On n-ary Algebras, Branes and Polyvector Gauge Theories in Noncommutative Clifford Spaces

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Abstract

Polyvector-valued gauge field theories in noncommutative Clifford spaces are presented. The noncommutative star products are associative and require the use of the Baker-Campbell-Hausdorff formula. Actions for p-branes in noncommutative (Clifford) spaces and noncommutative phase spaces are provided. An important relationship among the **n**-ary commutators of noncommuting spacetime coordinates $[X^1, X^2, ..., X^n]$ with the poly-vector valued coordinates $X^{123...n}$ in noncommutative Clifford spaces is explicitly derived $[X^1, X^2, ..., X^n] = n!\ X^{123...n}$. The large N limit of **n**-ary commutators of n hyper-matrices $\mathbf{X}_{i_1 i_2 .... i_n}$ leads to Eguchi-Schild p-brane actions for $p + 1 = n$. Noncommutative Clifford-space gravity as a poly-vector-valued gauge theory of twisted diffeomorphisms in Clifford-spaces would require quantum Hopf algebraic deformations of Clifford algebras.

1 Polyvector Gauge Field Theories in Noncommutative Clifford Spaces

Clifford algebras are deeply related and essential tools in many aspects in Physics. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory [3] whose generalized polyvector-valued coordinates are Clifford-valued quantities which incorporate lines, areas, volumes, hyper-volumes.... degrees of freedom associated with the collective particle, string, membrane, p-brane,... dynamics of p-loops (closed p-branes) in D-dimensional target spacetime backgrounds.

C-space Relativity naturally incorporates the ideas of an invariant length (Planck scale), maximal acceleration, non-commuting coordinates, supersymmetry, holography, higher derivative gravity with torsion and variable dimensions/signatures. It permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing. It resolves the ordering ambiguities in QFT, the problem of time in Cosmology and admits superluminal propagation (tachyons) without violations of causality. The relativity of signatures of the underlying spacetime results from taking different slices of C-space.

The conformal group in spacetime emerges as a natural subgroup of the Clifford group and Relativity in C-spaces involves natural scale changes in the sizes of physical objects without the introduction of forces nor Weyl's gauge field of dilations. A generalization of Maxwell theory of Electrodynamics of point charges to a theory in C-spaces involves extended charges coupled to antisymmetric tensor fields of arbitrary rank and where the analog of photons are tensionless p-branes. The Extended Relativity Theory in Born-Clifford Phase Spaces with a Lower and Upper Length Scales and the program behind a Clifford Group Geometric Unification was advanced by [5].

Furthermore, there is no EPR paradox in Clifford spaces [6] and Clifford-space tensorial-gauge fields generalizations of Yang-Mills theories and the Standard Model allows to predict the existence of new particles (bosons, fermions) and tensor-gauge fields of higher-spins in the 10 TeV regime [2]. Clifford-spaces can also be extended to Clifford-Superspaces by including both orthogonal and symplectic Clifford algebras and generalizing the Clifford super-differential exterior calculus in ordinary superspace to the full fledged Clifford-Superspace outlined in [8]. Clifford-Superspace is far richer than ordinary superspace and Clifford Supergravity involving polyvector-valued extensions of Poincare and (Anti) de Sitter supergravity (antisymmetric tensorial charges of higher rank) is a very relevant generalization of ordinary supergravity with applications in M-theory.

It was recently shown [1] how an unification of Conformal Gravity and a $U(4) \times U(4)$ Yang-Mills theory in four dimensions could be attained from a Clifford Gauge Field Theory in C-spaces (Clifford spaces) based on the (complex) Clifford $Cl(4, C)$ algebra underlying a complexified four dimensional spacetime (8 real dimensions). Tensorial Generalized Yang-Mills in C-spaces (Clifford spaces) based on poly-vector valued (anti-symmetric tensor fields) gauge fields $\mathcal{A}_M(\mathbf{X})$ and field strengths $\mathcal{F}_{MN}(\mathbf{X})$ have been studied in [2], [3] where $\mathbf{X} = X_M \Gamma^M$ is a C-space poly-vector valued coordinate

$$\mathbf{X} \ = \ s \ \mathbf{1} \ + \ x_\mu \ \gamma^\mu \ + \ x_{\mu_1 \mu_2} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \ + \ x_{\mu_1 \mu_2 \mu_3} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \ + \ ...... \ +$$

$$x_{\mu_1 \mu_2 \mu_3 ...... \mu_d} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \ \gamma^{\mu_3} .... \ \wedge \gamma^{\mu_d} \tag{1.1}$$

In order to match dimensions in each term of (1.1) a length scale parameter must be suitably introduced. In [3] we introduced the Planck scale as the expansion parameter in (1.1). The scalar component s of the C-space poly-vector valued coordinate $\mathbf{X}$ was interpreted by [4] as a Stuckelberg time-like parameter that solves the problem of time in Cosmology in a very elegant fashion.

A Clifford gauge field theory in the C-space associated with the ordinary $4D$ spacetime requires $\mathcal{A}_M(\mathbf{X}) = \mathcal{A}_M^A(\mathbf{X})\ \Gamma_A$ that is a poly-vector valued gauge field where M represents the poly-vector index associated with the C-space, and whose gauge group $\mathcal{G}$ is itself based on the Clifford algebra $Cl(3,1)$ of the tangent space spanned by 16 generators Γ_A. The expansion of the poly-vector Clifford-algebra-valued gauge field $\mathcal{A}_M^A$, for *fixed* values of A, is of the form

$$\mathcal{A}_M^A\ \Gamma^M\ =\ \Phi^A\ +\ \mathcal{A}_\mu^A\ \gamma^\mu\ +\ \mathcal{A}_{\mu_1\mu_2}^A\ \gamma^{\mu_1}\wedge\gamma^{\mu_2}\ +\ \mathcal{A}_{\mu_1\mu_2\mu_3}^A\ \gamma^{\mu_1}\wedge\gamma^{\mu_2}\wedge\gamma^{\mu_3}\ +\ .... \tag{1.2}$$

The index A spans the 16-dim Clifford algebra $Cl(3,1)$ of the tangent space such as

$$\Phi^A\ \Gamma_A\ =\ \Phi\ +\ \Phi^a\ \Gamma_a\ +\ \Phi^{ab}\ \Gamma_{ab}\ +\ \Phi^{abc}\ \Gamma_{abc}\ +\ \Phi^{abcd}\ \Gamma_{abcd}. \quad (1.3a)$$

$$\mathcal{A}_\mu^A\ \Gamma_A\ =\ \mathcal{A}_\mu\ +\ \mathcal{A}_\mu^a\ \Gamma_a\ +\ \mathcal{A}_\mu^{ab}\ \Gamma_{ab}\ +\ \mathcal{A}_\mu^{abc}\ \Gamma_{abc}\ +\ \mathcal{A}_\mu^{abcd}\ \Gamma_{abcd}. \quad (1.3b)$$

$$\mathcal{A}_{\mu\nu}^A\ \Gamma_A\ =\ \mathcal{A}_{\mu\nu}\ +\ \mathcal{A}_{\mu\nu}^a\ \Gamma_a\ +\ \mathcal{A}_{\mu\nu}^{ab}\ \Gamma_{ab}\ +\ \mathcal{A}_{\mu\nu}^{abc}\ \Gamma_{abc}\ +\ \mathcal{A}_{\mu\nu}^{abcd}\ \Gamma_{abcd}. \quad (1.3c)$$

etc......

In order to match dimensions in each term of (1.2) another length scale parameter must be suitably introduced. For example, since $\mathcal{A}_{\mu\nu\rho}^A$ has dimensions of $(length)^{-3}$ and $\mathcal{A}_\mu^A$ has dimensions of $(length)^{-1}$ one needs to introduce another length parameter in order to match dimensions. This length parameter does not need to coincide with the Planck scale. The Clifford-algebra-valued gauge field $\mathcal{A}_\mu^A(x^\mu)\Gamma_A$ in ordinary spacetime is naturally embedded into a far richer object $\mathcal{A}_M^A(\mathbf{X})\Gamma_A$ in C-spaces. The advantage of recurring to C-spaces associated with the $4D$ spacetime manifold is that one can have a (complex) Conformal Gravity, Maxwell and $U(4)\times U(4)$ Yang-Mills unification in a very geometric fashion as provided by [1]

Field theories in Noncommutative spacetimes have been the subject of intense investigation in recent years, see [12] and references therein. Star Product deformations of Clifford Gauge Field Theories based on ordinary Noncommutative spacetimes are straightforward generalizations of the work by [9]. The wedge star product of two Clifford-valued one-forms is defined as

$$\mathbf{A}\ \wedge_*\ \mathbf{A}\ =\ \big(\ (\mathcal{A}_\mu^A\ *\ \mathcal{A}_\nu^B)\ \Gamma_A\ \Gamma_B\ \big)\ dx^\mu\wedge dx^\nu\ =$$

$$\frac{1}{2}\ \big(\ (\mathcal{A}_\mu^A *_s \mathcal{A}_\nu^B)\ [\Gamma_A,\ \Gamma_B]\ +\ (\mathcal{A}_\mu^A *_a \mathcal{A}_\nu^B)\ \{\Gamma_A,\ \Gamma_B\}\ \big)\ dx^\mu\ \wedge\ dx^\nu. \tag{1.4}$$

In the case when the coordinates don't commute $[x^\mu, x^\nu] = \theta^{\mu\nu}$ (constants), the cosine (symmetric) star product is defined by [9]

$$f *_s g \equiv \frac{1}{2}(f*g + g*f) = f\,g + \left(\frac{i}{2}\right)^2 \theta^{\mu\nu}\,\theta^{\kappa\lambda}\,(\partial_\mu\,\partial_\kappa f)\,(\partial_\nu\,\partial_\lambda g) + O(\theta^4). \tag{1.5}$$

and the sine (anti-symmetric Moyal bracket) star product is

$$f *_a g \equiv \frac{1}{2}(f*g - g*f) = \left(\frac{i}{2}\right) \theta^{\mu\nu}\,(\partial_\mu f)\,(\partial_\nu g) +$$
$$\left(\frac{i}{2}\right)^3 \theta^{\mu\nu}\,\theta^{\kappa\lambda}\,\theta^{\alpha\beta}\,(\partial_\mu\,\partial_\kappa\,\partial_\alpha f)\,(\partial_\nu\,\partial_\lambda\,\partial_\beta g) + O(\theta^5). \tag{1.6}$$

Notice that both commutators *and* anticommutators of the gammas appear in the star deformed products in (1.4). The star product deformations of the gauge field strengths in the case of the $U(2,2)$ gauge group were given by [9] and the expressions for the star product deformed action are very cumbersome .

In this letter we proceed with the construction of Polyvector-valued Gauge Field Theories in *noncommutative* Clifford Spaces (C-spaces) which are polyvector-valued *extensions* and *generalizations* of the ordinary *noncommutative* space-times. We begin firstly by writing the commutators $[\Gamma_A, \Gamma_B]$. For $pq = odd$ one has [11]

$$[\ \gamma_{b_1 b_2 .....b_p},\ \gamma^{a_1 a_2 ......a_q}\ ] = 2\gamma^{a_1 a_2 ......a_q}_{b_1 b_2 .....b_p} -$$
$$\frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3 ....a_q]}_{b_3 .....b_p]} + \frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[a_1 ....a_4}_{[b_1 ....b_4}\ \gamma^{a_5 ....a_q]}_{b_5 .....b_p]} - ...... \tag{1.7}$$

for $pq = even$ one has

$$[\ \gamma_{b_1 b_2 .....b_p},\ \gamma^{a_1 a_2 ......a_q}\ ] = -\frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!}\ \delta^{[a_1}_{[b_1}\ \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]} -$$
$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!}\ \delta^{[a_1 ....a_3}_{[b_1 ....b_3}\ \gamma^{a_4 ....a_q]}_{b_4 .....b_p]} + ... \tag{1.8}$$

The anti-commutators for $pq = even$ are

$$\{\ \gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q}\ \} = 2\gamma^{a_1 a_2 ......a_q}_{b_1 b_2 .....b_p} -$$
$$\frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3 ....a_q]}_{b_3 .....b_p]} + \frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[a_1 ....a_4}_{[b_1 ....b_4}\ \gamma^{a_5 ....a_q]}_{b_5 .....b_p]} - ...... \tag{1.9}$$

and the anti-commutators for $pq = odd$ are

$$\{\ \gamma_{b_1 b_2 .....b_p},\ \gamma^{a_1 a_2 ......a_q}\ \} = -\frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!}\ \delta^{[a_1}_{[b_1}\ \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]} -$$

$$\frac{(-1)^{p-1}2p!q!}{3!(p-3)!(q-3)!} \; \delta^{[a_1....a_3}_{[b_1....b_3} \; \gamma^{a_4....a_q]}_{b_4.....b_p]} \; + \; ...... \qquad (1.10)$$

For instance,

$$\mathcal{J}^a_b \;=\; [\gamma_b, \gamma^a] \;=\; 2\gamma^a_b; \quad \mathcal{J}^{a_1 a_2}_{b_1 b_2}, \;=\; [\gamma_{b_1 b_2}, \gamma^{a_1 a_2}] \;=\; -\,8\;\delta^{[a_1}_{[b_1}\;\gamma^{a_2]}_{b_2]}. \qquad (1.11)$$

$$\mathcal{J}^{a_1 a_2 a_3}_{b_1 b_2 b_3} \;=\; [\gamma_{b_1 b_2 b_3}, \gamma^{a_1 a_2 a_3}] \;=\; 2\;\gamma^{a_1 a_2 a_3}_{b_1 b_2 b_3} \;-36\;\delta^{[a_1 a_2}_{[b_1 b_2}\;\gamma^{a_3]}_{b_3]}. \qquad (1.12)$$

$$\mathcal{J}^{a_1 a_2 a_3 a_4}_{b_1 b_2 b_3 b_4} \;=\; [\gamma_{b_1 b_2 b_3 b_4}, \gamma^{a_1 a_2 a_3 a_4}] \;=\; -\,32\;\delta^{[a_1}_{[b_1}\;\gamma^{a_2 a_3 a_4]}_{b_2 b_3 b_4]} \;+\; 192\;\delta^{[a_1 a_2 a_3}_{[b_1 b_2 b_3}\;\gamma^{a_4]}_{b_4]}. \qquad (1.13)$$

etc...

The second step is to write down the *noncommutative* algebra associated with the noncommuting poly-vector-valued coordinates in $D = 4$ and which can be obtained from the Clifford algebra (1.7-1.10) by performing the following replacements (and relabeling indices)

$$\gamma^\mu \leftrightarrow X^\mu, \quad \gamma^{\mu_1 \mu_2} \leftrightarrow X^{\mu_1 \mu_2}, \quad ....\; \gamma^{\mu_1 \mu_2 .....\mu_n} \leftrightarrow X^{\mu_1 \mu_2 ....\mu_n}. \qquad (1.14)$$

When the spacetime metric components $g_{\mu\nu}$ are *constant*, from the replacements (1.14) and the Clifford algebra (1.7-1.10) (after one relabels indices), one can then construct the following *noncommutative* algebra among the poly-vector-valued coordinates in $D = 4$, and *obeying* the Jacobi identities, given by the relations

$$[\; X^{\mu_1},\; X^{\mu_2}\;]_* \;=\; X^{\mu_1} \;*\; X^{\mu_2} \;-\; X^{\mu_2} \;*\; X^{\mu_1} \;=\; 2\; X^{\mu_1 \mu_2}. \qquad (1.15)$$

In most of the remaining commutators a suitable length scale parameter must be introduced in order to match units. We shall set this length scale (let us say the Planck scale) to *unity*. Also, by choosing the C-space coordinates to behave like anti-Hermitian operators we avoid the need to introduce i factors in the right hand side.

$$[\; X^{\mu_1 \mu_2},\; X^\nu\;]_* \;=\; 4\;(\; g^{\mu_2 \nu}\; X^{\mu_1} \;-\; g^{\mu_1 \nu}\; X^{\mu_2}\;). \qquad (1.16)$$

$$[\, X^{\mu_1 \mu_2 \mu_3},\, X^\nu\,]_* \;=\; 2\, X^{\mu_1 \mu_2 \mu_3 \nu}, \quad [\, X^{\mu_1 \mu_2 \mu_3 \mu_4},\, X^\nu\,]_* \;=\; -8\, g^{\mu_1 \nu}\, X^{\mu_2 \mu_3 \mu_4} \;\pm ...... \qquad (1.17)$$

$$[\; X^{\mu_1 \mu_2},\; X^{\nu_1 \nu_2}\;]_* \;=\; -\,8\; g^{\mu_1 \nu_1}\; X^{\mu_2 \nu_2} \;+\; 8\; g^{\mu_1 \nu_2}\; X^{\mu_2 \nu_1} \;+$$
$$8\; g^{\mu_2 \nu_1}\; X^{\mu_1 \nu_2} \;-\; 8\; g^{\mu_2 \nu_2}\; X^{\mu_1 \nu_1}. \qquad (1.18)$$

$$[X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2}]_* \ =\ 12\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\nu_2}\ \pm\ .... \tag{1.19}$$

$$[X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2\nu_3}]_* \ =\ -36\ G^{\mu_1\mu_2\ \nu_1\nu_2}\ X^{\mu_3\nu_3}\ \pm\ ...... \tag{1.20}$$

$$[X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}]_* \ =\ -16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ \pm ...... \tag{1.21}$$

$$[X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2}]_* \ =\ -16\ g^{\mu_1\nu_1}\ X^{\mu_2\mu_3\mu_4\nu_2}\ +\ 16\ g^{\mu_1\nu_2}\ X^{\mu_2\mu_3\mu_4\nu_1}\ -\ .... \tag{1.22}$$

$$[X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3}]_* \ =\ 48\, G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4} - 48\, G^{\mu_1\mu_2\mu_4\ \nu_1\nu_2\nu_3}\ X^{\mu_3} + ... \tag{1.23}$$

$$[X^{\mu_1\mu_2\mu_3\mu_4},\ X^{\nu_1\nu_2\nu_3\nu_4}]_* \ =\ 192\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4\nu_4}\ -\ .... \tag{1.24}$$

etc...... where

$$G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n}\ =\ g^{\mu_1\nu_1}\ g^{\mu_2\nu_2}\ ....\ g^{\mu_n\nu_n} + signed\, permutations \tag{1.25}$$

The metric components $G^{\mu_1\mu_2......\mu_n\ \nu_1\nu_2......\nu_n}$ in C-space can also be written as a determinant of the $n\times n$ matrix $\mathbf{G}$ whose entries are $g^{\mu_I\nu_J}$

$$det\ \mathbf{G}_{n\times n}\ =\ \frac{1}{n!}\ \epsilon_{i_1 i_2.....i_n}\ \epsilon_{j_1 j_2....j_n}\ g^{\mu_{i_1}\nu_{j_1}}\ g^{\mu_{i_2}\nu_{j_2}}\ ....\ g^{\mu_{i_n}\nu_{j_n}}. \tag{1.26}$$

$i_1, i_2, ..., i_n \subset I = 1,2,.....,D$ and $j_1, j_2, ..., j_n \subset J = 1,2,.....,D$. One must also include in the C-space metric G^{MN} the (Clifford) scalar-scalar component G^{00} (that could be related to the dilaton field) and the pseudo-scalar/pseudo-scalar component $G^{\mu_1\mu_2.....\mu_D\ \nu_1\nu_2......\nu_D}$ (that could be related to the axion field).

One must emphasize that when the spacetime metric components $g_{\mu\nu}$ are *no* longer *constant*, the noncommutative algebra among the poly-vector-valued coordinates in $D=4$, does *not* longer *obey* the Jacobi identities. For this reason we restrict our construction to a flat spacetime background $g_{\mu\nu} = \eta_{\mu\nu}$.

The noncommutative conditions on the polyvector coordinates in condensed notation can be written as

$$[X^M,\ X^N]_* \ =\ X^M * X^N - X^N * X^M\ =\ \Omega^{MN}(X)\ =\ f^{MN}{}_L\, X^L\ =\ f^{MNL}\, X_L \tag{1.27}$$

the structure constants f^{MNL} are antisymmetric under the exchange of polyvector valued indices. An immediate consequence of the noncommutativity of coordinates is

$$[\ \hat{X}^{\mu_1},\ \hat{X}^{\mu_2}\ ]\ =\ 2\ \hat{X}^{\mu_1\mu_2}\ \Rightarrow\ \Delta X^\mu\ \Delta X^\nu\ \geq\ \frac{1}{2}\mid\ <\ \hat{X}^{\mu\nu}\ >\ \mid=\ X^{\mu\nu} \quad (1.28)$$

Hence, the bivector area coordinates $X^{\mu\nu}$ in C-space can be seen as a measure of the noncommutative nature of the "quantized" spacetime coordinates $\hat{X}^\mu$.

The third step is to define the noncommutative star product of functions of X. The following naive noncommutative star product is *not* associative

$$(\ A_1\ *\ A_2\ )(Z)\ =\ exp\left(\ \frac{1}{2}\ \Omega^{MN}\ \partial_{X^M}\ \partial_{Y^N}\ \right)\ A_1(X)\ A_2(Y)|_{X=Y=Z}\ =$$

$$\sum_{n=0}^{\infty}\ \frac{(\frac{1}{2})^n}{n!}\ \Omega^{M_1N_1}\ \Omega^{M_2N_2}\ ....\ \Omega^{M_nN_n}\ (\partial^n_{M_1M_2......M_n}\ A_1)\ (\partial^n_{N_1N_2......N_n}\ A_2)+ ...... \quad (1.29)$$

where the ellipsis in (1.29) are the terms involving derivatives acting on Ω^{MN} and

$$\partial^n_{M_1M_2......M_n}\ A_1(Z)\ \equiv\ \partial_{M_1}\ \partial_{M_2}\ ......\ \partial_{M_n}\ A_1(Z). \quad (1.30a)$$

$$\partial^n_{N_1N_2......N_n}\ A_2(Z)\ \equiv\ \partial_{N_1}\ \partial_{N_2}\ ......\ \partial_{N_n}\ A_2(Z). \quad (1.30b)$$

Derivatives on Ω^{mn} appear in the ordinary Moyal star product when Ω^{mn} depends on the phase space coordinates. For instance, the Moyal star product when the symplectic structure $\Omega^{mn}(\vec{q},\vec{p})$ is *not* constant is given by

$$A\ *\ B=\ A\ exp\left(\ \frac{i\hbar}{2}\ \Omega^{mn}\ \overleftarrow{\partial}_m\ \overrightarrow{\partial}_n\ \right)\ B\ =$$

$$A\ B+i\hbar\ \Omega^{mn}\ (\partial_m A\ \partial_n B)+\ \frac{(i\hbar)^2}{2}\ \Omega^{m_1n_1}\ \Omega^{m_2n_2}\ (\partial^2_{m_1m_2}\ A)\ (\partial^2_{n_1n_2}\ B)\ +$$

$$\frac{(i\hbar)^2}{3}[\ \Omega^{m_1n_1}\ (\partial_{n_1}\ \Omega^{m_2n_2})\ (\partial_{m_1}\partial_{m_2}A\,\partial_{n_2}B-\partial_{m_2}A\,\partial_{m_1}\partial_{n_2}B\ )\ ]+O(\hbar^3). \quad (1.31)$$

Due to the derivative terms $\partial_{n_1}\Omega^{m_2n_2}$ the star product is associative up to second order only [10] $(f*g)*h\ =\ f*(g*h)+O(\hbar^3)$. Hence, due to the derivatives terms acting on $\Omega^{MN}(X)$ in (1.29), the star product will *no* longer be associative beyond second order.

The correct noncommutative and *associative* star product [16] corresponding to a Lie-algebraic structure for the noncommutative (C-space) coordinates requires the use of the Baker-Campbell-Hausdorff formula

$$exp(A)\,exp(B)\ =\ exp\left(\ A+B+\frac{1}{2}[A,B]+\frac{1}{12}(\ [A,[A,B]]-[B,[A,B]]\ )\ +.......\right). \quad (1.32a)$$

and is given by

$$(A_1 \ * \ A_2 \ )(X) \ = \ exp \left(\ \frac{i}{2} \ X^M \Lambda_M [\ i \ \partial_Y ; \ i \ \partial_Z \ ] \ \right) \ A_1(Y) \ A_2(Z) |_{X=Y=Z}. \tag{1.32b}$$

where the expression for the bilinear differential polynomial $\Lambda_M [i\partial_Y ; i\partial_Z]$ in eq-(1.32b) can be *read* from the Baker-Campbell-Hausdorff formula

$$e^{i \ K_M \hat{X}^M} \ e^{i \ P_N \hat{X}^N} \ = \ e^{i \ \hat{X}^M (\ K_M \ + \ P_M \ + \ \frac{1}{2} \ \Lambda_M [K,P] \ )}. \tag{1.32c}$$

and is given in terms of the structure constants $[X^N, X^Q] = f_M^{NQ} X^M$, after setting $K_N = i \ \partial_{Y^N}, \ P_Q = i \ \partial_{Z^Q}$, by the following expression

$$\Lambda_M [K, P] \ = \ i \ K_N \ P_Q \ f_M^{NQ} \ + \frac{i^2}{6} \ K_{N_1} \ P_{Q_1} \ (P_{N_2} - K_{N_2}) \ f_S^{N_1 Q_1} \ f_M^{S N_2} \ +$$

$$\frac{i^3}{24} \ (P_{N_2} \ K_{Q_2} \ + \ K_{N_2} \ P_{Q_2}) \ K_{N_1} \ P_{Q_1} \ f_{S_1}^{N_1 Q_1} \ f_{S_2}^{S_1 N_2} \ f_M^{S_2 Q_2} \ + \ .... \tag{1.32d}$$

When the star product is *associative* and noncommutative, with the fields and their derivatives vanishing fast enough at infinity, one has

$$\int \ A \ * B \ = \ \int \ A \ B \ + \ total \ derivative \ = \ \int \ A \ B. \tag{1.33a}$$

$$\int \ A \ * \ B \ * \ C \ = \ \int \ A \ (B \ * \ C) \ + \ total \ derivative \ = \ \int \ A \ (B \ * \ C) \ =$$

$$\int \ (B * C) \ A \ = \ \int \ (B * C) * A + total \ derivative \ = \ \int \ B * C * A \tag{1.33b}$$

therefore, when the star product is associative and the fields and their derivatives vanishing fast enough at infinity (or there are no boundaries) one has

$$\int \ A \ * \ B \ * C \ = \ \int \ B \ * C \ * \ A \ = \ \int \ C \ * A \ * \ B \ . \tag{1.33c}$$

The relations (1.33) are essential in order to construct invariant actions under star gauge transformations.

The C-space differential form associated with the polyvector-valued Clifford gauge field is

$$\mathbf{A} \ = \ \mathcal{A}_M \ dX^M \ = \ \Phi \ d\sigma \ + \ \mathcal{A}_\mu \ dx^\mu \ + \ \mathcal{A}_{\mu\nu} \ dx^{\mu\nu} \ + \ .... \ +$$

$$\mathcal{A}_{\mu_1 \mu_2 .....\mu_d} \ dx^{\mu_1 \mu_2 ....\mu_d}.. \tag{1.34a}$$

where $\Phi = \Phi^A \ \Gamma_A, \ \mathcal{A}_\mu = \mathcal{A}_\mu^A \ \Gamma_A, \ \mathcal{A}_{\mu\nu} = \mathcal{A}_{\mu\nu}^A \ \Gamma_A, ...$ The C-space differential form associated with the polyvector-valued field-strength is

$$\mathbf{F} \ = \ F_{MN} \ dX^M \ \wedge \ dX^N \ = \ F_{0 \ \mu} \ d\sigma \ \wedge \ dx^\mu \ + \ F_{0 \ \mu_1 \mu_2} \ d\sigma \ \wedge \ dx^{\mu_1 \mu_2} \ + \ ....$$

$$F_{0\ \nu_1\nu_2.....\nu_d}\ d\sigma \wedge dx^{\nu_1\nu_2......\nu_d} + F_{\mu\nu}\ dx^\mu \wedge dx^\nu + F_{\mu_1\mu_2\ \nu_1\nu_2}\ dx^{\mu_1\mu_2} \wedge dx^{\nu_1\nu_2} + ...$$

$$+\ F_{\mu_1\mu_2.....\mu_{d-1}\ \nu_1\nu_2.....\nu_{d-1}}\ dx^{\mu_1\mu_2......\mu_{d-1}}\ \wedge\ dx^{\nu_1\nu_2.......\nu_{d-1}}. \qquad (1.34b)$$

The field strength is antisymmetric under the exchange of poly-vector indices $F_{MN} = -F_{NM}$. For this reason one has $F_{00} = 0$ and $F_{12....d\ 12.....d} = 0$. Finally, given the noncommutative conditions on the poly-vector coordinates (1.27), the components of the Clifford-algebra valued field strength $F^C_{MN}\Gamma_C$ in *Noncommutative* C-spaces are

$$F_{[MN]}\ =\ F^C_{[MN]}\ \Gamma_C\ =\ (\ \partial_M\ \mathcal{A}^C_N\ -\ \partial_N\ \mathcal{A}^C_M\ )\ \Gamma_C\ +$$

$$\frac{1}{2}\ (\ \mathcal{A}^A_M * \mathcal{A}^B_N\ -\ \mathcal{A}^B_N * \mathcal{A}^A_M\ )\ \{\ \Gamma_A,\ \Gamma_B\ \} + \frac{1}{2}\ (\ \mathcal{A}^A_M * \mathcal{A}^B_N\ +\ \mathcal{A}^B_N * \mathcal{A}^A_M\ )\ [\ \Gamma_A,\ \Gamma_B\ ]. \qquad (1.35)$$

The commutators $[\ \Gamma_A,\ \Gamma_B\ ]$ and anti-commutators $\{\ \Gamma_A,\ \Gamma_B\ \}$ in eqs-(35), where A, B are polyvector-valued indices, can be read from the relations in eqs-(1.7-1.10) . Notice that both the standard commutators *and* anticommutators of the gammas appear in the terms containing the star deformed products of (1.35) and which define the Clifford-algebra valued field strength in noncommutative C-spaces; i.e. if the products of fields were to commute one would have had only the Lie algebra commutator $\mathcal{A}^A_M \mathcal{A}^J_B\ [\Gamma_A, \Gamma_B]$ pieces without the anti-commutator $\{\Gamma_A, \Gamma_B\}$ contributions in the r.h.s of eq-(1.35).

We should remark that one is *not* deforming the Clifford algebra involving $[\ \Gamma_A,\ \Gamma_B\ ]$ and $\{\ \Gamma_A,\ \Gamma_B\ \}$ in eq-(1.35) but it is the "point" product algebra $\mathcal{A}^A_M * \mathcal{A}^B_N$ of the fields which is being deformed. (Quantum) q-Clifford algebras have been studied by [13]. The symmetrized star product is

$$\mathcal{A}^A_M\ *_s\ \mathcal{A}^B_N\ \equiv\ \frac{1}{2}\left(\mathcal{A}^A_M * \mathcal{A}^B_N + \mathcal{A}^B_N * \mathcal{A}^A_M\right)\ =\ \mathcal{A}^A_M\ \mathcal{A}^B_N\ +$$

$$X^{\mu\nu}\ X^{\kappa\lambda}\ (\partial_\mu\ \partial_\kappa\ \mathcal{A}^A_M)\ (\partial_\nu\ \partial_\lambda\ \mathcal{A}^B_N)\ +\ ...... \qquad (1.36a)$$

the antisymmetrized (Moyal bracket) star product is

$$\mathcal{A}^A_M *_a \mathcal{A}^B_N\ \equiv\ \frac{1}{2}\left(\mathcal{A}^A_M * \mathcal{A}^B_N - \mathcal{A}^B_N * \mathcal{A}^A_M\right)\ =\ X^{\mu\nu}\ (\partial_\mu\ \mathcal{A}^A_M)\ (\partial_\nu\ \mathcal{A}^B_N)\ +$$

$$X^{\mu\nu}\ X^{\kappa\lambda}\ X^{\alpha\beta}\ (\partial_\mu\ \partial_\kappa\ \partial_\alpha\ \mathcal{A}^A_M)\ (\partial_\nu\ \partial_\lambda\ \partial_\beta\ \mathcal{A}^B_N)\ +\ .... \qquad (1.36b)$$

It is important to emphasize, as it is customary in Moyal star products, that the poly-vector coordinates appearing in the r.h.s of eqs-(35-36) are treated as c-numbers (as if they were commuting) while it is the product of functions appearing in the l.h.s of (1.35-1.36) which are *noncommutative*.

Star products in noncommutative C-space lead to *many more terms* in eqs-(1.35-1.36) than in ordinary noncommutative spaces. For example, there are derivatives terms involving polyvectors which do *not* appear in ordinary noncommutative spaces, like

$$-4\ g^{\mu_1\nu_1}\ X^{\mu_2\nu_2}\ \frac{\partial \mathcal{A}^A_M}{\partial X^{\mu_1\mu_2}}\ \frac{\partial \mathcal{A}^B_N}{\partial X^{\nu_1\nu_2}}\ \pm\ .... \qquad (1.37a)$$

$$2\ (\ g^{\mu_2\nu}\ X^{\mu_1}\ -\ g^{\mu_1\nu}\ X^{\mu_2}\ )\ \frac{\partial \mathcal{A}^A_M}{\partial X^{\mu_1\mu_2}}\ \frac{\partial \mathcal{A}^B_N}{\partial X^{\nu}}. \qquad (1.37b)$$

$$X^{\mu_1\mu_2\mu_3\nu}\ \frac{\partial \mathcal{A}^A_M}{\partial X^{\mu_1\mu_2\mu_3}}\ \frac{\partial \mathcal{A}^J_B}{\partial X^{\nu}}. \qquad (1.37c)$$

$$96\ G^{\mu_1\mu_2\mu_3\ \nu_1\nu_2\nu_3}\ X^{\mu_4\nu_4}\ \frac{\partial \mathcal{A}^A_M}{\partial X^{\mu_1\mu_2\mu_3\mu_4}}\ \frac{\partial \mathcal{A}^B_N}{\partial X^{\nu_1\nu_2\nu_3\nu_4}},\quad etc\ ...... \qquad (1.37d)$$

There is a *subalgebra* of the noncommutative polyvector-valued coordinates algebra (1.27) involving only X^μ and the bivector coordinates $X^{\mu\nu}$ when the spacetime metric components $g_{\mu\nu}$ are *constant*. However, because $[X^{\mu_1\mu_2}, X^\nu] \neq 0$ one must not confuse the algebra in this case with the ordinary Θ-noncommutative algebra $[X^{\mu_1}, X^{\mu_2}] = \Theta^{\mu_1\mu_2}$ where the components of $\Theta^{\mu_1\mu_2}$ are comprised of *constants* such that $[\Theta^{\mu_1\mu_2}, X^\nu] = 0$.

The analog of a Yang-Mills action in C-spaces when the background C-space flat metric G^{MN} is X-independent is given by

$$S\ =\ \int\ [DX]\ <\ F^A_{MN}\ \Gamma_A\ *\ F^B_{PQ}\ \Gamma_B\ >\ G^{MP}\ G^{NQ}. \qquad (1.38)$$

where $<\ \Gamma_A\ \Gamma_B\ >$ denotes the Clifford-scalar part of the Clifford geometric product of two generators. As mentioned in the introduction suitable powers of a length scale must be included in the expansion of the terms inside the integrand in order to have consistent dimensions (the action is dimensionless). The action (1.38) becomes

$$\int\ [DX]\ (\ F_{MN}\ *\ F^{MN}\ +\ F^a_{MN}\ *\ F^{MN}_a\ +$$

$$F^{a_1a_2}_{MN}\ *\ F^{MN}_{a_1a_2}\ +\ ....\ +\ F^{a_1a_2.....a_d}_{MN}\ *\ F^{MN}_{a_1a_2.....a_d}\ ). \qquad (1.39)$$

the measure in C-space is given by

$$DX\ =\ ds\ \prod dx^\mu\ \prod\ dx^{\mu_1\mu_2}\ \prod dx^{\mu_1\mu_2\mu_3}\ .....\ dx^{\mu_1\mu_2.......\mu_d}. \qquad (1.40a)$$

The Clifford-valued gauge field $\mathcal{A}_M$ transforms under star gauge transformations according to $\mathcal{A}'_M\ =\ U^{-1}_* * \mathcal{A}_M * U_* + U^{-1}_* * \partial_M U_*$. The field strength F transforms covariantly $F'_{MN} = U^{-1}_* * F_{MN} * U_*$ such that the action (1.39) is star gauge invariant. $U_* = exp_*(\xi(X)) = exp_*(\xi^A(X)\Gamma_A)$ is defined via a star power series expansion $U_* = \sum_n \frac{1}{n!}(\xi(X))^n_*$ where $(\xi(X))^n_* = \xi(X) * \xi(X) * ..... * \xi(X)$. The integral $\int F * F\ =\ \int F\ F\ +$ total derivatives. If the fields vanish fast enough at infinity and/or there are no boundaries, the contribution of the total derivative terms are zero.

When the star product is truly *associative* one has star gauge invariance of the action (1.39) under infinitesimal $\delta F = [F, \xi]_*$ transformations

$$\delta_\xi S \;=\; 2 \int \;<\; F * [F,\; \xi]_* \;>\; \;=\; 2 \int \;<\; F * F * \xi \;>\; -2 \int \;<\; F * \xi * F \;>. \qquad (40b)$$

If the star product is associative due to the relations in eqs-(33) one can show that eq-(1.40b) becomes (up to a trivial factor of 2)

$$\int \; F^A * F^B * \xi^C \;<\; \Gamma_A \, \Gamma_B \, \Gamma_C \;>\; - \int \; F^A * \xi^C * F^B \;<\; \Gamma_A \, \Gamma_C \, \Gamma_B \;>\; =$$

$$\int \; F^B * \xi^C * F^A \;<\; \Gamma_B \, \Gamma_C \, \Gamma_A \;>\; - \int \; F^A * \xi^C * F^B \;<\; \Gamma_A \, \Gamma_C \, \Gamma_B \;>\; = \; 0 \qquad (1.40c)$$

so one arrives at the zero result in (1.40c), assuring $\delta S = 0$, after using the *cyclic* property of the scalar part of the geometric product

$$<\; \Gamma_A \, \Gamma_B \, \Gamma_C \;>\; = \; <\; \Gamma_B \, \Gamma_C \, \Gamma_A \;>\; = \; <\; \Gamma_C \, \Gamma_A \, \Gamma_B \;> \qquad (1.40d)$$

and *relabeling* the indices $B \leftrightarrow A$ in the third term of (1.40c).

To finalize this section we will write the star product deformations of a scalar field theory $\Phi = \Phi(X^A)$ which depends on the poly-vector coordinates X^A . A typical Lagrangian is of the form

$$\frac{1}{2}(\partial_M \Phi) * (\partial^M \Phi) - \frac{m^2}{2} \, \Phi(X) * \Phi(X) - \frac{g^n}{n!} \, \Phi(X) * \Phi(X) * ....* \Phi(X). \quad (1.41)$$

and leads to the equations of motion

$$-\frac{\partial^2 \Phi(X)}{\partial X^2} \; - \; m^2 \; \Phi(X) \; - \; \frac{g^n}{(n-1)!} \; \Phi(X) * \Phi(X) * .... * \Phi(X) = 0. \quad (1.42)$$

Having provided the basic ideas and results behind polyvector gauge field theories in Noncommutative Clifford spaces, the construction of Noncommutative Clifford-space gravity as polyvector valued gauge theories of twisted diffeomorphisms in C-spaces will be the subject of future investigations. It would require quantum Hopf algebraic deformations of Clifford algebras [13]. Such theory is far richer than gravity in Noncommutative spacetimes [17].

2 Noncommutative p-branes

The Dirac-Nambu-Goto p-brane action is

$$S \; = \; T \int \; [d^{p+1}\sigma] \; \sqrt{\; |det \; (G_{ab})|} = \; T \int \; [d^{p+1}\sigma] \; \sqrt{|det \; [G_{\mu\nu} \; (\partial_a X^\mu) \; (\partial_b X^\nu)]|}. \tag{2.1}$$

where T is the p-brane tension. When the target spacetime background is $flat$, $G_{\mu\nu} \; = \; \eta_{\mu\nu}$, the determinant can be rewritten in terms of Nambu Poisson Brackets (NPB) as

$$det \; (G_{ab}) \; = \; \{ \; X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{p+1}} \; \} \; \{ \; X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{p+1}} \}_{NPB}. \tag{2.2}$$

However, when the target spacetime background is *curved*, $G_{\mu\nu} = G_{\mu\nu}(X^\rho(\sigma))$, the determinant is

$$\{ \; X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{p+1}} \; \} \; \{ \; X^{\nu_1}, X^{\nu_2}, ..., X^{\nu_{p+1}} \} \; G_{\mu_1\nu_1} G_{\mu_2\nu_2} ......G_{\mu_{p+1}\nu_{p+1}}. \tag{2.3}$$

and one cannot naively pull the metric factors $G_{\mu\nu}$ inside the brackets and perform the index contractions inside the brackets.

The simplest way to construct Noncommutative brane actions is to use star products. A star-product deformation of the Nambu-Poisson Brackets can be defined when $p + 1 = d = 2n = even$ as follows [20]

$$\{ \; X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{p+1}} \}_* =$$

$$\{ \; X_{\mu_1}, X_{\mu_2} \; \}_* \; * \; \{ \; X_{\mu_3}, X_{\mu_4} \; \}_* \; * \; ..... * \; \{ \; X_{\mu_p}, X_{\mu_{p+1}} \; \}_* \; \pm \; .... \tag{2.4}$$

where the ellipsis denotes signed permutations; i.e. the star-product deformations of the Nambu-Poisson-Brackets can be decomposed as a suitable anti-symmetrized sum of the star products of the Moyal brackets among *pairs* of variables. For instance

$$\{A, B, C, D\}_* \; = \; \{A, B\}_* * \{C, D\}_* + \{C, D\}_* * \{A, B\}_* + \{C, A\}_* * \{B, D\}_* +$$

$$\{B, D\}_* \; * \; \{C, A\}_* \; + \; \{D, A\}_* \; * \; \{C, B\}_* \; + \; \{C, B\}_* \; * \; \{D, A\}_* \tag{2.5}$$

Each term in (2.5) splits into 4 terms giving a total of $4 \times 6 = 24 = 4!$ terms out of which 12 have a positive sign and 12 have a negative sign. The Moyal brackets

$$\{X^{\mu_1}, \; X^{\mu_2}\}_* \; = \; X^{\mu_1} \; * \; X^{\mu_2} \; - \; X^{\mu_2} \; * \; X^{\mu_1}. \tag{2.6}$$

are defined in terms of the *noncommutative* and *associative* star product

$$(\; X^{\mu_1} * X^{\mu_2} \;)(\sigma) \; = \; exp\left(\; \frac{i}{2} \; \sigma^A \Lambda_A[\; i \; \partial_{\sigma'}; \; i \; \partial_{\sigma''} \;] \; \right) \; X^{\mu_1}(\sigma') \, X^{\mu_2}(\sigma'')|_{\sigma'=\sigma''=\sigma}. \tag{2.7}$$

where the expression for the bilinear differential polynomial $\Lambda_A[i\partial_{\sigma'}; i\partial_{\sigma''}]$ in eq-(2.7) is

$$\Lambda_A[k,p] \;=\; i\; k_B\; p_C\; f_A^{BC} \;+\; \frac{i^2}{6}\; k_{B_1}\; p_{C_1}\; (p_{B_2} - k_{B_2})\; f_D^{B_1C_1}\; f_A^{DB_2} \;+$$

$$\frac{i^3}{24}\; (p_{B_2}\; k_{C_2} \;+\; k_{B_2}\; p_{C_2})\; k_{B_1}\; k_{C_1}\; f_{D_1}^{B_1C_1}\; f_{D_2}^{D_1B_2}\; f_A^{D_2C_2} \;+\; .... \tag{2.8}$$

and is given in terms of the structure constants $[\sigma^B, \sigma^C] = f_A^{BC} \sigma^A$, after setting $k_B = i\; \partial_{\sigma'^B},\; p_C = i\; \partial_{\sigma''^C}$. The target Clifford-space background poly-vector coordinates $X^M(\sigma^A)$ are functions of the poly-vector valued coordinates corresponding to the poly-vector valued world manifold

$$\sigma^A\; \gamma_A \;=\; \sigma\; \mathbf{1} \;+\; \sigma^a\; \gamma_a \;+\; \sigma^{a_1a_2}\; \gamma_{a_1} \wedge \gamma_{a_2} \;+ ....\; +\; \sigma^{a_1a_2....a_d}\; \gamma_{a_1} \wedge \gamma_{a_2}\; ..... \wedge \gamma_{a_d}. \tag{2.9}$$

The commutators $[\sigma^B, \sigma^C] = f_A^{BC} \sigma^A$ are defined in the same manner as the noncommutative poly-vector coordinates algebra (1.15-1.24)

$$[\; \sigma^{a_1},\; \sigma^{a_2}\;]_* \;=\; \sigma^{a_1}\; *\; \sigma^{a_2} \;-\; \sigma^{a_2}\; *\; \sigma^{a_1} \;=\; 2\; \sigma^{a_1a_2}. \tag{2.10a}$$

$$[\; \sigma^{a_1a_2},\; \sigma^b\;]_* \;=\; \sigma^{a_1a_2}\; *\; \sigma^b \;-\; \sigma^b\; *\; \sigma^{a_1a_2} \;=$$
$$4\; \left(\; g^{a_2b}\; \sigma^{a_1} \;-\; g^{a_1b}\; \sigma^{a_2}\; \right). \tag{2.10b}$$

$$[\; \sigma^{a_1a_2a_3},\; \sigma^b\;]_* \;=\; \sigma^{a_1a_2a_3}\; *\; \sigma^b \;-\; \sigma^b\; *\; \sigma^{a_1a_2a_3} \;=\; 2\; \sigma^{a_1a_2a_3b}. \tag{2.10c}$$

$$[\; \sigma^{a_1a_2a_3a_4},\; \sigma^b\;]_* \;=\; \sigma^{a_1a_2a_3a_4} * \sigma^b - \sigma^b * \sigma^{a_1a_2a_3a_4} \;=\; -8\; g^{a_1b}\; \sigma^{a_2a_3a_4} \pm ...... \tag{2.10d}$$

etc.....

When $p + 1 = odd$, attempts have been made to introduce quantum deformations based on the Zariski star product deformations of the Nambu Poisson Brackets (NPB), but unfortunately these deformed brackets failed to obey all the required algebraic properties of a (quantum) bracket [20]. Therefore, to our knowledge, only when $p+1 = 2n$ is even one can perform a suitable star product deformations of the Nambu-Poisson Brackets (NPB).

Therefore, we construct the star deformed brane action in C-spaces by using the star products and brackets in the special case when $2^d = 2n = even$ and the target spacetime is $flat$. Secondly, one replaces the spacetime vector X^μ for the target C-space poly-vector coordinates X^M which are functions of σ^A

$$X^M(\sigma^A) \;=\; (\; s(\sigma^A),\; x^\mu(\sigma^A),\; x^{\mu_1\mu_2}(\sigma^A),\; ......,\; x^{\mu_1\mu_2.....\mu_D}(\sigma^A)\;). \tag{2.11}$$

Finally, the star deformed brane action in C-spaces when $D \geq d$ is

$$T\; \int [D^{2^d}\sigma] \sqrt{|\; \frac{1}{(2^d)!}\; \{\; X_{M_1}, X_{M_2}, ..., X_{M_{2^d}}\; \}_* \;*\; \{\; X^{M_1}, X^{M_2}, ..., X^{M_{2^d}} \}_*\; |}. \tag{2.12}$$

where the 2^d-dimensional Clifford-valued world-manifold measure is defined as

$$[D^{2^d}\sigma] \; = \; d\sigma \; \prod d\sigma^a \; \prod d\sigma^{a_1 a_2} \; ...... \; d\sigma^{a_1 a_2 .....d}. \qquad (2.13)$$

If one scales the poly-vector coordinates X^M and σ^A by suitable powers of a length scale (Planck scale for example) to render all the coordinates dimensionless, the star deformed brackets $\{ \; X_{M_1}, X_{M_2}, ..., X_{M_{2^d}} \; \}_*$ will be dimensionless and so will the tension parameter be dimensionless as well. In the ordinary p-brane action, the dimensions of the p-brane tension is that of $(mass)^{p+1}$. Instead of having a Clifford space of dimension 2^D as a target background one could have an ordinary spacetime with vector coordinates $X^\mu(\sigma^A)$ only, such that $\mu = 1, 2, ..., D$ and $D \geq 2^d$. However, it is more general to have a Clifford space of dimension 2^D as a target background for the poly-vector valued world manifold : $X^M(\sigma^A)$.

3 N-ary Algebras and Clifford Spaces

Ternary algebras have recently resurfaced with great intensity in the study of $M2$-brane duality where M theory on $AdS_4 \times S^7$ is dual to a superconformal field theory in three dimensions, with the supergroup $OSp(8|4)$, after Bagger-Lambert-Gustavsson (BLG) [22] constructed a Chern-Simons gauge theory in three dimensions with maximal supersymmety $\mathcal{N} = 8$. However, their construction only works for the $SO(4)$ gauge group and it does not provide the desired dual to M-theory on $AdS_4 \times S^7$ [23]. The authors [24] later have shown that the dual gauge theory is actually an $\mathcal{N} = 6$ superconformal Chern-Simons theory in three-dimensions and is associated to M-theory on $AdS_4 \times S^7/Z_k$, with N units of flux. The $M5$-brane duality is based on M theory on $AdS_7 \times S^4$ being dual to a six dimensional superconformal field theory whose super group is $OSp(6, 2|4)$. Recently it was shown by [25] how the $M5$ brane can be obtained from a mass deformed BLG theory which is realized by a Nambu bracket and such that a maximally supersymmetric Lagrangian for the fluctuation fields exists corresponding to a single $M5$ brane on $\mathbf{R^{1,2}} \times \mathbf{S^3}$.

N-ary algebras have been known for some time [20] since Nambu introduced his bracket (a Jacobian) in the study of branes and the generalizations of Hamiltonian mechanics based on Poisson brackets. In this section we shall show how poly-vector valued coordinates admit a very natural interpretation in terms of $n-$ary commutators.

The ternary commutator for noncommuting coordinates is defined as

$$[X^1, X^2, X^3] \; = \; X^1 \; [X^2, X^3] \; + \; X^2 \; [X^3, X^1] \; + \; X^3 \; [X^1, X^2] \; =$$

$$\frac{1}{2} \; \{ \; X^1, \; [X^2, X^3] \; \} \; + \; \frac{1}{2} \; [\; X^1, \; [X^2, X^3] \;] \; + \; cyclic \; permutations \qquad (3.1)$$

Due to the Jacobi identities, the terms

$$\frac{1}{2} \; [\; X^1, \; [X^2, X^3] \;] \; + \; cyclic \; permutations \; = \; 0. \qquad (3.2)$$

so that the ternary commutators become

$$[X^1, X^2, X^3] \ = \ \frac{1}{2} \ \{ \ X^1, \ [X^2, X^3] \ \} \ + \ cyclic \ permutations. \tag{3.3}$$

After using the relations, from eqs-(1.15-1.25),

$$[X^2, X^3] \ = \ 2 \ X^{23}, \quad \{ \ X^1, \ X^{23} \ \} \ = \ 2 \ X^{123}. \tag{3.4}$$

one gets finally

$$[X^1, X^2, X^3] \ = \ 2 \ X^{123} \ + \ cyclic \ permutations \ = \ 6 \ X^{123}. \tag{3.5}$$

since $X^{123} = X^{231} = X^{312} = -X^{132} = ...$

The 4-ary commutator is defined as

$$[X^1, X^2, X^3, X^4] \ = \ X^1 \ [X^2, X^3, X^4] \ - \ X^2 \ [X^3, X^4, X^1] \ + \\ X^3 \ [X^4, X^1, X^2] \ - \ X^4 \ [X^1, X^2, X^3] \ = \\ \frac{1}{2} \ \{ \ X^1, \ [X^2, X^3, X^4] \ \} \ + \ \frac{1}{2} \ [\ X^1, \ [X^2, X^3, X^4] \ ] \ - \ .... \ = \\ 3 \ \{ \ X^1, \ X^{234} \ \} \ + \ 3 \ [\ X^1, \ X^{234} \ ] \ - \ .... \ = \\ 6 \ X^{1234} \ + \ 18 \ (\ g^{12} \ X^{34} \ + \ g^{13} \ X^{42} \ + \ g^{14} \ X^{23} \ ) \ - \ .... \ = \ 24 \ X^{1234} \tag{3.6}$$

due to the cancellations

$$(\ g^{12} \ X^{34} \ + \ g^{13} \ X^{42} \ + \ g^{14} \ X^{23} \ ) \ - \ (\ g^{23} \ X^{41} \ + \ g^{24} \ X^{13} \ + \ g^{21} \ X^{34} \ ) \ + \\ (\ g^{34} \ X^{12} + g^{31} \ X^{24} + g^{32} \ X^{41} \ ) - (\ g^{41} \ X^{23} + g^{42} \ X^{31} + g^{43} \ X^{12} \ ) \ = \ 0. \tag{3.7}$$

resulting from the conditions $X^{\mu\nu} = -X^{\nu\mu}$, $g^{\mu\nu} = g^{\nu\mu}$ after recurring to the (anti) commutators

$$[X^1, X^{234}] \ = \ 2 \ X^{1234}, \quad \{X^1, X^{234}\} \ = \ 6 \ (g^{12} \ X^{34} \ + \ g^{13} \ X^{42} \ + \ g^{14} \ X^{23} \ ). \tag{3.8}$$

and the conditions $X^{1234} = -X^{2341} = X^{3412} = -X^{4123}$. For example, given a Noncommutative Clifford space in $D = 4$, one arrives at

$$[X^1, X^2] \ = \ 2 \ X^{12}, \ [X^1, X^2, X^3] \ = \ 6 \ X^{123}, \ [X^1, X^2, X^3, X^4] \ = \ 24 \ X^{1234}. \tag{3.9}$$

where X^1, X^2, X^3, X^4 is a shorthand notation for $X^{\mu_1}, X^{\mu_2}, X^{\mu_3}, X^{\mu_4}$. Therefore, one finds that the poly-vector coordinates $X^{\mu_1\mu_2}, X^{\mu_1\mu_2\mu_3}, X^{\mu_1\mu_2\mu_3\mu_4}$ can be seen, respectively, as the binary, ternary and 4-ary commutators of the noncommuting vector coordinates X^μ. In the general case, using the noncommutative algebra of eqs-(1.15-1.25) in Clifford spaces one arrives by recursion at

$$[\ X^1, \ X^2, \ ...., \ X^n \ ] \ = \ n! \ X^{123.....n}. \tag{3.10}$$

This n-ary commutator interpretation of the poly-vector valued coordinates of a noncommutative Clifford space warrants further investigation.

At this stage it is important to emphasize that the Noncommutative Clifford-valued poly-vector coordinates algebra given by eqs-(1.15-1.25) does *not* satisfy the Nambu-Filipov conditions which can be written as

$$\mathcal{D}_{[X^1,X^2]}\ [Y^1,\ Y^2,\ Y^3]\ =\ [\ X^1,\ X^2,\ [Y^1,\ Y^2,\ Y^3]\ ]\ =$$

$$[\ [X^1,X^2,Y^1],\ Y^2,\ Y^3\ ] + [\ Y^1,\ [X^1,X^2,Y^2],\ Y^3\ ] + [\ Y^1,\ Y^2,\ [X^1,X^2,Y^3]\ ]. \tag{3.11a}$$

$$[\ X^1,\ X^2,\ ....,X^{n-1},\ [\ Y^1,\ Y^2,\ ....,\ Y^n\ ]\ ]\ =$$

$$[\ [\ X^1,\ X^2,\ ....,X^{n-1},\ Y^1\ ],\ Y^2,\ ....,\ Y^n\ ]\ +$$

$$[\ Y^1,\ [\ X^1,\ X^2,\ ....,X^{n-1},\ Y^2\ ],\ Y^3,\ ....,Y^n\ ]\ +\ ....\ +$$

$$[\ Y^1,\ Y^2,\ ....,Y^{n-1},\ [\ X^1,\ X^2,\ ....,X^{n-1},\ Y^n\ ]\ ]. \tag{3.11b}$$

For n-ary brackets, Nambu showed that the Jacobian (the classical Nambu-Poison bracket)

$$\{X^1,\ X^2,........,\ X^n\}\ =\ \epsilon^{i_1 i_2 .....i_n}\ \partial_{i_1} X^1\ \partial_{i_2} X^2\ ....\ \partial_{i_n} X^n. \tag{3.12}$$

satisfies the Nambu-Filippov special conditions, [18], [20]. It is not difficult to see that

$$[\ X^1,\ X^2,\ [X^3,\ X^4,\ X^5]\ ]\ \neq$$

$$[\ [X^1,X^2,X^3],\ X^4,\ X^5\ ] + [\ X^3,\ [X^1,X^2,X^4],\ X^5\ ] + [\ X^3,\ X^4,\ [X^1,X^2,X^5]\ ]. \tag{3.13}$$

The main reason being that the ternary commutator

$$[X^1,X^2,X^3]\ =\ 6\ X^{123}\ \neq\ \sum_i\ f^{123}{}_i\ X^i. \tag{3.14}$$

Naturally, the Jacobi identity is satisfied

$$[\ X^1,\ [X^2,\ X^3]\ ]\ =\ [\ [X^1,\ X^2],\ X^3\ ]\ +\ [\ X^2,\ [X^1,\ X^3]\ ]. \tag{3.15}$$

n-ary algebras are relevant to the large N limit of covariant Matrix Models based on generalized n-th power matrices (hyper-matrices) [21] $\mathbf{X}_{i_1 i_2 ......i_n}$, that are extensions of square, cubic, quartic, ... matrices (hyper-matrices). These Matrix models bear a relationship to Eguchi-Schild p-brane actions for $p+1 = n$. The range of indices is $i_1, i_2, ..., i_n \subset I = 1, 2, ...N$. The n-ary commutator of n generalized n-th power matrices (hyper matrices) in the large $N \to \infty$ has a correspondence with the Nambu-brackets (NB) as follows

$$[\ \mathbf{X}^1,\ \mathbf{X}^2,\ ....,\ \mathbf{X}^n\ ]_{i_1 i_2 ......i_n}\ \to\ \{\ X^1,\ X^2,\ ....,\ X^n\ \}_{NB}. \tag{3.14}$$

by replacing the hyper matrix $\mathbf{X}_{i_1 i_2 ...... i_n}$ in the large $N \rightarrow \infty$ limit for the c-function of n-variables $X(\sigma^1, \sigma^2, ..., \sigma^n)$. The trace operation in the large N limit has a correspondence with the integral $\int d^n\sigma$ so that

$$Trace \ \left([\ \mathbf{X}^1, \ \mathbf{X}^2, \ ...., \ \mathbf{X}^n \ ]^2 \ \right) \ \rightarrow \ \int \ d^n\sigma \ \{ \ X^1, \ X^2, \ ...., \ X^n \ \}^2_{NB}. \tag{3.15}$$

recovering in this fashion the Eguchi-Schild p-brane actions for $p+1 = n$. The fermionic version of (3.15) is

$$\int \ d^n\sigma \ \bar{\Psi} \ \Gamma_{12....n-1} \ \{ \ X^1, \ X^2, \ ...., \ X^{n-1} \ , \Psi \}. \tag{3.16}$$

Covariant (super) brane actions based on n-ary structures and generalized matrix models have been recently constructed by [28]. The authors [26] have shown that the light-cone gauge-fixed action of a super p-brane belongs to a *new* kind of supersymmetric gauge theory of p-volume preserving diffeomorphisms (diffs) associated with the p-spatial dimensions of the extended object. These authors conjectured that this new kind of supersymmetric gauge theory must be related to an infinite-dim *nonabelian* antisymmetric gauge theory. It was recently shown in [27] how this new theory should be part of an underlying antisymmetric nonabelian tensorial gauge field theory of $p+1$-dimensional diffs (upon supersymmetrization) associated with the world volume evolution of the p-brane.

Ternary algebraic structures appearing in various domains of theoretical and mathematical physics were reviewed by [34], like the notion of quark algebraic confinement based on a Z_3 -graded matrix algebra over the complex field $\mathbf{C}$. A generalization of non-commutative geometry and gauge theories based on *ternary* Z_3-graded structures was constructed by [34]. The usual Z_2-graded structures such as Grassmann, Lie and Clifford algebras are generalized to the Z_3-graded case leading to *hypersymmetry* which is a Z_3 graded generalization of supersymmetry. The de Rham complex with the differential operator d satisfies the condition $d^3 = 0$ instead of $d^2 = 0$. Ternary generalizations of Clifford algebras were defined by the relations [34]

$$Q^a \ Q^b \ Q^c \ = \ \omega \ Q^b \ Q^c \ Q^a \ + \ \omega^2 \ Q^c \ Q^a \ Q^b \ + \ 3 \ \rho^{abc} \ \mathbf{1} \tag{3.17}$$

where ω is the cubic root of unity $e^{i2\pi/3}$ and ρ^{abc} is the analog of a cubic metric (a cubic matrix) obeying the conditions

$$\rho^{abc} \ + \ \omega \ \rho^{bca} \ + \ \omega^2 \ \rho^{cab} \ = \ 0. \tag{3.18}$$

Our whole construction of C-spaces [3] based on ordinary Clifford algebras can be extended to ternary Clifford algebras. By replacing the cubic roots of unity for the N-th roots of unity and the cubic metric for $\rho^{a_1 a_2 .....a_n}$ one can define the N-ary generalizations of Clifford algebras. In [19] and references therein one can find a generalization of n-ary Nambu algebras and beyond.

4 Branes in Noncommutative (Clifford) Phase spaces

Born's reciprocal relativity [29] in flat spacetimes is based on the principle of a *maximal* speed limit (speed of light) and a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) and where coordinates and momenta are unified on a single footing. We extended Born's theory to the case of curved spacetimes and constructed a noncommutative $deformed$ Born reciprocal general relativity theory in curved spacetimes [31] (without the need to introduce star products) as a local gauge theory of the $deformed$ Quaplectic group [30] that is given by the semi-direct product of $U(1,3)$ with the $deformed$ (noncommutative) Weyl-Heisenberg group corresponding to *noncommutative* generators $[Z_a, Z_b] \neq 0$. The Hermitian metric is complex-valued with symmetric and nonsymmetric components and there are *two* different complex-valued Hermitian Ricci tensors $\mathcal{R}_{\mu\nu}, \mathcal{S}_{\mu\nu}$. The deformed Born's reciprocal gravitational action linear in the Ricci scalars $\mathcal{R}, \mathcal{S}$ with Torsion-squared terms and BF terms was also provided [31].

Since phase spaces are an essential ingredient in Born's reciprocal relativity [29] where coordinates and momenta are interchangeable, we begin by providing a description of Noncommutative spaces based on Yang's Noncommutative phase space algebra [14]. There is a *subalgebra* of the C-space operator-valued coordinates which is *isomorphic* to the Noncommutative Yang's $4D$ spacetime algebra [14]. This can be seen after establishing the following correspondence between the C-space vector/bivector (area-coordinates) algebra, associated to the $6D$ angular momentum (Lorentz) algebra, and the Yang's spacetime algebra via the $SO(6)$ generators Σ^{ij} in $6D$ $(i,j = 1,2,3......,6)$ as follows [15]

$$i\,\hbar\,\Sigma^{\mu\nu} \;\leftrightarrow\; i\,\frac{\hbar}{\lambda^2}\,\hat{X}^{\mu\nu}, \quad i\,\Sigma^{56} \;\leftrightarrow\; i\,\frac{R}{\lambda}\,\mathcal{N}. \tag{4.1a}$$

$$i\,\lambda\,\Sigma^{\mu 5} \;\leftrightarrow\; i\,\hat{X}^{\mu}, \quad i\,\Sigma^{\mu 6} \;\leftrightarrow\; i\,\frac{R}{\hbar}\,\hat{P}^{\mu} \tag{4.1b}$$

where the indices $\mu, \nu = 1,2,3,4$. The scales λ and R are a lower and upper scale respectively, like the Planck and Hubble scale. The $SO(6)$ algebra $[\Sigma^{ij},\ \Sigma^{kl}] = -\eta^{ik}\Sigma^{jl} + ...$ can be recast in terms of a *noncommutative* phase space algebra as

$$[\hat{P}^{\mu},\ \mathcal{N}] \;=\; -\,i\,\eta^{66}\,\frac{\hbar}{R^2}\hat{X}^{\mu}, \quad [\hat{X}^{\mu},\ \mathcal{N}] \;=\; i\,\eta^{55}\,\frac{\lambda^2}{\hbar}\,\hat{P}^{\mu}. \tag{4.2a}$$

$$[\hat{X}^{\mu},\ \hat{X}^{\nu}] \;=\; -i\,\eta^{55}\,\hat{X}^{\mu\nu}, \quad [\hat{P}^{\mu},\ \hat{P}^{\nu}] \;=\; -i\,\eta^{66}\,\frac{\hbar^2}{R^2\,\lambda^2}\,\hat{X}^{\mu\nu}, \quad \hat{X}^{\mu\nu} \;=\; \lambda^2\,\Sigma^{\mu\nu}. \tag{4.2b}$$

$$[\hat{X}^\mu,\ \hat{P}^\mu]\ =\ i\,\hbar\,\eta^{\mu\nu}\,\frac{\lambda}{R}\,\Sigma^{56}\ =\ i\,\hbar\,\eta^{\mu\nu}\,\mathcal{N},\quad [\hat{X}^{\mu\nu},\ \mathcal{N}]\ =\ 0. \qquad (4.2c)$$

The last relation is the *modified* Weyl-Heisenberg algebra in $4D$ since $\mathcal{N}$ does *not* commute with X^μ nor P^μ. The remaining *nonvanishing* commutation relations are

$$[\Sigma^{\mu\nu},\ \hat{X}^\rho]\ =\ -\,i\,\eta^{\mu\rho}\,\hat{X}^\nu\ +i\,\eta^{\nu\rho}\hat{X}^\mu \qquad (4.3a)$$

$$[\Sigma^{\mu\nu},\ \hat{P}^\rho]\ =\ -\,i\,\eta^{\mu\rho}\,\hat{P}^\nu\ +i\,\eta^{\nu\rho}\hat{P}^\mu. \qquad (4.3b)$$

$$[\Sigma^{\mu\nu},\ \Sigma^{\rho\tau}]\ =\ -\,i\,\eta^{\mu\rho}\,\Sigma^{\nu\tau}\ +i\,\eta^{\nu\rho}\,\Sigma^{\mu\tau}\ -\ .... \qquad (4.3c)$$

Eqs-(4.2-4.3) are the defining relations of the Yang's Noncommutative $4D$ spacetime algebra involving the $8D$ phase-space variables X^μ, P^μ and the angular momentum (Lorentz) generators $\Sigma^{\mu\nu}$ in $4D$. The above commutators obey the Jacobi identities. An immediate consequence of Yang's noncommutative algebra is that now one has a modified products of uncertainties

$$\Delta X^\mu\ \Delta P^\nu\ \geq\ \frac{\hbar}{2}\,\eta^{\mu\nu}\,||<\ \Sigma^{56}\ >||;\quad \Delta X^\mu\ \Delta X^\nu\ \geq\ \frac{\lambda^2}{2}\,||<\ \Sigma^{\mu\nu}\ >||$$

$$\Delta P^\mu\ \Delta P^\nu\ \geq\ \frac{1}{2}(\frac{\hbar}{R})^2\,||<\ \Sigma^{\mu\nu}\ >||. \qquad (4.4)$$

Next we shall present how to construct noncommutative p-brane actions based on the Yang's noncommutative phase space algebra. The target spacetime $X^\mu(\sigma^{ij})$ coordinates depend on the variables σ^{ij} where the double-index notation stands for

$$\sigma^{ij}\ :\ q^m = \sigma^{m\ d+1};\ \ p^m = \sigma^{m\ d+2};\ \ \sigma^{m\ n};\ \ \sigma^{d+1\ d+2}. \qquad (4.5)$$

with $m, n = 1, 2, ..., d$ and $i, j = 1, 2, ..., d, d+1, d+2$. The star product is

$$(\ X^{\mu_1} * X^{\mu_2}\ )(\sigma^{ij})\ =\ exp\left(\ \frac{i}{2}\ \sigma^{ij}\Lambda_{ij}[\ i\ \partial_{\sigma'};\ i\ \partial_{\sigma''}\ ]\ \right)\ X^{\mu_1}(\sigma')\ X^{\mu_2}(\sigma'')|_{\sigma'=\sigma''=\sigma}. \qquad (4.6)$$

where the expression for the bilinear differential polynomial $\Lambda_{ij}[i\partial_{\sigma'}; i\partial_{\sigma''}]$ in eq-(4.6) is

$$\Lambda_{i_4j_4}[k,p]\ =\ i\,k_{i_1j_1}\,p_{i_2j_2}\,f^{i_1j_1\ \ i_2j_2}_{i_4j_4}\ +\frac{i^2}{6}\,k_{i_1j_1}\,p_{i_2j_2}\,(p_{i_3j_3}-k_{i_3j_3})\,f^{i_1j_1\ \ i_2j_2}_{kl}\,f^{kl\ \ i_3j_3}_{i_4j_4}\ +$$

$$\frac{i^3}{24}\,(p_{i_5j_5}\ k_{i_6j_6}\ +\ k_{i_5j_5}\ p_{i_6j_6})\ k_{i_1j_1}\ k_{i_2j_2}\ f^{i_1j_1\ \ i_2j_2}_{k_1l_1}\ f^{k_1l_1\ \ i_5j_5}_{k_2l_2}\ f^{k_2l_2\ \ i_6j_6}_{i_4j_4}\ +\ .... \qquad (4.7)$$

and is given in terms of the structure constants $[\sigma^{i_1 j_1}, \sigma^{i_2 j_2}] = f^{i_1 j_1 \ i_2 j_2}_{i_3 j_3} \sigma^{i_3 j_3}$, after setting $k_{ij} = i\, \partial_{\sigma'^{ij}}$, $p_{ij} = i\, \partial_{\sigma''^{ij}}$. The structure constants can be obtained from the $so(d+2)$ algebra

$$[\ \sigma^{i_1 j_1},\ \sigma^{i_2 j_2}\ ] = \ (-\eta^{i_1 i_2}\ \sigma^{j_1 j_2}\ \pm\ ....)\ =\ f^{i_1 j_1 \ i_2 j_2}_{i_3 j_3} \sigma^{i_3 j_3}\ \Rightarrow$$

$$f^{i_1 j_1 \ i_2 j_2}_{i_3 j_3}\ =\ (-\eta^{i_1 i_2}\ \sigma^{j_1 j_2}\ \pm\ ....)\ \sigma_{i_3 j_3}\ =\ (-\eta^{i_1 i_2}\ \delta^{j_1 j_2}_{i_3 j_3}\ \pm\ ....). \qquad (4.8)$$

Since one requires the dimension of the world manifold to be even $2n$, in order to define the star product of $2n$ entries as sums of pairwise star products of two entries, and the dimension of the angular momentum algebra $so(d+2)$ is $(d+2)(d+1)/2$, one should satisfy the condition $2n = (d+2)(d+1)/2$. For example, when $d = 3$, one has that the variables $\sigma^{ij} = q^a, p^a, \sigma^{ab}, \sigma^{45}$ for $a, b = 1, 2, 3$ span an underlying 10-dim space : $3 + 3 + 3 + 1 = 10$. The target spacetime coordinates are functions $X^\mu = X^\mu(q^a, p^a, \sigma^{ab}, \sigma^{45})$, the range of indices is $\mu, \nu = 1, 2, ...D$ and the target spacetime dimension must $D \geq 10$. It is interesting that $D = 10$ is the critical dimension of the superstring.

When $d = 4, 5$, the dimensions of the angular momentum algebra $so(d+2)$ given by $(d+2)(d+1)/2$ are both odd, $15, 21$ respectively. When $d = 6$ the dimension of the angular momentum algebra $so(d+2)$ given by $(d+2)(d+1)/2 = 28$ is even and allows one to define the star product of 28 entries as sums of pairwise star products of two entries. The target spacetime dimension must be in this case $D \geq 28$. It is interesting that $D = 28$ is the dimension of the bosonic version of F theory and also the dimension of the quaternionic Jordan algebra $J_4[\mathbf{H}]$ which can be recast as the 4×4 matrix algebra with quaternionic entries.

The deformed brane action for the target spacetime coordinates $X^\mu = X^\mu(\sigma^{ij})$, is

$$T \int\ [D\sigma^{ij}]\ \sqrt{|\frac{1}{(2n)!}\ \{\ X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{2n}}\ \}_*\ *\ \{\ X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{2n}}\}_*\ |}. \qquad (4.9)$$

The $2n$-dim world-manifold measure is

$$D\sigma^{ij}\ =\ d\sigma^{d+1\ d+2}\ \prod d\sigma^{ab}\ \prod dq^a\ \prod dp^a\ . \qquad (4.10)$$

where the range of indices is $a, b = 1, 2, 3, ..., d$, while d itself must obey the condition $4n = (d+2)(d+1)$ in order to define the star product of $2n$ entries $\{X^{\mu_1}(\sigma^{ij}), X^{\mu_2}(\sigma^{ij}), ..., X^{\mu_{2n}}(\sigma^{ij})\}_*$ as sums of pairwise star products of two entries as shown in eq-(2.5). If one scales the coordinates X^μ and σ^{ij} by suitable powers of a length scale to render all the coordinates dimensionless, the star deformed brackets $\{\ X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{2n}}\ \}_*$ will be dimensionless and so will the tension parameter be dimensionless as well. One could also replace the target spacetime X^μ coordinates in the action (4.9) for the bivector coordinates $X^{m_1 m_2}(\sigma^{ij})$ associated with a $SO(D+2)$ algebra.

The Noncommutative phase space Yang's algebra in $4D$ can be generalized to the Noncommutative Clifford phase space algebra associated to the $4D$

spacetime by invoking higher dimensions ($12D$ in this case instead of $6D$) as follows

$$X^\mu \leftrightarrow \lambda \, \Gamma^\mu \wedge \Gamma^5, \quad P^\mu \leftrightarrow \frac{\hbar}{R} \, \Gamma^\mu \wedge \Gamma^6. \tag{4.11}$$

$$X^{\mu_1\mu_2} \leftrightarrow \Upsilon^{[\mu_1\mu_2] \ [57]} \neq \lambda^2 \, \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^5 \wedge \Gamma^7$$

$$P^{\mu_1\mu_2} \leftrightarrow \Upsilon^{[\mu_1\mu_2] \ [68]} \neq (\frac{\hbar}{R})^2 \, \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^6 \wedge \Gamma^8. \tag{4.12}$$

$$X^{\mu_1\mu_2\mu_3} \leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3] \ [579]} \neq \lambda^3 \, \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^5 \wedge \Gamma^7 \wedge \Gamma^9$$

$$P^{\mu_1\mu_2\mu_3} \leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3] \ [6810]} \neq (\frac{\hbar}{R})^3 \, \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^6 \wedge \Gamma^8 \wedge \Gamma^{10}. \tag{4.13}$$

$$X^{\mu_1\mu_2\mu_3\mu_4} \leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3\mu_4] \ [57911]} \neq \lambda^4 \, \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^{\mu_4} \wedge \Gamma^5 \wedge \Gamma^7 \wedge \Gamma^9 \wedge \Gamma^{11}$$

$$P^{\mu_1\mu_2\mu_3\mu_4} \leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3\mu_4] \ [681012]} \neq (\frac{\hbar}{R})^4 \, \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^{\mu_4} \wedge \Gamma^6 \wedge \Gamma^8 \wedge \Gamma^{10} \wedge \Gamma^{12}. \tag{4.14}$$

The indices $\mu_1, \mu_2, \mu_3, \mu_4$ range from $1, 2, 3, 4$. The extra indices span 8 additional directions (dimensions) leaving a total dimension of $4 + 8 = 12$. The *noncommutative* Clifford phase space algebra commutators are defined in terms of the algebra

$$[\Upsilon^{MN}, \ \Upsilon^{PQ}] = -i \, G^{MP} \, \Upsilon^{NQ} + i \, G^{MQ} \, \Upsilon^{NP} + i \, G^{NP} \, \Upsilon^{MQ} - i \, G^{NQ} \, \Upsilon^{MP} \tag{4.15}$$

The generators obey $\Upsilon^{MN} = -\Upsilon^{NM}$, and $G^{MN} = G^{NM}$ under an exchange of multi-indices $M \leftrightarrow N$.

The algebra (4.15) has the same structure as a *generalized spin algebra* and satisfies the Jacobi identities. We must stress that

$$[\Upsilon^{MN}, \ \Upsilon^{PQ}] \neq [\ [\Gamma^M, \Gamma^N], \ [\Gamma^P, \Gamma^Q] \ ]. \tag{4.16}$$

except in the special case when M, N, P, Q are all *bivector* indices : hence we must *emphasize* that the generalized spin algebra (4.15) *is not isomorphic* to the noncommutative algebra of eqs-(1.15-1.24) ! For example, from the commutator

$$[\Upsilon^{[\mu_1\mu_2\mu_3] \ [579]}, \ \Upsilon^{[\nu_1\nu_2\nu_3] \ [6810]}] = -i \, G^{[\mu_1\mu_2\mu_3] \ [\nu_1\nu_2\nu_3]} \, \Upsilon^{[579] \ [6810]}. \tag{4.17}$$

one can infer the Weyl-Heisenberg algebra commutator

$$[X^{\mu_1\mu_2\mu_3}, \ P^{\nu_1\nu_2\nu_3}] = -i \, \hbar^3 \, G^{[\mu_1\mu_2\mu_3] \ [\nu_1\nu_2\nu_3]} \, \Upsilon^{[579] \ [6810]}. \tag{4.18}$$

From the commutator

$$[\Upsilon^{[\mu_1\mu_2\mu_3]\ [579]},\ \Upsilon^{[\nu_1\nu_2\nu_3]\ [579]}] \ = \ -\ i\ G^{[579]\ [579]}\ \Upsilon^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}. \qquad (4.19)$$

one can infer the commutator among the tri-vector coordinates

$$[X^{\mu_1\mu_2\mu_3},\ X^{\nu_1\nu_2\nu_3}] \ = \ -\ i\ \lambda^6\ G^{[579]\ [579]}\ \Upsilon^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}. \qquad (4.20)$$

where $\Upsilon^{[\mu_1\mu_2\mu_3]\ [\nu_1\nu_2\nu_3]}$ is a generalized angular momentum (spin) generator. From the commutator

$$[\Upsilon^{[\mu_1\mu_2\mu_3]\ [579]},\ \Upsilon^{[579]\ [6810]}] \ = \ i\ G^{[579]\ [579]}\ \Upsilon^{[\mu_1\mu_2\mu_3]\ [6810]}. \qquad (4.21)$$

one can infer the commutator

$$[X^{\mu_1\mu_2\mu_3},\ \Upsilon^{[579]\ [6810]}] \ = \ i\ \lambda^6\ \frac{1}{\hbar^3}\ G^{[579]\ [579]}\ P^{\mu_1\mu_2\mu_3}. \qquad (4.22)$$

which *exchanges* the $X^{\mu_1\mu_2\mu_3}$ for $P^{\mu_1\mu_2\mu_3}$, etc ..... Therefore, the above equations are the suitable tri-vector analog of Yang's algebra. Generalized star-deformed brane actions associated to a poly-vector-valued phase space world manifold of noncommuting (poly-vector) coordinates similar to those phase space variables in eqs-(4.11-4.14)

$$q^a,\ p^a,\ q^{a_1a_2},\ p^{a_1a_2},\ \ldots\ldots..,\ q^{a_1a_2\ldots.a_d},\ p^{a_1a_2\ldots.a_d},$$
$$\sigma^{a\ b},\ \sigma^{a_1a_2\ b_1b_2},\ \ldots\ldots..,\sigma^{a_1a_2\ldots.a_d\ b_1b_2\ldots..b_d},\ \sigma^{d+1\ d+2},\ \sigma^{d+1\ d+3\ d+2\ d+4},\ \ldots\ldots. \qquad (4.23)$$

can also be constructed. They have a similar structure as the star deformed brane actions in eq-(2.12).

To conclude we must address also the need for a Nonassociative geometry. The Octonionic Geometry (Gravity) developed long ago by Oliveira and Marques [32] was extended to Noncommutative and Nonassociative Spacetime coordinates associated with octonionic-valued coordinates and momenta [33]. The octonionic metric $\mathbf{G}_{\mu\nu}$ already encompasses the ordinary spacetime metric $g_{\mu\nu}$, in addition to the Maxwell $U(1)$ and $SU(2)$ Yang-Mills fields such that implements the Kaluza-Klein Grand Unification program *without* introducing extra spacetime dimensions. The color group $SU(3)$ is a subgroup of the exceptional G_2 group which is the automorphism group of the octonion algebra. It was shown [33] that the flux of the $SU(2)$ Yang-Mills field strength $\vec{\mathcal{F}}_{\mu\nu}$ through the area-momentum $\vec{\Sigma}^{\mu\nu}$ in the *internal isospin space* yields corrections $O(1/M^2_{Planck})$ to the energy-momentum dispersion relations without violating Lorentz invariance as it occurs with Hopf algebraic deformations of the Poincare algebra. Despite that Octonions are nonassociative, there are known Octonionic realizations of the Clifford $Cl(8), Cl(4)$ algebras, in terms of left and right products, which permit the construction of octonionic string actions that have a correspondence with ordinary string actions for strings moving in a curved Clifford-space target background associated with a $Cl(3,1)$ algebra.

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