedom that match the degrees of freedom of the three-dim worldvolume of a membrane in D-dim resulting from the three-dimensional worldvolume reparametrization invariance. Therefore, the $D-1$-dimensional $SU(\infty)$ Yang-Mills action, dimensionally reduced to one temporal dimension,

$$S_{YM} \ = \ - \ \frac{1}{4g^2_{YM}} \int d\tau \ Trace \ \left(\ (F^{i\tau})^2 \ + \ (F^{ij})^2 \ \right). \tag{9}$$

has the same form as the light-cone membrane action after the trace operation for $SU(\infty)$ is replaced by the integral $\int \ dq \ dp$. To proceed we shall make the following identificactions

$$\sigma_1 \ \leftrightarrow \ \frac{q}{\lambda_l}\ L; \quad \sigma_2 \ \leftrightarrow \ \frac{p}{\lambda_p}\ L, \quad \hbar \ \leftrightarrow \ L^2. \tag{10}$$

where λ_l, λ_p are a judicious length and momentum scale not necessarily the same as the Planck scale L and Planck momentum which are introduced to render σ_1, σ_2 with length dimensions. Therefore, the Moyal star products $A(\sigma_1, \sigma_2) * B(\sigma_1, \sigma_2)$ will be defined as in eq-(1) by replacing $q, p, \hbar$ for σ_1, σ_2, L^2, respectively, and expanding all the terms in suitable powers of L^2. The dimensions of $\mathcal{A}_\tau$ in (8) will be set to be those of *length* instead of the conventional *length*$^{-1}$.

The Yang-Mills formulation of the light-cone spherical membrane action moving in flat target spacetime backgrounds leads to the following deformations of the equations of motion $(D_\mu F^{\mu\nu})_* = 0$ dimensionally reduced to one temporal dimension

$$(\ D_\tau F^{\tau j} \ + \ D_i F^{ij} \ )_* \ = \ 0 \ \Rightarrow \ g^{\tau\tau} \ \partial_\tau \ (\ \partial_\tau X^j - (iL^2)^{-1} \ \{A_\tau, X^j\}_* \ ) \ -$$

$$g^{\tau\tau}\ (iL^2)^{-1}\ \{\ A_\tau,\ \partial_\tau X^j - (iL^2)^{-1}\ \{A_\tau, X^j\}_*\ \}_* - (iL^2)^{-2}\ \{\ X_i,\ \{X^i, X^j\}_*\ \}_*\ =\ 0. \tag{11a}$$

and the nonabelian analog of Gauss law

$$(D_i\ F^{i\tau})_*\ =\ 0\ \Rightarrow\ g^{\tau\tau}\ (iL^2)^{-1}\ \{\ X_i,\ \partial_\tau X^i - (iL^2)^{-1}\ \{A_\tau, X^i\}_*\ \}_*\ =\ 0. \tag{11b}$$

with $g_{\tau\tau} = -1; g_{ij} = \delta_{ij}$. We will firstly find solutions to the deformed light-cone membrane equations of motion (to *lowest* order in powers of L) of the expansion of the Moyal star products in eqs-(11), and afterwards we turn our attention to the solutions involving higher powers of L associated to the full fledged Moyal deformations.

In order to find a family of solutions to eqs-(11) to lowest order in powers of L, in the particular case when there are $D-2=3$ transverse direction X^i (corresponding to a membrane embedded in $D=5$) we will invoke the ansatz based on the separation of variables

$$X^i(\sigma_1, \sigma_2, \tau; L)\ =\ f^i(\tau)\ Y^i(\sigma_1, \sigma_2; L);\ \textit{no sum over repeated indices} \tag{12}$$

that leads to *real* solutions. The deformed equations of motion can be simplified considerably if : (i) one chooses the temporal gauge $\mathcal{A}_\tau = 0$, and (ii) if one invokes the additional $SU(2)$ Lie-Moyal algebraic relations

$$\{Y^i, Y^j\}_*\ =\ i\ L\ \epsilon^{ijk}\ Y^k;\quad i,j,k = 1,2,3. \tag{13}$$

where L is a suitable length parameter that must be introduced in order to match units in both sides of (13). After invoking the $SU(2)$ algebraic relations (13) in eqs-(11), setting $g^{\tau\tau} = -1$ and $\epsilon^{ijk}\epsilon_{ikl} = -2\delta^j_l$, the equations of motion of the deformed light-cone membrane (11a), to *lowest* order in L, become

$$Y^j\ \frac{d^2 f^j(\tau)}{d\tau^2}\ -\ \frac{2}{L^2}\ f^j(\tau)\ Y^j\ \sum_{i\neq j}\ (f_i)^2(\tau)\ =\ 0. \tag{14}$$

One can reabsorb the $\lambda^{-2} = (L/\sqrt{2})^{-2}$ factor in the second term of (14) into the temporal variable by a re-scaling $\tau \to \frac{\tau}{\lambda} = \tilde{\tau}$ such that $f^i(\tau) = f^i(\lambda\tilde{\tau}) = \tilde{f}^i(\tilde{\tau})$. Upon doing so, after factoring out $Y^j \neq 0$ and *dropping* the tilde symbols for convenience, eqs-(14) become

$$\frac{d^2 f_1(\tau)}{d\tau^2}\ -\ f_1(\tau)\ [\ (f_2)^2(\tau) + (f_3)^2(\tau)\ ]\ =\ 0. \tag{15a}$$

$$\frac{d^2 f_2(\tau)}{d\tau^2}\ -\ f_2(\tau)\ [\ (f_3)^2(\tau) + (f_1)^2(\tau)\ ]\ =\ 0. \tag{15b}$$

$$\frac{d^2 f_3(\tau)}{d\tau^2}\ -\ f_3(\tau)\ [\ (f_2)^2(\tau) + (f_1)^2(\tau)\ ]\ =\ 0. \tag{15c}$$

The other equation

$$\{\ f_i(\tau)\ Y_i(\sigma_1, \sigma_2; L),\ \frac{df^i(\tau)}{\partial\tau}\ Y^i(\sigma_1, \sigma_2; L)\ \}_*\ =\ 0. \tag{16}$$

is trivially satisfied $f_i(\tau)(df^i(\tau)/d\tau)\ \{ \ Y_i,\ Y^i \ \}_* = 0$ in a flat target background $\eta_{\mu\nu} = diag\ (-1,1,1,1)$ since

$$\{ \ Y_i,\ Y^i \ \}_* \ = \ \{\eta_{ik}\ Y^k,\ Y^i\}_* \ = \ \{Y^i,\ Y^i\}_* \ = \ 0. \tag{17}$$

The solutions to eqs-(15) are given in terms of the elliptic functions

$$f_1 \ = \ i\ k\ cn(\tau;k); \quad f_2 \ = \ k\ sn(\tau;k); \quad f_3 \ = \ -\ i\ dn(\tau;k). \tag{18}$$

where $0 < k < 1$ is the modulus. The functions f_1, f_2, f_3 obey similar equations as those obtained for the Euler-top equations of motion

$$\frac{df_1}{d\tau} \ = \ f_2\ f_3; \quad \frac{df_2}{d\tau} \ = \ f_3\ f_1; \quad \frac{df_3}{d\tau} \ = \ f_1\ f_2. \tag{19}$$

and which can be verified by a simple inspection due to the expressions for the derivatives of the elliptic functions given by

$$\frac{d(sn)}{d\tau} \ = \ cn\ dn; \quad \frac{d(cn)}{d\tau} \ = \ -\ sn\ dn; \quad \frac{d(dn)}{d\tau} \ = \ -\,k^2\ sn\ cn. \tag{20}$$

When one differentiates w.r.t τ the expressions on both sides of eq-(19) one arrives precisely at eqs-(15) which we intended to solve in the first place. Therefore, eqs-(15) can be *integrated* leading to the solutions (18) given in terms of the three elliptic functions. We notice that two of the solutions, f_1, f_3 are purely imaginary (for real modulus k) and f_2 is real-valued. This is no accident as we shall see below. The next step is to find a family of solutions to the classical $SU(2)$ Lie-Poisson-like algebraic relations obtained from the lowest order terms of the expression $(iL^2)^{-1}\{Y^i, Y^j\}_*$ and given by

$$\{Y^i, Y^j\}_{PB} \ = \ \epsilon^{ijk}\ \frac{Y^k}{L} \ \Rightarrow \ \{\tilde{Y}^i, \tilde{Y}^j\}_{PB} \ = \ \epsilon^{ijk}\ \tilde{Y}^k, \ \ \tilde{Y} = LY \tag{21}$$

The solutions are

$$Y^1 \ = \ \frac{i}{4}\ \frac{1}{L}\ [\ (\sigma_1)^2 \ - \ (\sigma_2)^2\ ]; \quad Y^2 \ = \ \frac{1}{4}\ \frac{1}{L}\ [\ (\sigma_1)^2 \ + \ (\sigma_2)^2\ ]$$

$$Y^3 \ = \ -\ \frac{i}{2}\ \frac{1}{L}\ \sigma_1\ \sigma_2 \tag{22}$$

The solutions (12) are *not* unique due to the symmetry of eqs-(11a,11b) under Moyal-deformed area-preserving diffs $\{q'(q,p;\hbar), p'(q,p;\hbar)\}_* = 1$. Hence, given a set of solutions (12) in the gauge $\mathcal{A}_\tau = 0$, one can find another set of solutions such that under infinitesimal Moyal-deformed area-preserving diffs the gauge field transforms as $\delta_\xi \mathcal{A}_\tau = \partial_\tau \xi + \{\xi, \mathcal{A}_\tau\}_*$ and the coordinate functions transform as $\delta_\xi X^k = \{\xi, X^k\}_*$.

Finally, the sought-after solutions after inserting the scaling $(\tau/\lambda) = \tilde{\tau}$ back into the solutions where $\lambda = \frac{L}{\sqrt{2}}$ (L is set to the Planck scale) are given by

$$X^1(\sigma_1, \sigma_2, \tau) \ = \ f_1(\frac{\tau}{\lambda}) \ Y^1(\sigma_1, \sigma_2) \ = \ ik \ cn(\frac{\tau}{\lambda}; k) \ \frac{i}{4} \ \frac{1}{L} \ [\ (\sigma_1)^2 \ - \ (\sigma_2)^2 \ ] \ =$$

$$- \ \frac{1}{4L} \ k \ cn(\frac{\tau}{\lambda}; k) \ [\ (\sigma_1)^2 \ - \ (\sigma_2)^2 \ ]; \ \ \lambda = \frac{L}{\sqrt{2}}. \tag{25a}$$

despite that $f_1(\frac{\tau}{\lambda})$ was imaginary one still obtains a *real*-valued solution for the coordinate X^1. Real-valued results for X^2, X^3 also occur

$$X^2(\sigma_1, \sigma_2, \tau) \ = \ f_2(\frac{\tau}{\lambda}) \ Y^2(\sigma_1, \sigma_2) \ = \ \frac{1}{4L} \ k \ sn(\frac{\tau}{\lambda}, k) \ [\ (\sigma_1)^2 \ + \ (\sigma_2)^2 \ ]. \tag{25b}$$

$$X^3(\sigma_1, \sigma_2, \tau) \ = \ f_3(\frac{\tau}{\lambda}) \ Y^3(\sigma_1, \sigma_2) \ = \ [- \ i \ dn(\frac{\tau}{\lambda}, k) \ ] \ [- \ \frac{i}{2} \ \frac{1}{L} \ \sigma_1 \ \sigma_2 \ ] \ =$$

$$- \ \frac{1}{2L} \ dn(\frac{\tau}{\lambda}, k) \ \sigma_1 \ \sigma_2. \tag{25c}$$

Therefore, to sum up, we have found *real*-valued solutions eqs-(25) to the (classical, lowest order) equations of motion associated to the light-cone spherical membrane action when the target $5D$ spacetime background is *flat* and the temporal gauge $\mathcal{A}_\tau = 0$ is chosen. When the temporal dependence for all the coordinates $X^i = f(\tau) Y^i$ $(i = 1, 2, 3)$ is fixed in terms of a *single* function $f = f_1 = f_2 = f_3$ one ends up with a differential equation of the form

$$\frac{d^2 f(\tau)}{d\tau^2} \ - 2 \ f^3(\tau) \ = \ 0. \tag{26}$$

Multiplying both sides of (26) by $f' = (df/d\tau)$ gives

$$f' \ \frac{df'}{d\tau} \ - 2 \ f^3 \ \frac{df}{d\tau} \ = 0 \ \Rightarrow \ f' \ df' \ - 2 \ f^3 \ df \ = 0. \tag{27}$$

Integrating (27) yields

$$\frac{1}{2} \ (\frac{df}{d\tau})^2 \ - \ \frac{1}{2} \ f^4 \ = \ \beta \ \Rightarrow \ . \tag{28}$$

$$\int \frac{df}{\sqrt{\ 2\beta \ + \ f^4}} \ = \ \int d\tau \ = \ \tau. \tag{29}$$

where β is an arbitrary constant of integration. The elliptic integral on the left is a very complicated expression of f leading to an implicit relation of the form $\tau = \tau(f)$ which must be inverted in order to obtain the expression for $f = f(\tau)$. Thus the expression for $f = f(\tau)$ is not analytic in this case. The integral (29) can be simplified enormously if $\beta = 0$, leading to $-(1/f) = \tau \Rightarrow f(\tau) = -(1/\tau)$. However, in this case one arrives at two purely imaginary solutions for X^1, X^3

and one real solution for X^2. To sum up, the real-valued solutions (25) involving the three separate elliptic functions is physically more appealing than the latter complex solutions. Elliptic functions also appeared in solutions to the $SU(\infty)$ Moyal-Nahm equations [9], [10].

Having found solutions to the deformed light-cone membrane equations of motion to *lowest* order in powers of L ("classical" limit) we turn our attention to the solutions involving higher powers of L associated to the full fledged Moyal deformations. In order to do so we shall perform an expansion in powers of L^2 (since the role of $\hbar$ corresponds to L^2 as explained in eq-(10)) of the form

$$X^i(\sigma_1, \sigma_2, \tau; L) \ = \ f^i(\frac{\tau}{\lambda})\, Y^i(\sigma_1, \sigma_2; L) \ = \ f^i(\frac{\tau}{\lambda})\, \frac{1}{L} \sum_{n=0}^{\infty} Y^i_n(\sigma_1, \sigma_2)\, (L^2)^n \quad (30)$$

the temporal functions $f^i(\tau/\lambda)$ are given in terms of the elliptic functions as shown above. The Y^i spatial functions are solutions to the equations

$$\frac{1}{iL^2} \ \{ \ Y^i(\sigma_1, \sigma_2; L), \ Y^j(\sigma_1, \sigma_2, L)\}_* \ = \ \frac{1}{iL^2} \ i \ \epsilon^{ijk} \ L \ Y^k(\sigma_1 \sigma_2; L). \quad (31)$$

Inserting the terms in the expansion (30) into the above equations (31) allows to *iteratively* solve for the components $Y^i_n(\sigma_1, \sigma_2)$ order by order in powers of L^{2n} as follows. The solutions to lowest order $Y^i_0(\sigma_1, \sigma_2)$ have already been obtained in eqs-(22) (they were quadratic in σ_1, σ_2). The solutions for $Y^i_1(\sigma_1, \sigma_2)$ are obtained by solving the equations

$$\epsilon^{ijk} \ Y^k_1 \ = \ (\partial_{\sigma_1} \ Y^i_1) \ (\partial_{\sigma_2} \ Y^j_0) \ + \ (\partial_{\sigma_1} \ Y^i_0) \ (\partial_{\sigma_2} \ Y^j_1) \ -$$
$$(\partial_{\sigma_2} \ Y^i_1) \ (\partial_{\sigma_1} \ Y^j_0) \ - \ (\partial_{\sigma_2} \ Y^i_0) \ (\partial_{\sigma_1} \ Y^j_1). \quad (32)$$

after inserting the known solutions $Y^i_0(\sigma_1, \sigma_2)$ obtained in eqs-(22). Having solved for $Y^i_1(\sigma_1, \sigma_2)$ in eqs-(32) ($i, j, k = 1, 2, 3$) the next order solutions $Y^i_2(\sigma_1, \sigma_2)$ are obtained by solving the equations

$$\epsilon^{ijk} \ Y^k_2 \ = \ -\frac{1}{24} \ \Big(\ \ (\partial^3_{\sigma_1} \ Y^i_0) \ (\partial^3_{\sigma_2} \ Y^j_0) \ + \ 3 \ (\partial_{\sigma_1} \partial^2_{\sigma_2} \ Y^i_0) \ (\partial_{\sigma_2} \partial^2_{\sigma_1} \ Y^j_0) \ \Big) - \sigma_1 \leftrightarrow \sigma_2 \ +$$
$$(\partial_{\sigma_1} \ Y^i_2) \ (\partial_{\sigma_2} \ Y^j_0) \ + \ (\partial_{\sigma_1} \ Y^i_0) \ (\partial_{\sigma_2} \ Y^j_2) \ + \ (\partial_{\sigma_1} \ Y^i_1) \ (\partial_{\sigma_2} \ Y^j_1) \ \ - \ \sigma_1 \leftrightarrow \sigma_2. \quad (33)$$

Inserting the solutions for $Y^i_0(\sigma_1, \sigma_2)$, $Y^i_1(\sigma_1, \sigma_2)$ into eqs-(33) allows one to solve for $Y^i_2(\sigma_1, \sigma_2)$. Repeating this process for the next order in powers of L^2, and so forth via an iterative procedure, one can solve in principle for Y^i_n. Hence, a knowledge of the coefficient functions $Y^i_n(\sigma_1, \sigma_2)$ in the expansion (30) of $Y^i(\sigma_1, \sigma_2; L)$ yields the solutions to the Moyal deformations of the $SU(2)$ Lie-algebraic equations in (31), and consequently to the solutions $X^i(\sigma_1, \sigma_2, \tau; L)$ of the full fledged Moyal deformations of the light-cone spherical membrane equations of motion in a $5D$ flat target spacetime background in the temporal gauge $\mathcal{A}_\tau = 0$. Solutions to the full-fledged Moyal deformed equations for X could turn out to be *complex*-valued despite that solutions to the lowest order in L (25) were found to be *real*-valued. Therefore one needs to verify this by an explicit computation.

To proceed with the Moyal quantization of the bosonic membrane one must find the quantum operators $\hat{X}^i$ corresponding to the c-numbers $X^i(\sigma_1, \sigma_2, \tau; L)$. Before doing so, to simplify matters, we will recall the correspondence

$$\sigma_1 \leftrightarrow \frac{q}{\lambda_l} L; \quad \sigma_2 \leftrightarrow \frac{p}{\lambda_p} L, \quad \hbar \leftrightarrow L^2. \tag{10}$$

so we can work once again with the usual variables q, p and $\hbar$ in the Moyal deformation quantization procedure in phase spaces. The operator $\hat{Y} \leftrightarrow Y(q, p; \hbar)$ correspondence is [3]

$$\hat{Y}^k(\hat{q}, \hat{p}) = \int d\xi \ d\eta \ dq \ dp \ e^{i[\ \xi(\hat{p}-p) \ + \ \eta(\hat{q}-q)\ ]} \ Y^k(q, p; \hbar) \ \Rightarrow$$

$$Y^k(q, p; \hbar) = \int d\xi \ d\eta \ Trace \ \left(e^{-i[\ \xi(\hat{p}-p) \ + \ \eta(\hat{q}-q)\ ]} \ \hat{Y}^k(\hat{q}, \hat{p}) \right) =$$

$$\int dy \ e^{\frac{-2i\pi py}{\hbar}} \ < q + y| \ \hat{Y}^k(\hat{q}, \hat{p}) \ |q - y > . \tag{34}$$

Using the completeness relation, the orthonormality conditions

$$\sum_{m=1} |\Psi_m >< \Psi_m| = \mathbf{1}; \quad \int dq \ \Psi^*_m(q) \ \Psi_n(q) = \delta_{mn}$$

and

$$< q + y| \ \Psi_m > = \Psi_m(q + y); \quad < \Psi_n| \ q - y > = \Psi^*_n(q - y). \tag{35}$$

One can rewrite

$$Y^k(q, p; \hbar) = \int dy \ < q + y| \ \hat{Y}^k \ |q - y > \ e^{\frac{-2\pi ipy}{\hbar}} =$$

$$\int dy \ < q + y|\Psi_m >< \Psi_m| \ \hat{Y}^k \ |\Psi_n >< \Psi_n|q - y > \ e^{\frac{-2\pi ipy}{\hbar}} =$$

$$\int dy \ \Psi_m(q + y) \ Y^k_{mn} \ \Psi^*_n(q - y) \ e^{\frac{-2\pi ipy}{\hbar}}. \tag{36}$$

The lower/upper limits of the definite integrals in (36) are $-\infty, +\infty$, respectively. The matrix elements are defined as

$$Y^k_{mn} = < \Psi_m| \ \hat{Y}^k(\hat{q}, \hat{p}) \ |\Psi_n > = \int_{-\infty}^{\infty} dq \ \Psi^*_m(q) \ \hat{Y}(q; -i\hbar\frac{\partial}{\partial_q}) \ \Psi_n(q). \tag{37}$$

by replacing the $\hat{p}$ operator inside $\hat{X}^k$ by the differential $-i\hbar\partial_q$ acting on $\Psi_n(q)$. Hence, to sum up, one can write write the coordinate X^k /matrix X^k_{mn} correspondence as

$$X^k(q,p,\tau;\hbar) \ = \ f^k(\tau)\, Y^k(q,p;\hbar) \ = \ f^k(\tau) \int_{-\infty}^{+\infty} dy\, \Psi_m(q+y)\, Y^k_{mn}\, \Psi^*_n(q-y)\, e^{\frac{-2\pi i p y}{\hbar}} =$$

$$\int_{-\infty}^{+\infty} dy\ \Psi_m(q+y)\ X^k_{mn}(\tau)\ \Psi^*_n(q-y)\ e^{\frac{-2\pi i p y}{\hbar}}; \quad X^k_{mn}(\tau) \ = \ f^k(\tau)\, Y^k_{mn}. \quad (38)$$

Despite that the solutions (38) have a formal *similarity* to the solutions of the $SU(\infty)$ Moyal-Nahm equations [10] they are very different since the infinite number of functions $\Psi_m(q)$ for $m = 1,2,3,......$ are *not* the entries of a column matrix with a $finite$ number of components, as it was in the case of [10] by using spinors. Secondly, the large $N \times N$ matrices ($N \rightarrow \infty$) X^k_{mn}, for $k = 1,2,3$, are *not* the gamma matrices γ^k in four dimensions.

By recurring to the relation [6]

$$e^{\frac{-2\pi i p y}{\hbar}} f(q) \ * \ e^{\frac{-2\pi i p y'}{\hbar}} g(q) \ = \ e^{\frac{-2\pi i p (y+y')}{\hbar}}\ f(q+y')\ g(q-y). \qquad (39)$$

in the evaluation of $Y^{i_1} \ * \ Y^{i_2}$, where $Y^i(q,p;\hbar)$ is given by eq-(36), after some algebra involving a change of integration variables from y, y' to the new set of variables $u = y + y'; v = q + y - y'$ (where q is interpreted as a parameter), and using the normalization condition

$$\int d(q+y-y')\ \Psi_m(q+y-y')\ \Psi^*_s(q+y-y') \ = \ \delta_{ms}. \qquad (40)$$

one can explicitly construct the Weyl-Wigner-Groenewold-Moyal (WWGM) map of the product of two Weyl ordered operators $\hat{Y}^{i_1}(\hat{q},\hat{p})\ \hat{Y}^{i_2}(\hat{q},\hat{p})$ onto the star product of their *symbols* $Y^{i_1}(q,p;\hbar) \ * \ Y^{i_2}(q,p;\hbar)$ as follows

$$\mathbf{W}\ [\ \hat{Y}^{i_1}(\hat{q},\hat{p})\ \hat{Y}^{i_2}(\hat{q},\hat{p})\ ] \ = \ (\ Y^{i_1} \ * \ Y^{i_2}\ )(q,p;\hbar) \ =$$

$$\int dy\, dy'\, \Psi_m(q+y-y')\, Y^{i_1}_{mn}\, \Psi^*_n(q-y-y')\, \Psi_r(q+y+y')\, Y^{i_2}_{rs}\, \Psi^*_s(q+y-y')\ e^{\frac{-2\pi i p (y+y')}{\hbar}} =$$

$$\int d(y+y')\ \Psi_r(q+y+y')\ (Y^{i_2}\ Y^{i_1})_{rn}\ \Psi^*_n(q-y-y')\ \ e^{\frac{-2\pi i p (y+y')}{\hbar}} \ =$$

$$\int dy''\ \ < q+y''|\ \hat{Y}^{i_1}(\hat{q},\hat{p})\ \hat{Y}^{i_2}(\hat{q},\hat{p})\ |q-y''>\ \ e^{\frac{-2\pi i p y''}{\hbar}}; \quad y+y' = y''. \quad (41)$$

By induction, one can prove that

$$Y^{i_1} \ * \ Y^{i_2} \ * \ Y^{i_3} \ = \ \int dy\ \ < q+y|\ \hat{Y}^{i_1}\ \hat{Y}^{i_2}\ \hat{Y}^{i_3}\ |q-y>\ \ e^{\frac{-2\pi i p y}{\hbar}} \ =$$

$$\int dz\ \Psi_m(q+z)\ (Y^{i_3}\ Y^{i_2}\ Y^{i_1})_{mn}\ \Psi^*_n(q-z)\ e^{\frac{-2\pi i p z}{\hbar}}. \qquad (42)$$

etc ...... Notice the *reversal* in the ordering of the matrices Y^i_{mn} in the r.h.s of (42) with respect to the ordering of the $Y^i(q,p;\hbar)$ in the l.h.s of (42).

The quantum equations of motion of the membrane in the light-cone gauge are

$$D_\tau \hat{F}^{\tau j} \ + \ D_i \hat{F}^{ij} \ = \ 0 \ \Rightarrow \ g^{\tau\tau}\partial_\tau \ (\ \partial_\tau \hat{X}^j - (i\hbar)^{-1} \ [\ \hat{A}_\tau, \hat{X}^j \ ] \ ) \ -$$

$$g^{\tau\tau}(i\hbar)^{-1} \ [\ \hat{A}_\tau, \ \partial_\tau \hat{X}^j - (i\hbar)^{-1} \ [\ \hat{A}_\tau, \hat{X}^j \ ] \ ] - (i\hbar)^{-2} \ [\ \hat{X}_i, \ [\ \hat{X}^i, \hat{X}^j \ ] \ ] \ = \ 0. \quad (43)$$

$$D_i \ \hat{F}^{i\tau} \ = \ 0 \ \Rightarrow \ g^{\tau\tau}(i\hbar)^{-1}[\ \hat{X}_i, \ \partial_\tau \hat{X}^i - (i\hbar)^{-1} \ [\ \hat{A}_\tau, \hat{X}^i \ ] \ ] \ = \ 0. \quad (44)$$

and the solutions $\hat{X}^j$ are given by the WWGM inverse map $\mathbf{W}^{-1}[X^k(q,p,\tau;\hbar)]$ defined explicitly by eq-(34) where $X^k(q,p,\tau;\hbar)$ are the solutions to the Moyal deformed membrane equations of motion (11a, 11b) constructed in eq-(30) when $\mathcal{A}_\tau = 0$.

The light-cone-gauge Hamiltonian for a spherical membrane moving in a flat $5D$ target spacetime background (omitting the zero modes) is [1]

$$H \ = \ \int \ d^2\sigma \ \left(\ \frac{1}{2} P^i \ P_i \ + \ \frac{1}{4} \{X^i, \ X^j\}_{PB} \ \{X^i, \ X^j\}_{PB} \ \right); \quad i,j = 1,2,3 \quad (45)$$

Notice that the integration variables in (45) correspond to the spatial σ_1, σ_2 ones since the Hamiltonian "charge" H is defined for fixed-times; i.e. it is defined over a spatial (hyper) surface. It has the same form as the Hamiltonian associated with a $SU(\infty)$ Yang-Mills theory in $4D$ dimensionally reduced to *one* temporal dimension

$$H \ = \ Trace \ \left(\ \frac{1}{2}(\frac{DA^i}{D\tau})^2 \ + \ \frac{1}{4} \ [\ A^i(\tau), \ A^j(\tau) \ ] \ [\ A_i(\tau), \ A_j(\tau) \ ] \ \right); \ i,j = 1,2,3. \quad (46)$$

The Trace operation for $SU(\infty)$ has a correspondence with the integral $\int d^2\sigma \leftrightarrow \int dq \ dp$. The $SU(\infty)$ Lie-algebra valued gauge fields given by $A_i(\tau) = A^a_i \ T_a$, with T_a being the $N^2 - 1$ generators of $SU(N)$, in the $N \to \infty$ limit have a one-to-one correspondence with the membrane coordinates $X^i(\tau,q,p)$; the $SU(\infty)$ Lie-algebra commutators $[A^i, A^j]$ correspond to the Poisson brackets $\{X^i, X^j\}_{PB}$, and $(\frac{DA^i(\tau)}{D\tau})^2$ correspond to the $P^i P_i$ terms. The above correspondence between (45) and (46) can be made more precise by invoking the WWGM correspondence between operators and their symbols in phase space

$$\mathbf{W} \ (\ \hat{P}^i \ ) \ = \ \mathbf{W} \ (\ \frac{D\hat{X}^i}{D\tau} \ ) \ = \ P^i(q,p,\tau;\hbar) \ =$$

$$\frac{dX^i(q,p,\tau;\hbar)}{d\tau} \ - \ \frac{1}{i\hbar} \ \{\mathcal{A}_\tau, \ X^i(q,p,\tau;\hbar) \ \}_* \quad (47)$$

in the temporal gauge $\mathcal{A}_\tau = 0 \Rightarrow D_\tau \hat{X}^k = \partial_\tau \hat{X}^k = \hat{P}^k$.

$$\mathbf{W} \Big(\ [\hat{X}^i, \ \hat{X}^j] \ [\hat{X}_i, \ \hat{X}_j] \ \Big) \ = \ \{X^i, \ X^j\}_* \ * \ \{X_i, \ X_j\}_*. \qquad (48)$$

From the relations (48) one can establish the WWGM correspondence between the quantum Hamiltonian (density) operator $\hat{H}(\hat{q}, \hat{p}, \tau)$ and its symbol $\mathcal{H}(q, p, \tau; \hbar)$

$$\hat{H}(\hat{q}, \hat{p}, \tau) \ = \ \Big(\ \frac{1}{2} \hat{P}^i \ \hat{P}_i \ + \frac{1}{4(i\hbar)^2} \ [\hat{X}^i, \ \hat{X}^j] \ [\hat{X}_i, \ \hat{X}_j] \ \Big) (\hat{q}, \hat{p}, \tau; \hbar). \qquad (50)$$

$$\mathbf{W}(\hat{H}) \ = \ \mathcal{H}(q, p, \tau; \hbar) \ = \ \frac{1}{2} P^i \ * \ P_i \ + \frac{1}{4(i\hbar)^2} \ \{X^i, \ X^j\}_* \ * \ \{X_i, \ X_j\}_*. \ (51)$$

The quantity (51) is being referred as a Hamiltonian "density" since it is its integral $\int d^2\sigma[......]$ in (45) that defines the light-cone membrane's Hamiltonian in the classical limit. Explicit solutions (when the temporal gauge $\mathcal{A}_\tau = 0$ is chosen) for the light-cone membrane coordinate functions $X^i(q, p, \tau; \hbar) = f^i(\tau/\lambda) Y^i(q, p; \hbar)$ were presented in eqs-(30); the expression for $f^i(\tau/\lambda)$ are given by the elliptic functions as displayed in eqs-(25) and the functions $Y^i(q, p, \hbar)$ obeying the $SU(2)$ Lie-Moyal equations (31) can be determined via an *iterative* procedure as outlined in eqs-(32,33). The momentum functions $P^i(q, p, \tau; \hbar) = (df^i(\tau/\lambda)/d\tau) Y^i(q, p; \hbar)$ are also determined. Hence, a time-dependent Moyal Hamiltonian (density) $\mathcal{H}(q, p, \tau; \hbar)$ in (51) can explicitly be constructed to any order in powers of $\hbar$ (in powers of L^2) based on the particular class of solutions found in (30) to the Moyal deformed membrane equations of motion (11a, 11b).

For example, to *lowest* order, the positive definite Hamiltonian density (51) based on the solutions (30) is

$$\mathcal{H}_0(q, p, \tau) \ = \ A(\tau) \ p^4 \ + \ B(\tau) \ q^4 \ + \ C(\tau) \ p^2 q^2. \qquad (52)$$

where $A(\tau), B(\tau), C(\tau)$ are positive definite functions involving sums of products of the squares of elliptic functions. The Hamiltonian density (52) has a similar form as a time-dependent *quartic* harmonic oscillator with the key difference in the quartic momentum p^4 term instead of the standard quadratic one p^2. Before writing down the Schrodinger equation associated with $\mathcal{H}_0$ in (52), one must first perform a Weyl-ordering of the $p^2 q^2$ term in $\mathcal{H}_0$ as follows

$$p^2 q^2 \rightarrow \frac{1}{6} \ (\ p^2 q^2 + q^2 p^2 + pqpq + qpqp + pq^2 p + qp^2 q \ ). \qquad (53a)$$

and afterwards replace $p \rightarrow -i\hbar \partial/\partial q$ in order to write the time-dependent Schrodinger equation $\hat{H}(q, p = -i\hbar\partial_q, \tau)\Psi = i\hbar(\partial\Psi/\partial\tau)$. For example, the term $p^2 q^2 \Psi \rightarrow -\hbar^2 \partial_q^2 (q^2 \Psi) = -\hbar^2 (2\Psi + q^2 \partial_q^2 \Psi + 4q\partial_q \Psi)$. In the Heisenberg formulation of QM the operator equations of motion are

$$\frac{d\hat{q}}{d\tau} \ = \ (i\hbar)^{-1} \ [\hat{H}_0, \ \hat{q}]; \quad \frac{d\hat{p}}{d\tau} \ = \ (i\hbar)^{-1} \ [\hat{H}_0, \ \hat{p}] \ \Rightarrow$$

$$\hat{q}(\tau) \ = \ e^{-i/\hbar \int \hat{H}_0 d\tau} \ \hat{q}(\tau = 0) \ e^{i/\hbar \int \hat{H}_0 d\tau}; \ \ \hat{p}(\tau) \ = \ e^{-i/\hbar \int \hat{H}_0 d\tau} \ \hat{p}(\tau = 0) \ e^{i/\hbar \int \hat{H}_0 d\tau}. \tag{53b}$$

In the WWGM formulation of QM, the time-dependent Wigner function (the ensemble's diagonal density matrix in phase space) for a *pure* state is defined as

$$\rho_{nn}(q, p, \tau; \hbar) \ = \ \int \ dy \ \Psi^*_n(q-y, \tau) \ \Psi_n(q+y, \tau) \ e^{\frac{-2\pi i p y}{\hbar}}; \ \ \textit{no sum over } n \tag{54}$$

The Moyal analog of the Liouville equations are

$$\frac{\partial \rho_{nn}(q, p, \tau; \hbar)}{\partial \tau} \ = \ (i\hbar)^{-1} \ \{\mathcal{H}, \ \rho_{nn}\}_* \ = \ (i\hbar)^{-1} \ (\mathcal{H} \ * \ \rho_{nn} \ - \ \rho_{nn} \ * \ \mathcal{H}). \tag{55}$$

and have a direct physical correspondence with the time-dependent Schrodinger equation $\hat{H}(\tau)\Psi = i\hbar(\partial\Psi/\partial\tau) \Rightarrow \Psi(q, \tau) = (e^{-i/\hbar \int \hat{H}(\tau) d\tau}) \ \Psi(q, \tau = 0)$. From (55) one can infer that the temporal evolution of ρ in terms of the star-exponential is

$$\rho(q, p, \tau; \hbar) \ = \ (e_*^{-i/\hbar \int \mathcal{H}(\tau) \ d\tau}) \ * \ \rho(q, p, \tau = 0; \hbar) \ * \ (e_*^{i/\hbar \int \mathcal{H}(\tau) \ d\tau}). \tag{56}$$

where the star-exponential is $e^F_* = 1 + F + \frac{1}{2!} F * F \ + \ \frac{1}{3!} F * F * F$ ....

The expectation values of the quantum Hamiltonian operator can be written in terms of an integral involving the time-dependent Wigner function $\rho_{nn}(q, p, \tau; \hbar)$ as

$$< \Psi_n | \ \hat{H}(\tau) \ |\Psi_n > \ = \ \frac{\int \ dq \ dp \ \mathcal{H}(q, p, \tau; \hbar) \ \rho_{nn}(q, p, \tau; \hbar)}{\int \ dq \ dp \ \rho_{nn}(q, p, \tau; \hbar)}. \tag{57}$$

the diagonal elements $< \Psi_n | \ \hat{H}(\tau) \ |\Psi_n >$ correspond to the *expectation* values of the light-cone membrane's energy (density) configurations associated to the quantum states $|\Psi_n >$ at a given instant of time τ. In order to evaluate the integrals in (57) one needs to solve the Moyal-Liouville equations (55). If, and only if, the ρ_{nn} were time *independent*, the solutions would have simplified enormously because if $\partial_\tau \rho_{nn} = 0 \Rightarrow \{\mathcal{H}, \rho_{nn}\}_* = 0 \Rightarrow \mathcal{H} * \rho_{nn} = \rho_{nn} * \mathcal{H} = E_n \ \rho_{nn}$ and one would have recovered the energy (density) eigenvalues (real numbers) corresponding to the eigenfunctions of the double-star differential equations. However, in the time *dependent* case the situation changes considerably. In this case one may propose the following solutions to the differential equations

$$\mathcal{H}(q, p, \tau; \hbar) * \rho_{nn}(q, p, \tau; \hbar) \ = \ \lambda_{1,n}(\tau) \ \rho_{nn}(q, p, \tau; \hbar); \ \textit{no sum over } n.$$

$$\rho_{nn}(q, p, \tau; \hbar) * \mathcal{H}(q, p, \tau; \hbar) \ = \ \lambda_{2,n}(\tau) \ \rho_{nn}(q, p, \tau; \hbar); \ \ \lambda_{1,n}(\tau) \ \neq \ \lambda_{2,n}(\tau) \tag{58}$$

such that the solutions to the Moyal-Liouville eqs-(55) are given in terms of the initial-valued density $\rho_{nn}(q,p,\tau=0;\hbar)$ and the right/left "spectral" functions $\lambda_1(\tau), \lambda_2(\tau)$ as follows

$$\rho_{nn}(q,p,\tau;\hbar) \;=\; \left(e^{-i/\hbar \int \; (\; \lambda_{1,n}(\tau)-\lambda_{2,n}(\tau) \;) \; d\tau} \right) \rho_{nn}(q,p,\tau=0;\hbar). \quad (59)$$

To finalize one should mention that the proper treatment of star products in *curved* phase spaces involves the Fedosov star products [8]. Since the spherical membrane is not flat one should replace the Moyal star products for Fedosov star products and/or properly defined covariant star products compatible with the curved two-dimensional spatial surface. Furthermore, a Moyal treatment in field theory [11] is also possible here to deal with the membrane deformation quantization procedure by working with the canonical pair of conjugate field variables P^i, X^i; the Hamiltonian functional $H[P,X]$ of eq-(45) and the Wigner density functional $\rho[P,Q]$. By replacing $P \rightarrow -i(\delta/\delta X)$ ($\hbar=1$) in eq-(45), the Schrodinger wave-functional differential equation will be of the form

$$\int d^2\sigma' \left(- \frac{\delta^2}{\delta X^i(\sigma',\tau)^2} \;+\; \{X^i,\; X^j\}_{\sigma'} \; \{X^i,\; X^j\}_{\sigma'} \right) \Psi[X^i(\sigma',\tau)] \;=\; E\; \Psi[X^i(\sigma,\tau)] \quad (60)$$

where the integration domain in the l.h.s of (57) is of the form $\int_0^{\sigma_1} d\sigma_1' \int_0^{\sigma_2} d\sigma_2'$. For a spherical membrane one may choose the dimensionless coordinates $\sigma_1 = cos(\theta); \sigma_2 = \phi$. To solve (60) is not an easy task. For this reason, we opted to tackle the membrane deformation quantization procedure by working with the Hamiltonian density (51) and following the standard steps of the WWGM formalism of Quantum Mechanics. The relationship between our proposal presented here and the Matrix formalism of M-theory [12] warrants further investigation. Furthermore, rather than working with a non-covariant light-cone gauge and non-covariant Matrix models, it is desirable to begin with a fully *covariant* Matrix model description of membranes based on *ternary* algebraic structures and Nambu brackets [13], [14], [15], [16]. In order to move forward with the membrane quantization program, a starting point will be to find solutions to the time-dependent Schrodinger equation associated with the time-dependent quartic oscillator with q^4, p^4, q^2p^2 terms and that displays an important $p \leftrightarrow q$ *duality* symmetry (52).

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