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Title
Electroweak forces acting on TE, TM, TEM.

Abstract
In a previous paper [1] we showed that the energy impulse four vector of the propagation of electromagnetic fields into a waveguide and in free space can be described by a Dirac spinor ψ.
This suggest an analogy with for example TE-electron, TM-positron and possibly TEM-neutrino.
Aim of this work is an interpretation of the action, if any, of the electroweak gauge group $SU(2)\otimes U(1)$ on the before mentioned e.m. fields (TE, TM, TEM modes). This is based on the following observation: the energy impulse four vector is invariant under a global transformation of $SU(2)\otimes U(1)$**,** so ψ can be "gauged" in order to verify the effect not only of the electromagnetic force but also of the weak forces.
In other words, what are "weak forces", if any, on TE, TM and TEM?
Obviously this requires "a modification of the Dirac equation to accomodate the larger gauge group" (Hestenes, [2]).
This is in fact done here, and it is shown that the analogous of the "weak forces" can be roughly interpreted in the following way: the W boson acts as a horn antenna (receiving or transmitting), performing the transformation TEM ←→ TE, TM, giving or subtracting mass to the field; the Z° boson is as a radar target acting on the TEM (neutrinos) with a doppler frequency. Those objects have a mathematical counterpart in gauge fields.
No Higgs boson is needed in the theory.

[1] G. Bettini, "Algebra di Clifford ed equazione di Dirac per i campi in guida", available in vixra.

[2] D. Hestenes, "Clifford Algebra and the interpretation of quantum mechanics", in "Clifford Algebras and their Applications in Mathematical Physics", NATO ASI Series, Reidel (1986)

Electroweak forces acting on TE, TM, TEM.

Introduction and summary

What we intend to deal here (except suggestive asides) is not particle physics and the implications and consequences, but the description of the propagation of any electromagnetic field through its energy momentum vector.
Useful references (electromagnetism, Clifford algebra, etc..) are in [1]...... [7], and on electron theory in [8]......[14].
I think I have well demonstrated in [15] “Clifford Algebra and Dirac equation for TE, TM in waveguide" that the usual description of the fields in terms of V and I involve the Dirac equation. Conversely, I have shown that the Dirac equation provides (and distinguished) TE and TM fields and describes the polarization. As a byproduct, of course, are included in the same description the TEM.
Avoiding such considerations as Bohm would say "ontological", from a epistemological point of view that we face and we can study is at least a complete formal analogy. We say a Dirac equation which admits fields TE and TM each with both polarizations, and idem TEM fields at the speed of light. The formal analogy is with electrons positrons and neutrinos.
I identify the mass m with the cutoff frequency of a waveguide which allows the TE and TM fields in it. I finally identify the action and the effect of a gauge field generated by $\psi \to \psi' = \psi e^{-i\varphi(t)}$ (i.e., electrical potential), with a variation in size of the waveguide.
At this point there is an extension of the analysis that can only arouse curiosity, and that is by definition a problem from electronic engineer. It happens that:

1°) the group that leaves unchanged the total energy momentum vector of the electromagnetic field is not only the 'electromagnetic gauge group" $\psi \to \psi' = \psi e^{i\varphi}$ (Hestenes) but the group $SU(2) \otimes U(1)$;
2°) the application of a local transformation $SU(2) \otimes U(1)$ in physics is in the way we produce and describes the "unification of the electromagnetic forces and the electroweak force”.
Well in short you can not be satisfied until it is shown the effect of a local transformation $SU(2) \otimes U(1)$ on the electromagnetic field or the effect of the "electroweak force" (if any) on the TM and TE, TEM modes.

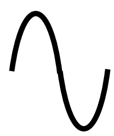
The two main difficulties are as follows.
Firstly, the difficulty of a Dirac equation that admits the group $e^{\mathrm{T} ji\beta + ij\nu - i\Phi + j\rho}$ i.e. with exponential from right as global transformation. This is necessary to introduce gauge fields of $SU(2) \otimes U(1)$.

Secondly, the difficulty to express mathematically the shift from the speed of light (or TEM) to a velocity v (i.e. TE TM) and vice versa, or "give a mass” to TEM.

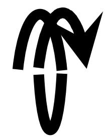
The weak force

I have repeatedly expressed (....to myself) the idea that the action of physical objects on a incident signal might be interpreted as the action of the electroweak force, or the action of the particles γ Z° e W, thus making them "physically visible”.
As for the electromagnetic force, which is responsible for the particle "photon", I have already given in [15] an interpretation in terms of TE and TM fields in a waveguide.
I tried several times but it seemed almost impossible to see the action of the particles W and Z °, assimilating to the action of objects operating on TE TM TEM modes.
Let us think for a moment the analogy "electron" ← → "electromagnetic signal, or if we want the analogy" methods of quantum mechanics "← →" electromagnetism and radar. "
Deviation or deceleration or acceleration of a signal in waveguide ([15]) match in physics or in quantum mechanics with the so-called " e.m. force " or " e.m. interaction”. This force is exerted by the particle photon, carrier of “e.m. force "
Particles W and Z ° carriers of the "weak force" should find their interpretation in the action of a radar target, or an object in waveguide, or similar.
You can support this point of view?
You can view the action of an object on an electromagnetic signal incident on it saying, 'Look, this is the action of the Z ° particle "or "This is like the action of the W particle "?
Now it seems that this is possible.
I intend to show that there are physical objects that operate on the TM and TE TEM similar to the action of W and Z ° particles on electrons and neutrinos.
For the moment confine ourselves to a qualitative examination.
I remember the similarities and help me with the drawings.

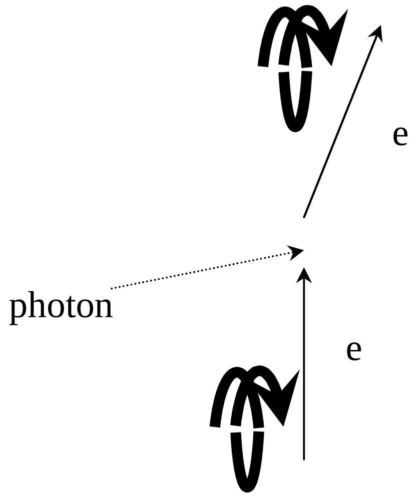
neutrino:
(TEM)
circular polarization, speed c



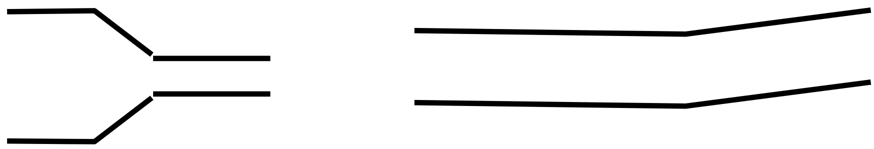
electron (and positron)
(TE or TM)
circular polarization, speed V



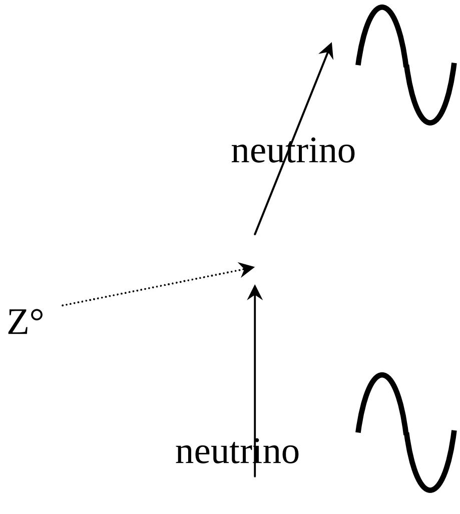
We begin to summarize the action of the photon. It slows down or speed up or diverts electrons



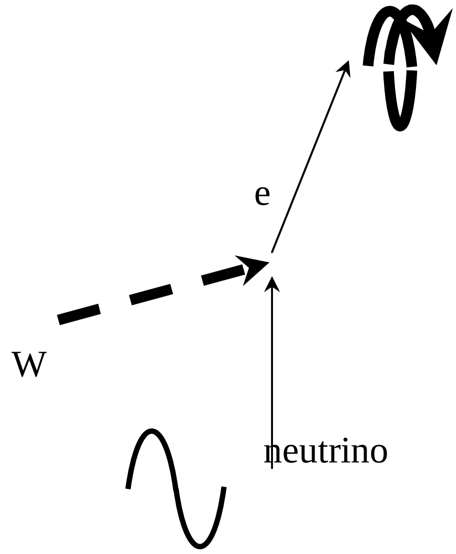
In a radar analogy is a fictitious " equivalent waveguide" that slows down or diverts TE or TM modes.



(I have given this interpretation in [15] In the following I give a simple interpretation of it).
We try instead to interpret the actions of the particles W and Z ° carriers of the "weak force". No photon is able to act (to divert or slow down) a neutrino, and this is consistent with the fact that no guide acts on a TEM being by definition a TEM free from any waveguide. Neutrino may act conversely the Z °, whose action, seen in terms of action of a radar target can be represented by a scattering from TEM to TEM, diverted or delayed (or accelerated) to equal polarization.



The object that can do this could simply be a radar target in any motion approaching or going away.
The more complex the action of the W particle. It is able to transform into electron neutrinos or vice versa (and is therefore a particle with charge).



In a radar analogy an object that transforms TE and TM in TEM or vice versa exists and is ...... a horn antenna. It operates the transition free space – waveguide and then the above transformations.

eletctron neutrino neutrino electron

The challenge is to translate the action of these objects in these mathematical operators to operate in a similar manner to the Z ° and W particles in quantum mechanics. Or conversely, to translate the mathematical formalism of the action of W and Z° particles in quantum mechanics in mathematical operators representing the electromagnetic action of the various objects.
This is the task that I will try to solve by studying the action of the group $SU(2)\otimes U(1)$ on TE TM and TEM fields.

The invariance with $SU(2)\otimes U(1)$ of the energy momentum vector

The basic observation from which to start is as follows.
I have seen in [15] that ψ was ultimately responsible for providing the four-vector $\psi\hat{\mathrm{T}}\psi^*$. It is assigned a spinor with 8 parameters while 4 are enough to assign a four vector. So there is a fourfold arbitrariness in ψ (Hestenes, [3], [4]), which is represented by the 4 parameters transformation:

(1) $\psi' \to \psi e^{\mathrm{T}ji\beta} e^{ij\nu} e^{-i\Phi} e^{j\rho}$

(or rather $\psi' \to \psi e^{\mathrm{T}ji\beta + ij\nu - i\Phi + j\rho}$ being the exponential non-commutative, except $e^{\mathrm{T}ji\beta}$).
It is indeed significant that an arbitrary transformation of this kind leaves $\psi\hat{\mathrm{T}}\psi^*$ unchanged (see also Appendix 1).
Now the group $e^{\mathrm{T}ji\beta + ij\nu - i\Phi + j\rho}$ is the group $SU(2)\otimes U(1)$. Specifically i, j, ij is $SU(2)$, the group of all space rotations, which leave $\hat{T}$ unchanged . The same thing does Tji. It follows that in the description of the electromagnetic field with a four-vector $\psi\hat{\mathrm{T}}\psi^*$ you can experiment on ψ a global transformation $SU(2)\otimes U(1)$ without altering the four velocity (or the energy momentum vector).
What happens if the transformation from global becomes local? If we impose the invariance of the Dirac equation for local transformations $SU(2)\otimes U(1)$ gauge fields arise following the usual techniques of gauge theories (covariant derivative and so on).
The problem, for continuing, however, is that (1) is first accepted as a legitimate global transformation into a new equation ("a modification of the Dirac equation to accommodate the larger gauge group", Hestenes [4]).
The Dirac equation actually is formulated as it only accepts the “electromagnetic gauge group" $e^{-i\Phi}$.

A modified Dirac equation that admits $SU(2)$

The starting points that made me arrive at the result are, say, three.
One is the following. I have always given some sort of disturbance, especially after I rewrote my way the Dirac equation, the fact that it was written:

(3) $\partial^* \psi = -\hat{i} m \psi i \hat{\mathrm{T}}$

(4) $\psi = \psi_1 + j\psi_2 + \mathrm{T}j\psi_3 + \mathrm{T}\psi_4$

with a sign (-) and not with the sign (+). Why (-) and not (+)? Among other things I had originally written with the (+), then I have to correct to derive the equations written with the signs as well as in many books.

(5)
$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_4 + \frac{\partial}{\partial z}\psi_3 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_1 = 0$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_3 - \frac{\partial}{\partial z}\psi_4 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_2 = 0$$

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_2 + \frac{\partial}{\partial z}\psi_1 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_3 = 0$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_1 - \frac{\partial}{\partial z}\psi_2 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_4 = 0$$

This is the first premise. I will say then as I worked.
Now the second premise. In (3) who prevents $SU(2)$ as global symmetry (j, ij from right) is the irritating position of i . If there was not, or was switched to the left, everything would be solved.
For the part ψ_+ of ψ, I can write the equation in the way I wanted to say

(6) $\partial^* \psi_+ = -\hat{i} m i \psi_+ \hat{\mathrm{T}}$

which means to take instead of ψ only the part:

(7) $\psi = \psi_+ = \psi_1 + \mathrm{T}j\psi_3$

So, it is apparent but illusory to say that (6) accepts $SU(2)$.
(For example if I change ψ with an exponential from right $\psi \to \psi' = \psi e^{ji\rho}$, substituting in (7) shows that terms arise in undesirable components $j\psi_2$ that are unacceptable, because ψ no longer commute with i which is the initial hypothesis).
This is the second premise. I will say then what I have got.

Third observation.
Solutions at rest of the Dirac equation $\partial^* \psi = -\hat{i} m \psi i \hat{\mathrm{T}}$ are:

(8)
$$\psi = e^{-i\omega t} \qquad \psi_1 \neq 0$$
$$\psi = je^{-i\omega t} \qquad \psi_2 \neq 0$$
$$\psi = \mathrm{T}jie^{+i\omega t} \qquad \psi_3 \neq 0$$
$$\psi = \mathrm{T}ji(je^{+i\omega t}) \qquad \psi_4 \neq 0$$

In my representation of radar polarization, the solution with only $\psi_1 \neq 0$ viz

(9) $$\psi = \psi_1 = e^{-i\omega t}$$

gives a TE at rest with an electric field

rotating in the x, y plane from x to y.
The solution with only $\psi_2 \neq 0$ i.e. $\psi = j\psi_2 = je^{-i\omega t}$ represents, as in the electron theory, the opposite rotation.
But all this happens just because I have arbitrarily chosen to represent the TE (satisfying the condition that we give the Klein Gordon equation), the Dirac equation $\partial^* \psi = -\hat{i} m \psi i \hat{\mathrm{T}}$. If I had chosen the equation with the opposite sign for m I would have solutions with opposite ω and therefore this time was rather $\psi = je^{+i\omega t}$ ($\psi_2 \neq 0$ solution) to represent the same rotation as before to the electric field, viz:

(11) $$\vec{E} = \psi_+ \hat{i} + \psi_- (-j)\hat{i} = \psi_- (-j)\hat{i} = je^{+\omega t}(-j)\hat{i} = e^{-i\omega t}\hat{i}$$

So you can use to represent the same rotation as before to the electric field with the Dirac equation (5), but with the opposite sign of m in the equations that provide ψ_2. We can also use democratically both components, combined in quadrature with sin and cos, so as to maintain the condition:

(12) $\qquad \left|\vec{E}\right| = 1$

That said, I have solved the whole issue leaning on these three facts, arriving the following proposal. Write the equations for the single direction of rotation $\vec{E} = e^{-\omega t}\hat{\imath}$ of the electric field, using the same equation (6)

(13) $\qquad \partial^{*}\psi = -\hat{\imath}mi\psi\hat{\mathrm{T}}$

but now also in agreement with the acceptability of the term ψ_2 and more generally of all, in the usual form:

(14) $\qquad \psi = \psi_1 + j\psi_2 + \mathrm{T}j\psi_3 + \mathrm{T}\psi_4$

Now not only apparent but it is also possible that (13) admits $SU(2)$.
(In fact now if for example changes ψ with an exponential from right $\psi \to \psi' = \psi e^{ji\rho}$, substituting in (13) arise terms in $j\psi_2$ components what time are acceptable).
Developed (13) get the equations (I write for plane wave in z for a reason which I'll explain later):

(15)
$$+\frac{\partial}{\partial z}\psi_3 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_1 = 0$$
$$-\frac{\partial}{\partial z}\psi_4 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_2 = 0$$
$$+\frac{\partial}{\partial z}\psi_1 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_3 = 0$$
$$-\frac{\partial}{\partial z}\psi_2 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_4 = 0$$

that coincide with the Dirac equation (5) for plane wave in z, but with the opposite sign of m in terms $m\psi_2$ and $m\psi_4$.

We interpret the solution with (10) and (11) which should be amended in $\vec{E} + Tji\vec{H} = \psi_+\hat{\imath} + \psi_-(-j)\hat{\jmath}$, with a $\hat{\jmath}$ in the part ψ_-, to maintain the condition:

(16) $$\left|\vec{E}\right|^2 = 1$$

(For additional considerations and details, see the 'Appendix 2).
I repeat that equation (13) represents the only rotations $e^{-\omega t}\hat{\imath}$ of electric field, namely that in the conventions IEEE is a polarization R ("right").
You can see that the equation gives the R TE and instead the TM L.
A similar equation gives the polarizations TE L (left) and TM R with a change of sign of m in the second member.
I am writing to remind you, with the appropriate index R and L:

(13 a) $$\partial^*\psi_R = -\hat{\imath}mi\psi_R\hat{\mathrm{T}}$$
(13 b) $$\partial^*\psi_L = +\hat{\imath}mi\psi_L\hat{\mathrm{T}}$$

Obviously then $m = 0$ is integrated into a single equation valid for both TEM R and TEM L:

(13c) $$\partial^*\psi = 0$$

Analysis

Eq. (13) is no longer the Dirac equation, but it is a "restricted" (or expanded?) Dirac equation. But proves easily (skip steps) that:
- is invariant under rotations around the z axis;
- is relativistic-invariant for any velocity along the z axis;
- does not satisfy the Klein Gordon equation in general, but satisfy for $\partial * = j\frac{\partial}{\partial z} + \mathrm{T}\frac{\partial}{\partial \tau}$, i.e. for plane waves in z.

The whole than enough to study TE TM TEM along z.
I put the (13) to a long series of tests on which I do not want to dwell. I recall that is (perhaps) the most significant and that the effect of a $SU(2)$ gauge field namely that generated by

(17) $$\psi \to \psi' = \psi e^{+ijWt} .$$

Equation (13) with the introduction of a suitable "covariant derivative" (for details see Appendix 3), becomes:

(18) $$\partial^* \psi + \hat{i} mi \psi \hat{\mathrm{T}} - T\psi ijW = 0$$

With long, but canonical developments:

(19)
$$+\frac{\partial}{\partial z}\psi_3 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_1 - \psi_2 * iW = 0$$
$$-\frac{\partial}{\partial z}\psi_4 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_2 + \psi_1 * iW = 0$$
$$+\frac{\partial}{\partial z}\psi_1 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_3 + \psi_4 * iW = 0$$
$$-\frac{\partial}{\partial z}\psi_2 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_4 - \psi_3 * iW = 0$$

$W = 0$ yelds the following TE solution:

(20)
$$\psi_1 = e^{-i\omega t + ik_z z} \qquad \psi_2 = e^{+i\omega t - ik_z z}$$
$$\psi_3 = Be^{-i\omega t + ik_z z} \qquad \psi_4 = -Be^{+i\omega t - ik_z z}$$
$$B = \frac{\sqrt{\omega - \omega_0}}{\sqrt{\omega + \omega_0}} \qquad k_z^2 = \omega^2 - \omega_0^2 \qquad \omega_0 \equiv m$$

A similar solution applies to the TM.
For $m=0$ and $W=0$ instead, you have (among many) the solution TEM (TEM "right"):

$$\psi_1 = e^{-i\omega t+ik_z z} \qquad \psi_2 = e^{+i\omega t-ik_z z}$$

(21) $$\psi_3 = e^{-i\omega t+ik_z z} \qquad \psi_4 = -e^{+i\omega t-ik_z z}$$

$$k_Z = \omega$$

In short (19) represent for $W=0$ TE and TM fields in a waveguide with cutoff $\omega_0 = m$, and propagate in the way that you would expect.
For $m=0$ and $W=0$ are TEM fields.
More complex the situation $W \neq 0$. Here I have had various problems, until I realized that it must be assumed that the coupling of magnetic components with the field created by $\psi \to \psi' = \psi e^{+ijWt}$ be through a "coupling charge " of opposite sign. Under these conditions in (19) $\psi_4 * iW$ and $\psi_3 * iW$ changes sign and so the equations are "right":

$$+\frac{\partial}{\partial z}\psi_3 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_1 - \psi_2 * iW = 0$$

$$-\frac{\partial}{\partial z}\psi_4 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_2 + \psi_1 * iW = 0$$

(22)
$$+\frac{\partial}{\partial z}\psi_1 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_3 - \psi_4 * iW = 0$$

$$-\frac{\partial}{\partial z}\psi_2 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_4 + \psi_3 * iW = 0$$

Examined the effect of gauge fields.
Seeking a solution as an attempt (to be checked a posteriori) that it has:

$$\psi_1 = \psi_2 *$$

$$\psi_2 = \psi_1 *$$

(23)
$$\psi_3 = -\psi_4 *$$

$$\psi_4 = -\psi_3 *$$

Equations (22) becomes

$$+\frac{\partial}{\partial z}\psi_3+\left(\frac{\partial}{\partial\tau}+i(m-W)\right)\psi_1=0$$

$$-\frac{\partial}{\partial z}\psi_4+\left(\frac{\partial}{\partial\tau}-i(m-W)\right)\psi_2=0$$

(24)

$$+\frac{\partial}{\partial z}\psi_1+\left(\frac{\partial}{\partial\tau}-i(m-W)\right)\psi_3=0$$

$$-\frac{\partial}{\partial z}\psi_2+\left(\frac{\partial}{\partial\tau}+i(m-W)\right)\psi_4=0$$

of clear solution:

$$\psi_1=e^{-i\omega t+ik_z z}\qquad\qquad \psi_2=e^{+i\omega t-ik_z z}$$

(25) $$\psi_3=Be^{-i\omega t+ik_z z}\qquad\qquad \psi_4=-Be^{+i\omega t-ik_z z}$$

$$B=\frac{\sqrt{\omega-(m-W)}}{\sqrt{\omega+(m-W)}}\qquad\qquad k_z^2=\omega^2-(m-W)^2$$

But (25) also fulfill the assumptions attempt (23) and therefore are the solution.
In (22) then the solutions $\psi_1,\psi_2\neq 0$ at rest are TE fields which propagate in a waveguide with cutoff $\omega_0=m-W$.
Also solutions with $\psi_3,\psi_4\neq 0$ at rest are TM fields which propagate in the same waveguide with cutoff $\omega_0=m-W$.
In essence, the application of this $SU(2)$ gauge field is equivalent to the transition in a waveguide with cutoff $\omega_0=m-W$, with $k_z^2=\omega^2-(m-W)^2$ and group velocity:

(26) $$v_g=\frac{d\omega}{dk_z}=\sqrt{1-\frac{\omega_{0,2}^2}{\omega^2}}$$ where $\omega_{0,2}=(m-W)$

The particularity of this waveguide is that for $W=m$ becomes a "horn antenna", that disappears, i.e. the field becomes a TEM.
The conclusion is that the gauge transformations $\psi\rightarrow\psi'=\psi e^{+ijWt}$ in equations (13) of TE TM is equivalent to the action of a "horn antenna" in transmission.

From a TEM is obtained exactly the opposite result, from TEM to TE, which in quantum mechanics is like saying “from neutrino to electron / positron”.
Do the steps about.
We start from the equations of a TEM

$$+\frac{\partial}{\partial z}\psi_3+\frac{\partial}{\partial\tau}\psi_1=0$$

$$-\frac{\partial}{\partial z}\psi_4+\frac{\partial}{\partial\tau}\psi_2=0$$

(27)

$$+\frac{\partial}{\partial z}\psi_1+\frac{\partial}{\partial\tau}\psi_3=0$$

$$-\frac{\partial}{\partial z}\psi_2+\frac{\partial}{\partial\tau}\psi_4=0$$

Consider, for example, the solution TEM R (“right”)

$$\psi_1=e^{-i\omega t+ik_z z} \qquad \psi_2=e^{+i\omega t-ik_z z}$$

(28)

$$\psi_3=Be^{-i\omega t+ik_z z} \qquad \psi_4=-Be^{+i\omega t-ik_z z}$$

$$B=1 \qquad k_z^2=\omega^2$$

We now introduce in (27) a gauge field as in (22) but opposite sign.
(I assume it obvious that if the gauge transformation $\psi\to\psi'=\psi e^{+ijWt}$ was equivalent to the action of a "horn antenna" in transmission, will serve the opposite transformation $\psi\to\psi'=\psi e^{-ijWt}$ for the action of a receiving antenna). The equations of TEM under the action of gauge field thus become the following:

$$+\frac{\partial}{\partial z}\psi_3+\frac{\partial}{\partial\tau}\psi_1+\psi_2*iW=0$$

$$-\frac{\partial}{\partial z}\psi_4+\frac{\partial}{\partial\tau}\psi_2-\psi_1*iW=0$$

(29)

$$+\frac{\partial}{\partial z}\psi_1+\frac{\partial}{\partial\tau}\psi_3+\psi_4*iW=0$$

$$-\frac{\partial}{\partial z}\psi_2+\frac{\partial}{\partial\tau}\psi_4-\psi_3*iW=0$$

Is it possible that these equations provide a TE R ("right")?
Look for a possible solution once again in an effort hypothesis (to be checked a posteriori):

(23)
$$\psi_1 = \psi_2 *$$
$$\psi_2 = \psi_1 *$$
$$\psi_3 = -\psi_4 *$$
$$\psi_4 = -\psi_3 *$$

In this case (29) become:

(30)
$$+\frac{\partial}{\partial z}\psi_3 + \frac{\partial}{\partial \tau}\psi_1 + \psi_1 iW = 0$$
$$-\frac{\partial}{\partial z}\psi_4 + \frac{\partial}{\partial \tau}\psi_2 - \psi_2 iW = 0$$
$$+\frac{\partial}{\partial z}\psi_1 + \frac{\partial}{\partial \tau}\psi_3 - \psi_3 iW = 0$$
$$-\frac{\partial}{\partial z}\psi_2 + \frac{\partial}{\partial \tau}\psi_4 + \psi_4 iW = 0$$

namely those of a TE "right" in a waveguide with cutoff $\omega_0 = W$, with solution:

(31)
$$\psi_1 = e^{-i\omega t + ik_z z} \qquad \psi_2 = e^{+i\omega t - ik_z z}$$
$$\psi_3 = Be^{-i\omega t + ik_z z} \qquad \psi_4 = -Be^{+i\omega t - ik_z z}$$
$$B = \frac{\sqrt{\omega - \omega_o}}{\sqrt{\omega + \omega_0}} \qquad k_z^2 = \omega^2 - \omega_0{}^2 \qquad \omega_0 = W$$

There is a posteriori verified the (23), so this is the solution sought.
The conclusion is that the gauge transformation $\psi \to \psi' = \psi e^{-ijWt}$ in equations (27) of TEM is equivalent to the action of a receiving "horn antenna", closed on a waveguide with cutoff $\omega_0 = W$.
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Interpretations: the analogy with the W particle

We have seen that the TEM (28) with frequency ω is reported in a waveguide with $\omega_0 = W$, transforming in TE (31).
With easy but lengthy calculations (Appendix 4) and with the normalization:

(32) $$\psi\psi^* = \frac{\omega_0}{\omega}$$

the four-velocity is:

(33) $$\psi(-\hat{T})\psi^* = -\hat{T} + \frac{k_Z}{\omega}\hat{k}$$

It varies between $(-\hat{T})$ at rest and $(-\hat{T} + \hat{k})$.in the extreme case of TEM, speed c (Note: c = 1 in the units used).
Appears explicitly the group velocity of the wave along the z axis:

(34) $$v_g = \frac{d\omega}{dk_z} = \sqrt{1 - \frac{\omega_0^2}{\omega^2}} = \frac{k_Z}{\omega}$$

But there is some further interesting interpretation that are worth highlighting.
Compute $\psi\hat{k}\psi^*$ resulting:

(35) $$\psi\hat{k}\psi^* = \frac{\omega_0}{\omega} e^{-2i\omega t + 2ik_Z z}\hat{i}$$

We can combine (33) and (35) in a single expression to obtain:

(36) $$\psi(-\hat{T} + \hat{k})\psi^* = -\hat{T} + \frac{k_Z}{\omega}\hat{k} + \frac{\omega_0}{\omega} e^{-2i\omega t + 2ik_Z z}\hat{i} = -\hat{T} + \hat{\ell}$$

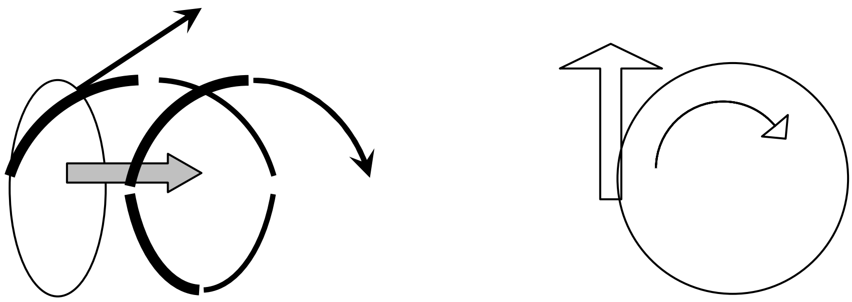
where $\hat{\ell}$ is a spacelike unit vector, as a result of (31):

(37) $$\hat{\ell} = \frac{k_Z}{\omega}\hat{k} + \frac{\omega_0}{\omega} e^{-2i\omega t + 2ik_Z z}\hat{i} \rightarrow \hat{\ell}\hat{\ell} = 1$$

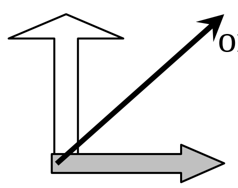
$\hat{\ell}$ parametrically describes a helix, and $(-\hat{T} + \hat{\ell})$ is a four-velocity, which describes a propagation at speed of light along the helix.

We are therefore faced with a fact and a possible interpretation.
The fact is the following:
transformation (36) transforms the four-velocity of a e.m. field along z moving at the speed of light, changing it in the four-velocity of an e.m. field at the speed of light in motion on a helix.
Possible interpretation is this:
gauge transformation $\psi \rightarrow \psi' = \psi e^{+ijWt}$ simulates the action of the receiving antenna for which a TEM is captured in a waveguide in the form of TE, traveling from moment to moment on a helix at the speed of light.

Graphically the field that travels the helix at the speed of light is indicated by the black arrow in the figures, the arrow represents the velocity on helix.



The actual speed of translation along the axis of the waveguide is the gray arrow.
The speed of rotation in the circulation is the white arrow.
Among the three velocities $v_g{}^2 + (\frac{\omega_0}{\omega})^2 = 1$ applies at all times (Note: c = 1 is the speed of light in the units used).
Among the omega, omega cutoff and k a similar relationship holds, namely
$\omega^2 = \omega_0{}^2 + k_z{}^2$.
Formulas are consistent with waveguide propagation.



Accordingly, we interpret the effect of W appearing in (25).
I will speak for brevity and more or less appropriate of "weak interaction".
Examine the emission of a TEM (from TE to TEM), or “how to free trapped light”, or what makes a "horn antenna" in transmission.

I start from equations (22) which solution is a TE field that propagates in a waveguide with cutoff $\omega_{0,2} = m - W$ and / or a TM that propagates in the same waveguide always with cutoff $\omega_{0,2} = m - W$.

Therefore (26) apply and also

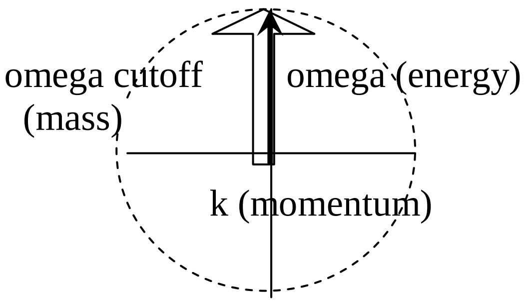
$$\omega^2 = (m - W)^2 + k_z{}^2$$

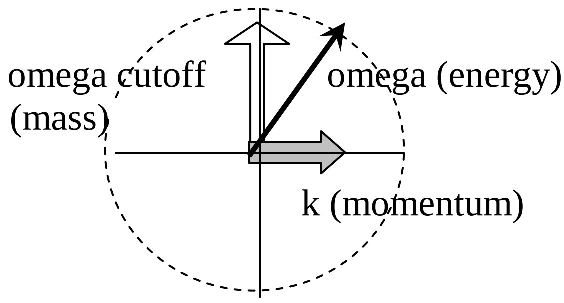
Use this as a key to reading.

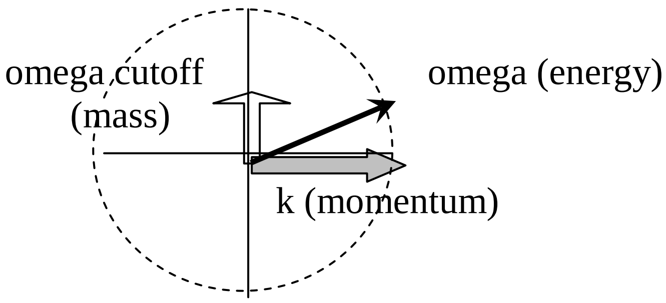
Before applying W the field was in a guide with omega cutoff equal to m (especially if $\omega = m$ the field was at cutoff).

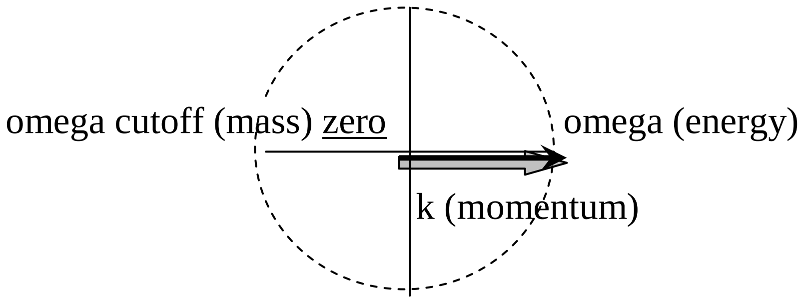
Let $\omega = Cost.$ as it is for a TE in waveguide.

The drawings that represent the effect of the weak interaction for increasing values of W are therefore these:





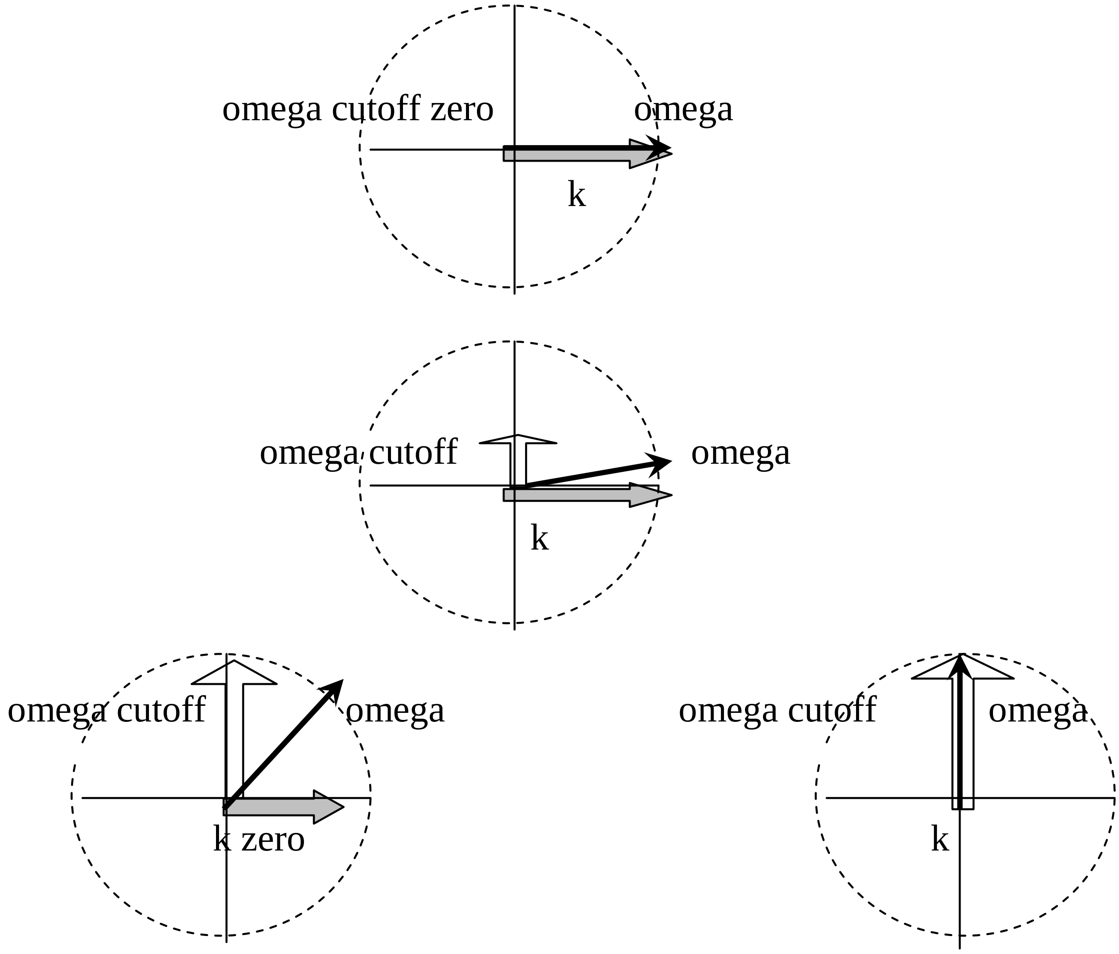




If $W = m$ the field becomes a TEM.

We note that the weak interaction does not yield and does not absorb energy but redistributes the energy which from energy in a circle becomes traveling energy.

Now examines the transformation from TEM to TE or "how to give mass" to a TEM, namely what makes a receiving horn antenna.
I always start from the equations (22) with $m = 0$ having as solution "before treatment" ($W = 0$) a TEM field.
"After the treatment", i.e. after the application of W switch to TE with "mass" W (note: to do so is to be understood that the horn antenna in reception to act opposite to the transmission).
The key to understanding this time $\omega^2 = W^2 + k_z^2$ with constant omega.
W appears in the formula as "omega cutoff" and is zero for $W = 0$.
The drawings below represent the effect of the weak interaction for increasing values of W, from $W = 0$ to $W = \omega$:
It starts from a TEM and you come to TE with omega cutoff equal to W.



If as extreme case $W = \omega$ a TE forms at rest, at the cutoff frequency.
Even in this case the weak interaction does not yield and does not absorb energy but redistributes energy from traveling energy to energy in a circle.

Another analogy: the γ particle

We show that the TE (20) can be slowed or speeded up in waveguide through the "electromagnetic force" represented by the gauge field created by the $SU(2)$ transformation $\psi \to \psi' = \psi e^{-iUt}$.
Because the field (20) satisfies (13) and this admits $SU(2)$ and in particular accept the transformation $\psi \to \psi' = \psi e^{-iUt}$, I can introduce this as a gauge field.
The introduction of a suitable "covariant derivative" leads to:

$$\partial^* \psi + \hat{i} m i \psi \hat{\mathrm{T}} + T \psi i U = 0 \tag{38}$$

and in extended form:

$$
\begin{aligned}
&+\frac{\partial}{\partial z}\psi_3 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_1 + \psi_1 iU = 0 \\
&-\frac{\partial}{\partial z}\psi_4 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_2 + \psi_2 iU = 0 \\
&+\frac{\partial}{\partial z}\psi_1 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_3 + \psi_3 iU = 0 \\
&-\frac{\partial}{\partial z}\psi_2 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_4 + \psi_4 iU = 0
\end{aligned} \tag{39}
$$

As you will see as a posteriori justification, the coupling of components ψ_2, ψ_4 with the field created by $\psi \to \psi' = \psi e^{-iUt}$ must be assumed through a " coupling charge" of opposite sign. Under these conditions in (38) $\psi_2 iU$ and $\psi_4 iU$ changes sign and so the equations "right":

$$
\begin{aligned}
&+\frac{\partial}{\partial z}\psi_3 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_1 + \psi_1 iU = 0 \\
&-\frac{\partial}{\partial z}\psi_4 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_2 - \psi_2 iU = 0 \\
&+\frac{\partial}{\partial z}\psi_1 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_3 + \psi_3 iU = 0 \\
&-\frac{\partial}{\partial z}\psi_2 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_4 - \psi_4 iU = 0
\end{aligned} \tag{39}
$$

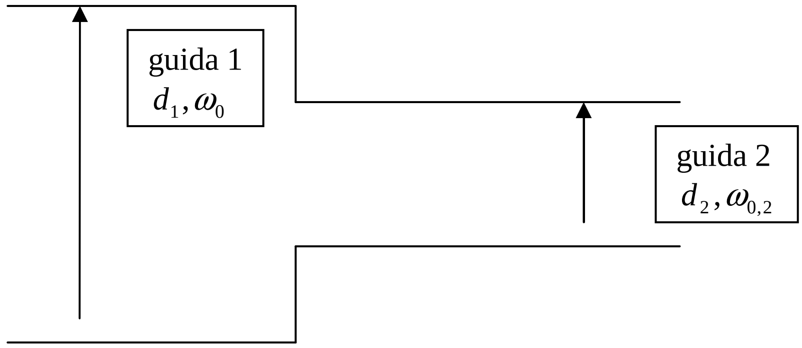
We seek a solution in the form:

(40)
$$\psi_1 = e^{-i\omega t + ik_z z} \qquad \psi_2 = e^{+i\omega t - ik_z z}$$
$$\psi_3 = Be^{-i\omega t + ik_z z} \qquad \psi_4 = -Be^{+i\omega t - ik_z z}$$

with B unknow.
Substituting with some passage we find that the solution exists and has:

(41)
$$B = \frac{\sqrt{(\omega - U) - \omega_0}}{\sqrt{(\omega + U) + \omega_0}} \qquad k_z^2 = (\omega - U)^2 - \omega_0^2$$

In short, the field is proceeding in waveguide with decreased ω.
I've interpreted in [15] this fact with the propagation of TE in a guide 2, "equivalent waveguide" with a different cutoff $\omega_{0,2}$ (i.e. size d_2).



For this we use the $v_g = \dfrac{d\omega}{dk_z}$, formula for the group velocity in the waveguide.
From (41) is obtained

(42) $$k_z = \sqrt{(\omega - U)^2 - \omega_0^2}$$ so

(43) $$v_g = \frac{d\omega}{dk_z} = \sqrt{1 - \frac{\omega_{0,2}^2}{\omega^2}}$$ where

(44) $$\omega_{0,2} = \frac{\omega_0}{1 - \frac{U}{\omega}}$$

Rather than considering an "equivalent waveguide" with cutoff frequency $\omega_{0,2}$, let's instead to find according to (41) that the field is moving in the same guide, which has cutoff ω_0, but someone or something has changed its ω in ($\omega - U$).
That something is detectable by the mathematical point of view in the operation $\psi \to \psi' = \psi e^{-iUt}$, who acted on ψ decreasing the ω.
From the electromagnetic point of view such an action is produced by the interaction with an object in the waveguide that gives a Doppler frequency (here negative).
And 'certainly this interpretation is easier and more immediate than the last.
Anyway , the resulting action is that of "electromagnetic force" exerted by the γ particle.
We return to equations (39) and try to get a TM solution.
From the electromagnetic point of view with the same parameters ω, ω_0 makes no difference whether it is a TM or a TE, in the sense that the final ω result of the interaction with an object in waveguide that gives a Doppler frequency producing ($\omega - U$), must be the same for a TE or a TM. However to achieve this is to be assumed in place of (39) other equations in which U must change of sign, as happens with the change of sign of electric charge ($U = \pm eV$) in the Dirac equationfor the positron.
Precisely the equations must become:

(45)
$$\begin{aligned}
&+\frac{\partial}{\partial z}\psi_3 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_1 - \psi_1 iU = 0 \\
&-\frac{\partial}{\partial z}\psi_4 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_2 + \psi_2 iU = 0 \\
&+\frac{\partial}{\partial z}\psi_1 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_3 - \psi_3 iU = 0 \\
&-\frac{\partial}{\partial z}\psi_2 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_4 + \psi_4 iU = 0
\end{aligned}$$

They actually have TM solution even with:

(46)
$$B = \frac{\sqrt{(\omega - U) - \omega_0}}{\sqrt{(\omega - U) + \omega_0}} \qquad k_z^2 = (\omega - U)^2 - \omega_0^2$$

From this follows the same velocity in the guide that would have a TE.

Visual interpretation

How can we "see" the effects of W and γ considered above? How can we more generally portray the effect of the various gauge fields?
We take a few steps back.
From a mathematical point of view the relationship between the electromagnetic interactions and weak interactions is expressed by the fact that all arise from gauge fields generated by $SU(2)\otimes U(1)$transformations.
For each transformation corresponds to a field.
The changes involved are those of the type:

$\psi \to \psi' = \psi e^{-iUt}$

$\psi \to \psi' = \psi e^{-jWt}$

$\psi \to \psi' = \psi e^{-ijWt}$

$\psi \to \psi' = \psi e^{-TjiZt}$

therefore exponential generators of rotations $\hat{i}\hat{j} = i$, or $\hat{i}\hat{k} = j$, or $ij = \hat{k}\hat{j}$ and where necessary $\hat{i}\hat{j}\hat{k}\hat{T} = Tji$.
Now, as he says Hestenes [12] in the Weinberg Salam theory of electroweak interactions $SU(2)\otimes U(1)$ appears as an internal symmetry in an <u>abstract</u> space.
Instead, always says Hestenes (I translate freely his thoughts) should be possible to give a geometric interpretation in <u>real</u> spacetime.
I have proposed this and something more: it must be possible to give a geometric interpretation also on the TE, TM and TEM.
I mean to interpret the effect of the electroweak force not only on elementary particles (electrons, neutrinos, etc..) but also on normal fields TE, TM and TEM.
Us refer now specifically to the action of the generators of rotations $\hat{i}\hat{j} = i$, or $\hat{i}\hat{k} = j$, or $ij = \hat{k}\hat{j}$ and where necessary $\hat{i}\hat{j}\hat{k}\hat{T} = Tji$.
Why the generators $\hat{i}\hat{j} = i$, $\hat{i}\hat{k} = j$, $ij = \hat{k}\hat{j}$ and $\hat{i}\hat{j}\hat{k}\hat{T} = Tji$ are involved?
Let's step back here, too, running from $\psi\hat{\mathrm{T}}\psi^*$.
The expression

(47) $\qquad \psi\hat{\mathrm{T}}\psi^*$

provides the energy momentum vector of the body in question (here the <u>mode</u> under consideration, TE, TM, TEM) as described by the spinor ψ .
We can interpret the action of ψ in (47) as that of "kick-start" the body, describing both the correct values of energy and momentum.
We operate one of the $SU(2)\otimes U(1)$transformations, for example $\psi \to \psi' = \psi e^{-i\Phi}$.

The (47) become:

(48) $\psi'\hat{T}\psi'^* = \psi(e^{-i\Phi}\hat{T}e^{+i\Phi})\psi^*$

From this we understand that if the transformation is such that:

(49) $(e^{-i\Phi}\hat{T}e^{+i\Phi}) = \hat{T}$

i.e. "leave $\hat{T}$ unchanged", nothing has changed in (47) because:

(50) $\psi'\hat{T}\psi'^* = \psi(e^{-i\Phi}\hat{T}e^{+i\Phi})\psi^* = \psi\hat{T}\psi^*$

$SU(2)\otimes U(1)$ is precisely "the group of all transformations that leave $\hat{T}$ unchanged".
Specifically $SU(2)$ which includes the generators $\hat{i}\hat{j} = i$, $\hat{i}\hat{k} = j$, $ij = \hat{k}\hat{j}$, is the group of all spacelike rotations that leave $\hat{T}$ unchanged.
As a result, they do not change anything in the energy and momentum of the body, for all conditions of motion.
So you might think are useless?
The fact is that they do not change energy and momentum of the body if they are global changes, as in (47), i.e. with constant angle, independent from coordinates.
We interpret this fact graphically.
The unit vectors $\hat{i}$ $\hat{j}$ $\hat{k}$ $\hat{\mathrm{T}}$ are the unit vectors of the axes x, y, z and of the time axis but, following Hestenes, consider it as axes "stuck" to the body. So they are also a trio of unit vector $\hat{i}$ $\hat{j}$ $\hat{k}$ which indicates the attitude, while $\hat{\mathrm{T}}$ indicates the "proper time". The spinor ψ determines the rotations on them. Particularly if ψ is unitary the unit vectors

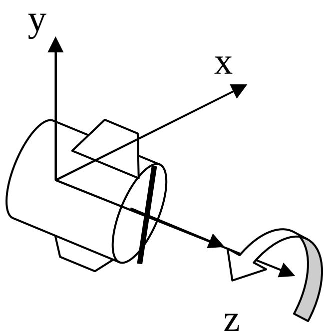
(51)
$$\psi\hat{i}\psi^* = \hat{e}_1$$
$$\psi\hat{j}\psi^* = \hat{e}_2$$
$$\psi\hat{k}\psi^* = \hat{e}_3$$
$$\psi\hat{\mathrm{T}}\psi^* = \hat{e}_0 = \hat{u}$$

form a set of axes rotated with respect to $\hat{i}$ $\hat{j}$ $\hat{k}$ $\hat{\mathrm{T}}$.
(Note: if ψ is a Lorentz rotation, "set in motion" the body. If, however, is one of the $SU(2)\otimes U(1)$rotations, which is precisely "the group of all transformations that leave $\hat{T}$ unchanged, nothing happens, at least for $\hat{T}$).

With reference to (47) and (48) a moment's reflection shows us that any of the $SU(2)\otimes U(1)$ rotations, eg $e^{-i\Phi}$, can be interpreted as a rotation applied before ψ acting , and before has set in motion the body.
This aspect is very important.
Confine ourselves to $SU(2)$ spacelike rotations , with generators $\hat{i}\hat{j}=i$, $\hat{i}\hat{k}=j$, $ij=\hat{k}\hat{j}$:

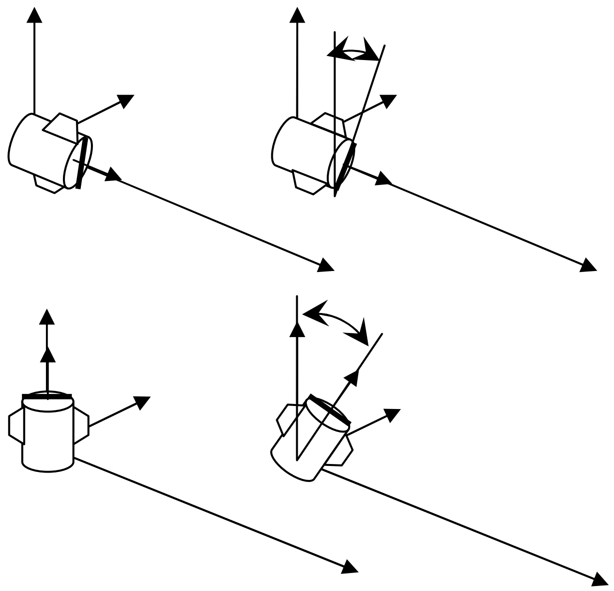
we can identify any of the $SU(2)$ rotations leaving $\hat{T}$ unchanged as a change in attitude of the body at rest.

We proceed from here, for an explanation of course approximate, but which has the merit to provide a visual picture of the action of gauge fields and how they determine the electromagnetic force and weak force.
Reasons for an electromagnetic field in circular polarization.
We take a first image of the field as a body to which it is stuck a system of axes $\hat{i}$ $\hat{j}$ $\hat{k}$ and especially $\hat{k}$ represents the axis of rotation. Implicitly admits that the body or the field "whisk" around the axis $\hat{k}$. In the Hestenes interpretation $\hat{k}$ is the spin axis.
A figure representing the body as a small satellite "spinning" around its axis.

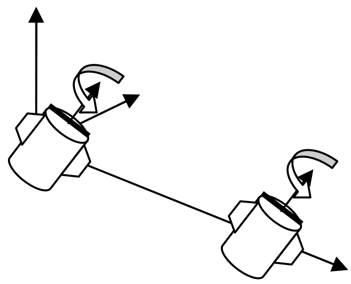


Consider the fixed angle rotations, global transformations of $SU(2)$.
We hypothesized to be able to assimilate these $SU(2)$ rotations leaving $\hat{T}$ unchanged (and then do not interact with the energy momentum vector) to a "change in attitude" of the body at rest.

Is quite reasonable to think that when the body "spinning" has however changed attitude



does not change its total energy, when it is stationary, nor his momentum, when it is in motion.
In practice will continue its motion with the conservation of momentum, energy and angular momentum.



But the situation changes (and changes to interpretation) when the angles are for example a function of time.
Let us begin with the simplest that is .... the electromagnetic force, in contrast to the next we'll see who is the weak force.
It is generated by a transformation $\psi \to \psi' = \psi e^{-i\varphi(t)}$ or more explicitly $\psi \to \psi' = \psi e^{-iUt}$.
This included in the Dirac equation produces a decrease of ω to $(\omega - U)$.
Neglecting to retrace all the technical details have already been examined elsewhere, in the end U appears as an energy additive or a ω additive (or subtractive as here) that someone has communicated to the field (the body).
(note: we should not simplistically think that to $e^{-i\omega t}$ already present in ψ addition e^{-iUt} because that is written in the formula $\psi \to \psi' = \psi e^{-iUt}$. In fact just this happens, but tells us only the math solving the Dirac equation).

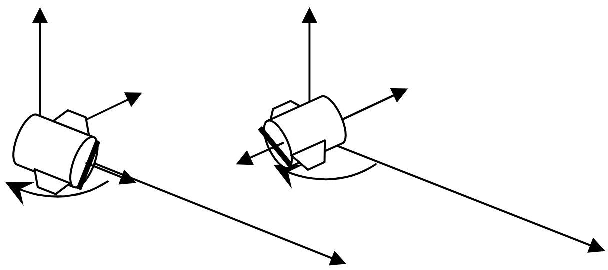
So here we have an immediate interpretation that even needs to be done because .... is ready.

An additional rotation, bringing the rotation to be faster or slower, change the energy of the body.

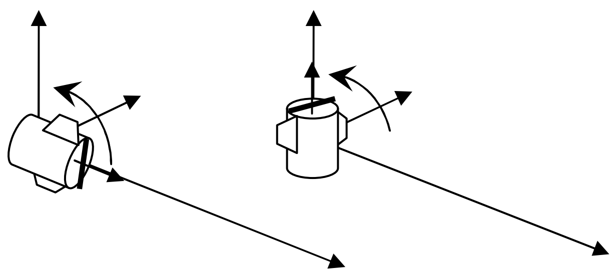
precisely what the effect of potential on a charged particle in quantum mechanics, the effect of photon γ on the electron.

Let now the generators $\hat{i}\hat{k} = j$ and $ij = \widehat{kj}$.
These, as stated Hestenes (I translate) "does not leave $\hat{k}$ unchanged".
We can clearly see the effects of a gauge transformation with $\hat{i}\hat{k} = j$ and $ij = \widehat{kj}$.
We recall first that in the theory of weak interactions to these two generators is attributed to the action of the W particle and therefore it is this fact that we explain. Stresses that the action of the W particle is, among others, can transform into electron neutrinos, or vice versa.
In an analogy with TE, TM, TEM briefly and succinctly as we would say "give mass to TEM" or vice versa "bring TE and TM at the speed of light". From a purely electromagnetic point of view is to provide a TEM of a "cutoff" (which it did not), or rather "free TE and TM" from its cutoff frequency thereby transforming them into TEM.
Well let's see to interpret geometrically the action of $e^{-\hat{i}\hat{k}\rho} = e^{-j\rho}$ and $e^{\hat{j}\hat{k}\nu} = e^{-ij\nu}$ (the signs are of convenience) on $\hat{k}$, still considered as the axis of rotation of the body.
For $\rho = \nu = \frac{\pi}{2}$ we see immediately that $e^{-\hat{i}\hat{k}\rho}$ brings $\hat{k}$ on $(-\hat{i})$ and $e^{\hat{j}\hat{k}\nu}$ brings $\hat{k}$ on $\hat{j}$.

$e^{-\hat{i}\hat{k}\rho}\hat{k} = -\hat{i}$ $\quad\rightarrow$



$e^{\hat{j}\hat{k}\nu}\hat{k} = \hat{j}$ $\quad\rightarrow$



We can consider equivalent the action of two generators in that both lead $\hat{k}$ in the transverse plane (as well as would a combination thereof).

Obviously for $\rho, \nu \leq \frac{\pi}{2}$ an intermediate situation occurs..

For constant ρ, ν these are constant changes in attitude and this does not change the total energy of the body, when it is stationary, nor his impulse, when it is in motion. The body continues its motion with the conservation of momentum, energy and angular momentum.

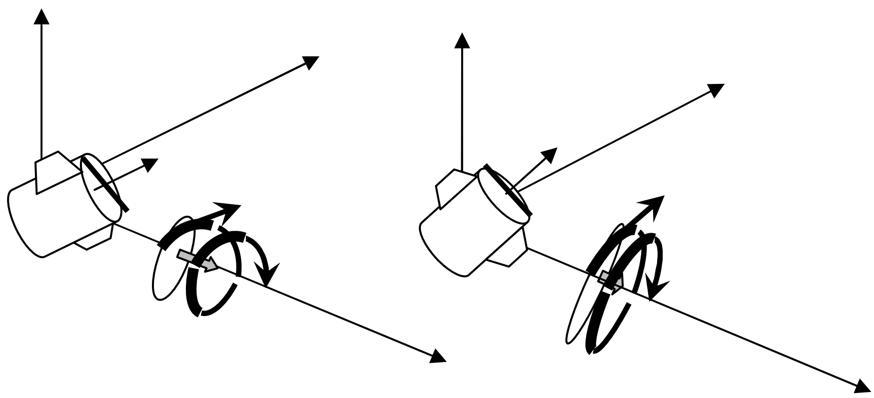
Let instead for example $e^{\hat{j}\hat{k}\nu} = e^{-ij\nu} = e^{-ijWt}$ i.e. the angle of rotation around the axis x becomes a function of time $\nu = Wt$.

We can see that something will happen more complicated.

Mathematics provides us with the answer, which is suggested also by reasonably intuition:

<u>the satellite slows its movement along z and acquires a precession motion of $\hat{k}$ around z axis.</u>

Essentially part of its energy of motion goes into energy of rotation.



It's what happens to a field in a waveguide: the field has gained mass or energy at rest.

For $W = \omega$ motion stops completely and energy is all set in rotation (in the waveguide the field is at cutoff frequency).

There is so explained, albeit in primitive form indeed, as the transformation from TEM to TE, TM and vice versa, due to a "horn antenna" is produced by the gauge transformations $\psi \rightarrow \psi' = \psi e^{-ijWt}$.

We have so highlighted a possible interpretation in terms of analogy with the action of W on neutrinos.

Another analogy: the Z° particle

In the theory of weak interactions, the action of the Z ° particle is expressed through the joint action of the gauge fields generated by transformations with generators i and Tji. I plan to see if there is a similar situation on TEM.
Consider on a TEM the action of a transformation

(47) $$\psi \to \psi' = \psi e^{-TjiZt - iUt}$$

This involves the introduction of an appropriate covariant derivative that leads to the equation:

(48) $$\partial^* \psi + ji\psi Z + T\psi iU = 0$$

Developing in full you get the same terms in U already calculated in (39), to which you add new terms in Z. With the necessary calculations we obtain:

(49)
$$+\frac{\partial}{\partial z}\psi_3 + \frac{\partial}{\partial \tau}\psi_1 + i\psi_3 Z + \psi_1 iU = 0$$

$$-\frac{\partial}{\partial z}\psi_4 + \frac{\partial}{\partial \tau}\psi_2 + i\psi_4 Z - \psi_2 iU = 0$$

$$+\frac{\partial}{\partial z}\psi_1 + \frac{\partial}{\partial \tau}\psi_3 + i\psi_1 Z + \psi_3 iU = 0$$

$$-\frac{\partial}{\partial z}\psi_2 + \frac{\partial}{\partial \tau}\psi_4 + i\psi_2 Z - \psi_4 iU = 0$$

These equations provide, in the absence of gauge fields, a TEM solution that can be both right and left.
Let's see what possible solutions exist in the presence of gauge fields.
Seeking first a solution in the form (21) (TEM "right").
Substituting (21) in (49) with k_Z and ω indeterminate there are actually solutions of the form (21) with the condition:

(50) $$(k_Z + Z)^2 = (\omega - U)^2$$

So from an initial condition in the absence of fields with $k_Z = \omega$ must happen that k_Z and ω e undergo a change as to satisfy (50).
From a physical point of view the ω of a TEM can increase or decrease through the interaction with an object (or a "target").

For example, consider the following situation: a TEM that propagates around z interacts with a moving target that communicates a Doppler ω_d and continuing in "forward scattering" with increasing frequency from ω to $\omega+\omega_d$.
However, if we consider the problem from a physical point of view the ω of a TEM can increase or decrease, but k_Z must do the same, maintaining the condition of equality between ω and k (which means speed c = 1).
It follows from (50) that the action of U and Z is not permissible with the signs that appear there, that is (for positive U and Z) with an increase of k_Z and a decrease of ω.
Therefore the only possible hypothesis is that under the transformation (47):

a) U and Z appear both and not separately, not only the one and only the other;
b) U and Z have equal value and opposite sign and then
c) there are "coupling charges" to U and Z opposite.
Let us appear in (49) the presence of "coupling charges" to U and Z in square brackets.
Quite <u>subtle</u> and biased I use the following arbitrary names:
call $\left[Y/2\right]$ the coupling charge to Z;
call $[T3]$ the coupling charge to U.
(49) thus becomes:

(51)

$$+\frac{\partial}{\partial z}\psi_3+\frac{\partial}{\partial\tau}\psi_1+\left[Y/2\right]i\psi_3 Z+[T3]\psi_1 iU=0$$

$$-\frac{\partial}{\partial z}\psi_4+\frac{\partial}{\partial\tau}\psi_2+\left[Y/2\right]i\psi_4 Z-[T3]\psi_2 iU=0$$

$$+\frac{\partial}{\partial z}\psi_1+\frac{\partial}{\partial\tau}\psi_3+\left[Y/2\right]i\psi_1 Z+[T3]\psi_3 iU=0$$

$$-\frac{\partial}{\partial z}\psi_2+\frac{\partial}{\partial\tau}\psi_4+\left[Y/2\right]i\psi_2 Z-[T3]\psi_4 iU=0$$

Solve with:

52) $$\left[\frac{Y}{2}\right]=+\frac{1}{2}$$

and:

(53) $$[T3]=-\frac{1}{2}$$

The solution is thus (TEM "right"):

(54) $$\left(k_Z + \frac{1}{2}Z\right)^2 = \left(\omega + \frac{1}{2}U\right)^2$$

This solution is physically compatible and is the action a "moving target" which gives a Doppler ω_d with an increased frequency of TEM from ω to $\omega + \omega_d$. The action of this object is so identified with the field produced by the gauge transformation (47).
Now consider the solution TEM "left" in the absence of fields:

$$\psi_1 = e^{+i\omega t - ik_z z} \qquad \psi_2 = e^{-i\omega t + ik_z z}$$

(55) $$\psi_3 = e^{+i\omega t - ik_z z} \qquad \psi_4 = -e^{-i\omega t + ik_z z}$$

$$k_Z = \omega$$

Interacting with the same target above and then under the action of gauge field produced by the transformation (47), under hypothesis (52) (53) the following solution of (51) is found:

(56) $$\left(k_Z - \frac{1}{2}Z\right)^2 = \left(\omega - \frac{1}{2}U\right)^2$$

This leads to the absurd situation where the same target communicate a positive doppler to TEM "right" and a negative doppler to TEM "left", which is not physically reasonable.
So we must suppose to "coupling charges" of the TEM "left" to U and Z equal to:

(57) $$\left[\frac{Y}{2}\right] = -\frac{1}{2}$$

(58) $$[T3] = +\frac{1}{2}$$

therefore opposite to those of TEM "right".
This will find the correct solution (54).
The equations (51) with the relevant specifications (52) (53) or (57) (58) allow a classification of modes TEM "right" and TEM "left", in relation to their coupling charges with respect to the gauge fields.

The following table of classification of modes appears:

	$\left[Y/2\right]$	$[T3]$
TEM "left"	$\frac{1}{2}$	$-\frac{1}{2}$
TEM "right"	$-\frac{1}{2}$	$\frac{1}{2}$

which clearly recalls the classification of neutrinos in the Standard Model (obviously without being able to assign any meaning to the symbols, which I chose to art so subtle):

	$\left[Y/2\right]$	$[T3]$
ν_L	$\frac{1}{2}$	$-\frac{1}{2}$
$\bar{\nu}_R$	$-\frac{1}{2}$	$\frac{1}{2}$

In conclusion, and distinguishing fact from interpretation, we saw it as a fact what is the action on the TEM of gauge transformations (47) and have revealed a possible interpretation in terms of analogy with the action of Z° on neutrinos.

TE TM TEM modes classification

Consider on a TEM the action of a general transformation:

(59) $$\psi \to \psi' = \psi e^{-TjiZt-iUt-ijWt}$$

This involves the introduction of an appropriate covariant derivative that leads to the equation:

(60) $$\partial^* \psi + ji\psi Z + T\psi iU + T\psi ijW = 0$$

Developing in full you get the same terms in W and U already calculated in (29) and (39), to which you add the term in Z. With the necessary calculations we obtain:

(61)
$$+\frac{\partial}{\partial z}\psi_3 + \frac{\partial}{\partial \tau}\psi_1 + i\psi_3 Z + \psi_1 iU + \psi_2 {}^* iW = 0$$
$$-\frac{\partial}{\partial z}\psi_4 + \frac{\partial}{\partial \tau}\psi_2 + i\psi_4 Z - \psi_2 iU - \psi_1 {}^* iW = 0$$
$$+\frac{\partial}{\partial z}\psi_1 + \frac{\partial}{\partial \tau}\psi_3 + i\psi_1 Z + \psi_3 iU + \psi_4 {}^* iW = 0$$
$$-\frac{\partial}{\partial z}\psi_2 + \frac{\partial}{\partial \tau}\psi_4 + i\psi_2 Z - \psi_4 iU - \psi_3 {}^* iW = 0$$

This is to all intents and purposes a single equation that can represent all the possible modes TEM "right", TEM "left", TM "right", TM "left", TE "right", TE "left". Starts from (61) in the absence of gauge fields, which as a solution TEM "right" and / or TEM "left", you can reach all other modes depending on the presence or absence of gauge fields and coupling charges relevant to each mode. This process not only long, it also contains the degrees of arbitrariness and so I would simply mention the possibility.

We can at least show that (61) despite their seemingly impenetrable have solution. We seek such a solution in the form (20) but with B and k_Z indeterminate. Substituting in (61) the first two give the common condition:

(62) $$B = \frac{(\omega - U) - W}{k_Z + Z}$$

The latter two both give the condition:

(63) $$B = \frac{k_Z + Z}{(\omega - U) + W}$$

For the solution must therefore be:

(64) $$(k_Z + Z)^2 = (\omega - U)^2 - W^2$$

and you have a solution with:

(65) $$B = \frac{\sqrt{(\omega - U) - W}}{\sqrt{(\omega - U) + W}}$$

A further discussion is outside of what I propose at this time.

Conclusions

We interpreted the action of the gauge fields of $SU(2)\otimes U(1)$ on the e. m. modes TEM "right", TEM "left", TM "right", TM "left", TE "right", TE "left".
The interpretation was made by studying the following effects:
effect of $SU(2)$ transformations on equations:

(13 a) $\partial^*\psi_R = -\hat{\imath}mi\psi_R\hat{\mathrm{T}} \rightarrow$ for TM "left", TE "right"
(13 b) $\partial^*\psi_L = +\hat{\imath}mi\psi_L\hat{\mathrm{T}} \rightarrow$ for TM "right", TE "left";

effect of $SU(2)\otimes U(1)$ transformations on the equation:

(13c) $\partial^*\psi = 0 \rightarrow$ for TEM "left", TEM "right".

Have identified the physical objects that implement these actions.
Were then shown similarities, all obviously questionable and require study, with the action of γ, W, Z°.
I would insist and clearly distinguish the analysis on the TE fields etc. from the similarities.
a - We saw it as a fact which is the action on the modes TEM "right" TEM "left", TM "right", TM "left", TE "right", TE "left of the $SU(2)$ and $SU(2)\otimes U(1)$ transformations, and have identified the physical objects that implement these actions.
b - We have highlighted the possible similarities with the action on neutrinos and electrons.

Appendix 1

$\psi\hat{\mathrm{T}}\psi^*$ **invariance with** $SU(2)\otimes U(1)$

Do to deserve extended passages that show the $\psi\hat{\mathrm{T}}\psi^*$ invariance.
By:

(1) $\psi' \to \psi e^{\mathrm{T}ji\beta + ij\nu - i\Phi + j\rho}$

immediately with some passage:

(2) $(\psi')\hat{\mathrm{T}}(\psi')^* = \psi e^{Tji\beta}\hat{T}(e^{Tji\beta})^*\psi^*$

To continue we note that the element $\mathrm{T}ji$ enjoys the properties:

(3) $(\mathrm{T}ji)^* = \mathrm{T}ji$

(4) $(\mathrm{T}ji)^2 = -1$

and thus for (3):

$$(\psi')\hat{\mathrm{T}}(\psi')^* = \psi e^{Tji\beta}\hat{T}e^{Tji\beta}\psi^*$$

Since Tji anticommutes with $\hat{T}$ still follows:

$$(\psi')\hat{\mathrm{T}}(\psi')^* = \psi e^{Tji\beta}e^{-Tji\beta}\hat{T}\psi^* = \psi\hat{T}\psi^*$$

finally showing the invariance of $\psi\hat{\mathrm{T}}\psi^*$.

Appendix 2

Relation between spinors and fields

Let:

(1) $\psi = \frac{1}{\sqrt{2}}(e^{-i\omega t} + je^{-i\omega t}) = \psi_1 + j\psi_2$

or rather

(2) $\psi = \frac{1}{\sqrt{2}}(e^{-i\omega t} + je^{i\omega t}) = \psi_1 + j\psi_2$

If ψ_1 and $j\psi_2$ represent opposite rotations as in (1), is relatively unimportant to associate the vector

(3) $\vec{E} = \psi_+ \hat{i} + \psi_- (-j)\hat{i}$

or rather the vector

(4) $\vec{E} = \psi_+ \hat{i} + \psi_- (-j)\hat{j}$

In fact in (3) and (4) changes only the initial position of the $\psi_-(-j)$ contribution (along $\hat{i}$ or along $\hat{j}$). This initial position is rapidly absorbed because the two vectors is in (3) in (4) are counterotating, and then the two different choices are equivalent only to a different initial position for $\vec{E}$.
If instead ψ_1 and $j\psi_2$ represent the same direction of rotation as in (2), then the initial position of the two vectors is maintained over time and therefore is not irrelevant to their position: they must be in quadrature.
The choice to do is therefore (4), which is suitable for both cases.
Of course even in the presence of the magnetic field (4) reads:

(5) $\vec{E} + Tji\vec{H} = \psi_+ \hat{i} + \psi_- (-j)\hat{j}$

As an exercise calculate with (5) the field that corresponds to the solution (20) of text (a TE R):

$$\psi_1 = e^{-i\omega t+ik_z z} \qquad \psi_2 = e^{+i\omega t-ik_z z}$$

$$\psi_3 = Be^{-i\omega t+ik_z z} \qquad \psi_4 = -Be^{+i\omega t-ik_z z} \tag{20}$$

$$B = \frac{\sqrt{\omega-\omega_0}}{\sqrt{\omega+\omega_0}} \qquad k_z^2 = \omega^2 - \omega_0^2 \qquad \omega_0 \equiv m$$

From:

$$\psi = \psi_1 + j\psi_2 + \mathrm{T}j\psi_3 + \mathrm{T}\psi_4$$

we have:

$$\psi_+ = \psi_1 + \mathrm{T}j\psi_3$$
$$\psi_- = j\psi_2 + \mathrm{T}\psi_4$$

and substituting expressions (20) yields:

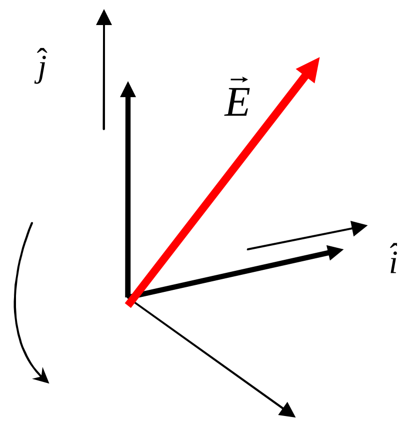
$$\vec{E} + Tji\vec{H} = \psi_+\hat{i} + \psi_-(-j)\hat{j} = (e^{-i\omega t+ik_Z z} + TjBe^{-i\omega t+ik_Z z})\hat{i} + (je^{+i\omega t-ik_Z z} - TBe^{+i\omega t-ik_Z z})(-j)\hat{j}$$

From here, with some step, paying attention to the commutative or anticommutative property the various terms, we obtain

$$\vec{E} + Tji\vec{H} = e^{-i\omega t+ik_Z z}\hat{i} + TjiBe^{-i\omega t+ik_Z z}\hat{j} + e^{-i\omega t+ik_Z z}\hat{j} - TjiBe^{-i\omega t+ik_Z z}\hat{i}$$

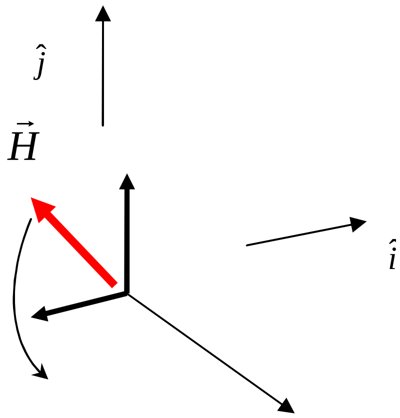
Recognize two terms of the electric field in quadrature:

$$\vec{E} = e^{-i\omega t+ik_Z z}\hat{i} + e^{-i\omega t+ik_Z z}\hat{j}$$

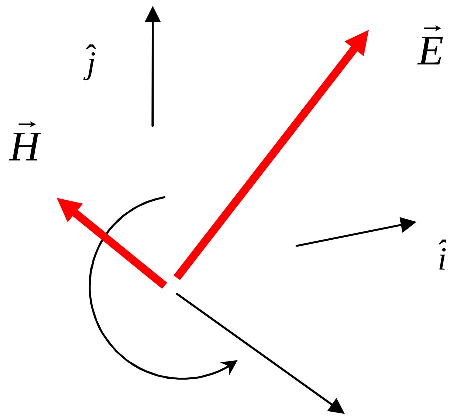


and two terms of magnetic field:

$$Tji\vec{H} = TjiBe^{-i\omega t+ik_z z}\,\hat{j} - TjiBe^{-i\omega t+ik_z z}\,\hat{i}$$



Altogether we obtain an electric field and a magnetic field in circular polarization, with amplitude (normalizing (20) with $\sqrt{2}$) respectively 1 and B. The Poynting vector $\vec{E}\times\vec{H}$ is directed toward positive z and the field has right polarization (R) in the IEEE conventions.



Appendix 3

Gauge fields and covariant derivative

I recall briefly the way in which gauge fields "born".
"If an equation that expresses a physical law admits a global transformation that must still be true after a transformation of the same type but local."
Consider as an example (13) of the text that I rewrite here:

(1) $$\partial^{*}\psi + \hat{\imath} m i \psi \hat{\mathrm{T}} = 0$$

The equation admits the global transformation

(2) $$\psi \to \psi' = \psi e^{+ij\Phi}$$

which means "if ψ satisfies (1), even ψ' satisfies (1)".
(The verification is straightforward by replacing (2) in (1) and a simplification of an exponential from right).
But suppose instead that the transformation becomes local, especially a function of time:

(3) $$\psi \to \psi' = \psi e^{+ijWt}$$

Now replace in (1) and being

$$\partial^*\psi' = \partial^*(\psi e^{+ijWt}) = (\partial^*\psi)e^{+iWt} + T(\psi)\frac{\partial}{\partial\tau}e^{+ijWt} = (\partial^*\psi)e^{ijWt} + T\psi ijWe^{+ijWt}$$

the result (after simplification of an exponential from right) is:

$$\partial^{*}\psi + \hat{\imath} m i \psi \hat{\mathrm{T}} + T\psi ijW = 0$$

So if (1) is true this is no longer true. To ensure that (1) is still satisfied we introduce a suitable "covariant derivative"

(4) $$\partial^*\psi \to D^*\psi = \partial^*\psi - T\psi ijW$$

that makes the extra term $+T\psi ijW$ simplify . The (1) becomes:

(5) $$\partial^{*}\psi + \hat{\imath} m i \psi \hat{\mathrm{T}} - T\psi ijW = 0$$

The new term expresses the presence of a gauge field.

Appendix 4

Normalizing ψ

We have seen that the TEM (28) with frequency ω is reported in a waveguide guide with $\omega_0 = W$, transforming in TE (31).
For continuity reasons we believe that TE keeps the frequency ω.
Rewrite it in full in the complete

(1) $$\psi = \psi_1 + j\psi_2 + \mathrm{T}j\psi_3 + \mathrm{T}\psi_4$$

and with the normalization:

(2) $$\psi\psi^* = 1$$

Appears, with easy but lengthy calculations:

(3) $$\psi = \frac{(1+BTj)e^{-i\omega t+ik_Z z} + j(1-BTj)e^{+i\omega t-ik_Z z}}{\sqrt{2}\sqrt{1-B^2}} \quad \rightarrow \quad \psi\psi^* = 1$$

The energy momentum vector, or four-velocity with the normalization (2), results with the same length calculations (taking $(-\hat{T})$ as four-velocity at rest):

(4) $$\psi(-\hat{T})\psi^* = -\frac{1+B^2}{1-B^2}\hat{T} + \frac{2B}{1-B^2}\hat{k} \quad \rightarrow \quad \psi\psi^* = 1$$

This can be further clarified.
From:

(5) $$B = \frac{\sqrt{\omega - \omega_o}}{\sqrt{\omega + \omega_0}}$$

is

(6) $$\psi(-\hat{T})\psi^* = -\frac{\omega}{\omega_0}\hat{T} + \frac{k_Z}{\omega_0}\hat{k} \quad \rightarrow \quad \psi\psi^* = 1$$

This goes to ∞ for $\omega_0 \rightarrow 0$.

Con una moltiplicazione per With a multiplication $\frac{\omega_0}{\omega}$ (see below) we could make appear explicitly the group velocity of the wave along the z axis:

(7) $$v_g = \frac{d\omega}{dk_z} = \sqrt{1 - \frac{\omega_0^2}{\omega^2}} = \frac{k_Z}{\omega}$$

We calculate also the position of:

(8) $$\hat{e}_3 = \psi\hat{k}\psi^*$$

namely that in the Hestenes interpretation of the motion of a "small rigid body" should be the position of $\hat{k}$ while in motion or, on the Hestenes interpretation of the electron, the position of spin. It appears, again with long calculations:

(9) $$\hat{e}_3 = \psi\hat{k}\psi^* = e^{-2i\omega t + 2ik_Z z}\hat{i} \quad \rightarrow \quad \psi\psi^* = 1$$

difficult to interpret as spin.
This is all you get with a normalization $\psi\psi^* = 1$.
This type of normalization, seemingly intelligent, is not the most suitable. Just think that in the limit situation of TEM field $\psi\psi^* = 0$ and then dividing by $\psi\psi^*$ in order to normalize is not possible.
It is more convenient the normalization:

(10) $$\psi\psi^* = \frac{\omega_0}{\omega}$$

so we have the four velocity:

(11) $$\psi(-\hat{T})\psi^* = -\hat{T} + \frac{k_Z}{\omega}\hat{k}$$

It varies between $(-\hat{T})$ at rest and $(-\hat{T} + \hat{k})$ in the extreme case of TEM speed c (Note: c = 1 in the units used).
Likewise the remaining formulas are best interpreted as stated in the text.

Appendix 5

Clifford Algebra

Algebra here is based on 4 elements $\hat{i}\ \hat{j}\ \hat{k}\ \hat{\mathrm{T}}$, unit vectors in spacetime (sometimes referred to the authors e_1, e_2, e_3, e_0). They have the following properties:

(1) $\hat{i}^2 = 1 \quad \hat{j}^2 = 1 \quad \hat{k}^2 = 1 \quad \hat{\mathrm{T}}^2 = -1 \quad \hat{j}\hat{i} = -\hat{i}\hat{j}$ etc

and I use the symbols $i\ \ j\ \ \mathrm{T}$ to generalize the usual imaginary unit i of the xy plane

(2) $i = \hat{i}\hat{j} \quad j = \hat{i}\hat{k} \quad \mathrm{T} = \hat{i}\hat{\mathrm{T}}$

All this, combined with the rule concerning the conjugates

(3) $(AB)^* = B^* A^*$

generates all properties of interest.
In fact is enough to admit that fact $\hat{i}\ \hat{j}\ \hat{k}$ do not change by conjugation (as it is intuitive that it should be) to derive for example, or rediscover, the usual rule for the conjugate i^*:

(4) $i^* = (\hat{i}\hat{j})^* = \hat{j}^*\hat{i}^* = \hat{j}\hat{i} = -\hat{i}\hat{j} = -i$

and so is obtained

(5) $j^* = -j \qquad \mathrm{T}^* = -\mathrm{T}$

Apply, as a consequence of (1) and (2),

(6) $i^2 = -1 \quad j^2 = -1 \quad \mathrm{T}^2 = 1$

(7) $ij = -ji \qquad i\mathrm{T} = -\mathrm{T}i \qquad j\mathrm{T} = -\mathrm{T}j$

The 16 elements algebra

1, $\hat{i}\ \hat{j}\ \hat{k}\ \hat{\mathrm{T}}$ (4 elements), $\hat{i}\hat{j}\ \ \hat{i}\hat{\mathrm{T}}$ etc. (6 elements), $\hat{i}\hat{j}\hat{k}$ etc. (4 elements), $\hat{i}\hat{j}\hat{k}\hat{\mathrm{T}}$

contains a subalgebra of 8 elements ("even subalgebra of a Clifford algebra", Hestenes)

1, $\hat{i}\hat{j}\ \ \hat{i}\hat{\mathrm{T}}$ etc. (6 elements), $\hat{i}\hat{j}\hat{k}\hat{\mathrm{T}}$

rewritten at will as consisting of all possible products between

$$1\ ,\ i\ ,\ \ j\ ,\ \mathrm{T}\ \ ,\ ij\ \ ,\ i\,\mathrm{T}\ \ ,\ \ j\,\mathrm{T}\ \ ,\ \mathrm{T}\ \ ji$$

Element $\mathrm{T}ji$ hence the previous property benefits of:

(8) $$(\mathrm{T}ji)^* = \mathrm{T}ji$$

(9) $$(\mathrm{T}ji)^2 = -1$$

The complex

(10) $$z = x + iy \quad (\quad \vec{x} = \hat{i}z = x\hat{i} + y\hat{j}\ )$$

generalizes in spacetime with

(11) $$z = x + iy + jz + \mathrm{T}\tau \quad (\vec{x} = \hat{i}z = x\hat{i} + y\hat{j} + z\hat{k} + \tau\hat{\mathrm{T}})$$

(not confuse z in first and second member, sorry).
We have

(12) $$zz^* = x^2 + y^2 + \cancel{z}^2 - \tau^2 \quad (\vec{x}^2 = \vec{x}\vec{x} = zz^*)$$

On xy plane symbols or operators

(13) $$\partial = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$$

$$\partial^* = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

are, respectively, to express the derivative and the Cauchy Riemann conditions.
These are generalized in

(14) $$\partial = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} - j\frac{\partial}{\partial z} - \mathrm{T}\frac{\partial}{\partial \tau}$$

$$\partial* = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z} + \mathrm{T}\frac{\partial}{\partial \tau}$$

and the property is

(15) $$\partial\partial^* = \partial^*\partial = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2}$$

Alternatively to the symbol or operator ∂^* used to express the analyticity one can use the operator that is obtained by multiplying by $\hat{i}$ from left
(Note: if $\partial^* f = 0$ also $\hat{i}\partial^* f = 0$ and vice versa).
The operator thus obtained

(16) $$\hat{i}\partial^* = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} + \frac{\partial}{\partial \tau}\hat{\mathrm{T}} = \vec{\partial}_V$$

is formally a four-vector, as $\vec{x}$.
So on.
This algebra differs from the STA for the choice of the base with the properties (1).
The STA choice is for spacelike unit vectors $\gamma_k (k = 1,2,3)$ having square (-1). Thus there is a basis in spacetime that instead of (1) has the properties:

(17) $$\gamma_k^2 = -1, \gamma_0^2 = 1$$

So doing to obtain a unit vector basis x, y, z in space should be defined three bivectors (Hestenes, [3]):

(18) $$\sigma_k = \gamma_k \gamma_0$$

Hestenes note explicitly the opportunities of either choice ([3], p.25):
“If instead we had chosen $\gamma_k^2 = 1, \gamma_0^2 = -1$ we could entertain the solution $\sigma_k = \gamma_k$, which may seem more natural, because...”,
because vectors in spacetime would also be vectors in space.
I prefer to keep this option best suits to engineers (unit vectors $\hat{i}$ $\hat{j}$ $\hat{k}$ with square +1, imaginary unit i, complex number $x + iy$, etc.).
Plus (Doran, [2]) for any of the two choices the even algebras are isomorphic, so working in even algebra there is no change in anything.
I should also note that all the conditions that I used as a vector, complex number, imaginary unit and so on recall mnemonically concepts of the past and we can sometimes help but are materially misleading. All the entities we have introduced are simply numbers, and we can correctly call "Clifford numbers", simple underlying rules, sum product and division, of the Clifford algebra. The same goes for symbols such as asterisk or the arrow for vectors etc., here have the sole function of mnemonic recall. What matters are only the properties of algebra I have briefly summarized.

References

[1] D. Hestenes "A unified language for Mathematics and Physics", in "Clifford Algebras and their Applications in Mathematical Physics", NATO ASI Series, Reidel (1986)

[2] C. Doran et al.,"States and Operators in the Spacetime Algebra", Found. Phys. 23(9), 1239 (1993)

[3] D. Hestenes, "Space time algebra", Gordon and Breach (1966)

[4] S. Gull et al., "Imaginary numbers are not real – the geometric algebra of spacetime", Found. Phys. 23(9):1175, (1993)

[5] W. Pauli, "Teoria della relatività", Boringhieri (1958)

[6] M.I. Skolnik, "Radar Handbook", Mc Graw Hill (1970)

[7] S. Ramo, J. R. Whinnery, T. van Duzer, "Fields and Waves in Communication Electronics", John Wiley (1994)

[8] D. Hestenes, "Quantum mechanics from self – interaction", Found. Phys. 15, 63-87 (1985)

[9] D. Hestenes, "Zitterbewegung modeling", Found. Phys. 23, 365-387,(1993)

[10] D Hestenes, "The zitterbewegung interpretation of quantum mechanics", Found. Phys. 20, 1213-1232 (1990)

[11] D. Hestenes, "Clifford Algebra and the interpretation of quantum mechanics", in "Clifford Algebras and their Applications in Mathematical Physics", NATO ASI Series, Reidel (1986)

[12] D. Hestenes, "Space-time structure of weak and electromagnetic interactions", Found. Phys. 12, 153-168 (1982)

[13] D. Hestenes, "Mysteries and Insights of Dirac Theory", available on line at http://modelingnts.la.asu.edu .

[14] D. Hestenes, "Spacetime calculus", available on line at http://modelingnts.la.asu.edu .

[15] G. Bettini, "Algebra di Clifford ed equazione di Dirac per i campi in guida", available in vixra.