On Time dependent Black Holes and Cosmological Models from a Kaluza-Klein mechanism

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Abstract

Novel static, time-dependent and spatial-temporal solutions of Einstein field equations, displaying singularities, with and without horizons, and in several dimensions are found based on a dimensional reduction procedure widely used in Kaluza-Klein type theories. The Kerr-Newman black-hole entropy as well as the Reissner-Nordstrom, Kerr and Schwarzschild black-hole entropy are derived from the corresponding Euclideanized actions. A very special cosmological model based on the dynamical interior geometry of a Black Hole is found that has *no* singularities at $t = 0$ due to the smoothing of the mass distribution. We conclude with another cosmological model equipped also with a dynamical horizon and which is related to Vaidya's metric (associated with the Hawking-radiation of black holes) by interchanging $t \leftrightarrow r$ which might render our universe as a dynamical black hole.

1 Introduction

The static spherically symmetric solutions to Einstein's field (vacuum) equations have been known for a long time and were found by Schwarzschild, Hilbert and which furnished the modern concept of a black-hole. The formation of trapped surfaces in spherically symmetric gravitational collapse leading to black holes can be viewed in terms of how much mass is there within a given area-radius of the matter cloud. In order to avoid trapped surface formation (a horizon) there must be a mechanism available to throw away and radiate the mass so that the total mass in a shell of comoving radius r, at an epoch t, does not exceed the size of the physical area radius determining the size of the apparent horizon, at any given time t. It was found by [5] that the physical mechanism for the formation of a naked singularity was due to the presence of *shear* which delays the formation of the apparent horizon. It was also found by [6] that loss of matter due to heat flux prevents the trapped surface formation and a naked singularity is formed at the end state of the gravitational collapse. The latter authors considered a scenario where the interior spacetime, described by a heat conducting fluid sphere, is matched to a Vaidya metric in higher dimensions. The non-occurrence of a horizon is due to the fact that the rate of mass loss is exactly counterbalanced by the decrease of the boundary area-radius. These results posed a counter example to the so-called cosmic censorship hypothesis [44] that still remains a mathematically unproven conjecture to our knowledge.

In this work we shall construct static, time-dependent and spatial-temporal solutions, with/without horizons and displaying singularities, based on the dimensional reduction procedure widely used in Kaluza-Klein type theories. Motivated by the above observations, we firstly review the work of of [1], [2], [3],[4] where it was shown why Gravity in $D = d+n$ dimensions can be interpreted as a d-dim Yang-Mills-like gauge theory of diffeomorphisms of an internal n-dim space interacting with a gauged non-linear sigma model field. We explain how the "vacuum" solutions $A_\mu = 0$ yield the same functional form of the Schwarzschild metric in the advanced and retarded temporal Eddington-Finkelstein coordinates form, and which in turn, lead to the Fronsdal-Kruskal-Szekeres expression of the metric in the region $r < 2GM$.

In the next sections **3, 4, 5, 6** we find static, time-dependent and spatial-temporal vacuum solutions, simplifying *enormously* the earlier calculations by [1], [2], [3], [4] when the metric can be written in block diagonal form resulting from setting the gauge field of diffeomorphisms $A_\mu = 0$. In section **3** we study the decomposition of the Einstein-Hilbert action in $D = d + (D - d)$ dimensions, with $d = p + q$ and $n = D - d$ followed by section **4** where we employ the methods of section **3** in the particular case when $d = 1+1$ and show that our results agree with the $1 + 1$ dilaton-gravity action reported in the literature. Vacuum solutions to Einstein's field equations in spacetimes with signature $(+, +, -, -)$ which are the hyperbolic version of Schwarzschild's solution containing a conical singularity at $r = 0$ were found by [8]. In section **5** we solve the vacuum field equations in a very $straightforward$ fashion compared to the standard methods in the literature leading directly to the most general version of the Schwarzschild solution in any dimension. In particular, we review the derivation [20], [21] of the Black-Hole area-entropy relation starting from the Euclidean gravitational action associated with a point-mass delta function source and provide the action-entropy relations in the case of the Reissner-Nordstrom and Kerr-Newman solutions.

In section **6** novel spatial-temporal vacuum solutions to Einstein's field equations are found and their physical interpretation is discussed. It is shown that *horizonless* solutions occur when $\varphi_o > 0,\ t \pm r \geq 0$ or when $\varphi_o < 0,\ t \pm r \leq 0$, where φ_o is an integration constant of length dimensions. The singularity at $t \pm r = 0$ is *timelike* and is visible to an observer. The existence of *horizons* occurs when $\varphi_o > 0,\ t \pm r \leq 0$ or when $\varphi_o < 0,\ t \pm r \geq 0$. The singularity at $t \pm r = 0$ is *spacelike* and would be hidden behind the horizons at $t \pm r = -\varphi_o$ for observers in the regions $|t \pm r| > |\varphi_o|$. In particular, we also recover the Kantowski-Sachs metric which is the t-relative of the Schwarzschild solution and find a $4D$ metric inspired from the Janus geometry in $3D$.

In the final section **7** a very peculiar cosmological model based on the dynamical interior geometry of a Black Hole is found and that has no singularities at $t = 0$ due to the smoothing of the mass distribution. We conclude with another cosmological model equipped also with a dynamical horizon and which is related to the Vaidya's metric (associated with the Hawking-radiation of black holes) by interchanging $t \leftrightarrow r$ which might render our universe as a dynamical black hole [43]. The physical implications of our novel cosmological solutions in section **7** that are well behaved at $t = 0$ deserve further investigation.

2 Gravity as Gauge Theory of the Diffeomorphism Group

Some time ago, it was shown by [1], [2], [3], [4] that a Kaluza-Klein formalism of Einstein's theory, based on the $(2,2)$-fibration of a generic 4-dimensional spacetime, describes General Relativity as a Yang-Mills gauge theory on the 2-dimensional base manifold, where the local gauge symmetry is the group of the diffeomorphisms of the 2-dimensional fibre manifold. They found the Schwarzschild solution by solving the field equations after a very laborious procedure. Their formalism was valid for any (d,n) fibration of the the D-dim spacetime $D = d + n$ and allowed [11], [12] to provide a new realization of the Maldacena-Susskind-'t Hooft holographic principle.

The line element of [1], [2], [3], [4]

$$ds^2 \ = \ g_{ab}\ dy^a\ dy^b \ + \ (g_{\mu\nu} + g_{ab}\ A^a_\mu\ A^b_\nu)\ dx^\mu\ dx^\nu \ + \ 2 g_{ab}\ A^b_\mu\ dx^\mu\ dy^a. \tag{2.1}$$

in light cone coordinates

$$u \ = \ \frac{1}{\sqrt{2}}\ (t + r), \quad v \ = \ \frac{1}{\sqrt{2}}\ (t - r). \tag{2.2}$$

$$A^a_u \ = \ \frac{1}{\sqrt{2}}\ (A^a_t \ + \ A^a_r), \quad A^a_v \ = \ \frac{1}{\sqrt{2}}\ (A^a_t \ - \ A^a_r). \tag{2.3}$$

after using the Polyakov ansatz :

$$g_{\mu\nu} = \begin{pmatrix} -2\ h(t,r) & -1 \\ -1 & 0 \end{pmatrix}, \tag{2.4}$$

becomes

$$\begin{aligned} ds^2 \ = \ g_{ab}\ dy^a\ dy^b \ - 2du\ dv \ - 2h(u)\ du^2 \ + \\ g_{ab}\ (A^a_u du + A^b_v dv)\ (A^b_u du + A^b_v dv) \ + 2 g_{ab}\ (A^a_u du + A^b_v dv)\ dy^a. \end{aligned} \tag{2.5}$$

Upon setting $g_{ab} = e^\sigma\ \rho_{ab}$ such that $det\ \rho_{ab} = 1$ and after a very laborious calculation Yoon [2] arrived finally at the expression for the scalar curvature in light-cone coordinates

$$\mathcal{R} \;=\; -\frac{1}{2}\, e^{2\sigma}\, \rho_{ab}\, F^a_{+-}\, F^b_{+-} \;+\; e^{\sigma}\, \mathcal{R}_2 \;+\; e^{\sigma}\, D_+\sigma\, D_-\sigma \;-$$
$$\frac{1}{2}\, e^{\sigma}\, \rho^{ab}\, \rho^{cd}\, (D_+\rho_{ab})\, (D_-\rho_{cd}) \;+\; \frac{1}{2}\, e^{\sigma}\, \rho^{ab}\, \rho^{cd}\, (D_+\rho_{ac})\, (D_-\rho_{bd}) \;+$$
$$2h_{++}\, e^{\sigma}\, [\; D^2_-\sigma \;+\frac{1}{2}(D_-\sigma)^2 \;+\frac{1}{4}\, \rho^{ab}\, \rho^{cd}\, (D_-\rho_{ac})\, (D_-\rho_{bd})\;] \tag{2.6}$$

plus surface terms. The Lie-bracket is

$$[\; A_\mu \;,\; g_{ab}\;] \;=\; (\partial_a\, A^c_\mu(x^\mu, y^a))\, g_{bc}(x^\mu, y^a) \;+\; (\partial_b\, A^c_\mu(x^\mu, y^a))\, g_{ac}(x^\mu, y^a) \;+$$
$$A^c_\mu(x^\mu, y^a)\, \partial_c\, g_{ab}(x^\mu, y^a). \tag{2.7}$$

the Yang-Mills-like field strength is

$$F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu \;-\partial_\nu A^a_\mu \;-\; [A_\mu, A_\nu]^a \;=$$
$$\partial_\mu A^a_\nu \;-\partial_\nu A^a_\mu \;-\; A^c_\mu \partial_c A^a_\nu \;+A^c_\nu \partial_c A^a_\mu. \tag{2.8}$$

The covariant derivative of a tensor *density* ρ_{ab} with weight 1 is

$$D_\mu\, \rho_{ab} \;=\; \partial_\mu\, \rho_{ab} \;-[\; A_\mu \;,\; \rho\;]_{ab} \;+(\partial_c A^c_\mu)\, \rho_{ab} \;=$$
$$\partial_\mu\, \rho_{ab} \;-A^c_\mu \partial_c\, \rho_{ab} \;-(\partial_a A^c_\mu)\, \rho_{cb} \;-(\partial_b A^c_\mu)\, \rho_{ac} \;+(\partial_c A^c_\mu)\, \rho_{ab}. \tag{2.9}$$

the covariant derivative on the scalar density $\Omega = e^\sigma$ of weight -1 is

$$D_\mu \Omega \;=\; \partial_\mu \Omega \;-A^a_\mu \partial_a \Omega \;-(\partial_a A^a_\mu)\Omega \;\Rightarrow . \tag{2.10}$$

$$D_\mu \sigma \;=\; \partial_\mu \sigma \;-A^a_\mu \partial_a \sigma \;-(\partial_a A^a_\mu). \tag{2.11}$$

after factoring the e^σ terms.

The authors [1] were able to solve the equations of motion associated with the Einstein-Hilbert action

$$S \;=\; \int du\; dv\; d^2y\; \mathcal{R}. \tag{2.12}$$

by varying the Einstein-Hilbert action $before$ imposing the gauge fixing conditions (2.4) and $det\ \rho_{ab} = 1$ giving a total of 10 equations for the 10 fields $\sigma, h_{++}, h_{--}, A^a_+, A^a_-, \rho_{ab}$ with $a, b = 1, 2$. These 10 fields match the same number of idependent components of the metric $g_{\mu\nu}$ in $4D$. After a very laborious procedure the authors found solutions for the "vacuum" field configurations $A^a_+ = 0, A^a_- = 0$ given by

$$ds^2 \;=\; 2\; du\; dv \;-(1 - \frac{2GM}{u})\; dv^2 \;+u^2\; d\Omega^2. \tag{2.13}$$

$$ds^2 \;=\; -\,2\; du\; dv \;-(1 - \frac{2GM}{v})\; du^2 \;+v^2\; d\Omega^2. \tag{2.14}$$

which have the $same\ functional\ form$ as the Schwarzschild solution in the retarded and advanced temporal Eddington-Finkelstein coordinates, $u = t - r_*, v = t + r_*$

$$ds^2 \; = \; 2 \; dr \; dv \; - (1 - \frac{2GM}{r}) \; dv^2 \; + r^2 \; d\Omega^2. \qquad (2.15)$$

$$ds^2 \; = \; - \, 2 \; du \; dr \; - (1 - \frac{2GM}{r}) \; du^2 \; + r^2 \; d\Omega^2. \qquad (2.16)$$

with the subtle technicality that r_* appearing in the definitions $u = t - r_*, v = t + r_*$ in eqs-(2.15, 2.16) is the *tortoise* radial coordinate $r_*(r)$ given by

$$\int dr_* \; = \; \int \frac{dr}{1 - 2GM/r} \; \Rightarrow \; r_* \; = \; r + 2GM \; ln \; |\frac{r}{2GM} - 1|. \qquad (2.17)$$

It is well known [7] that one can introduce afterwards the Fronsdal-Kruskal-Szekeres coordinates [16], [17], [18] in terms of the Eddington-Finkelstein coordinates after the series of steps

$$u \; = \; t - r_*, \quad v \; = \; t + r_*, \; W \; = \; - \, e^{-u/4GM}, \quad Z \; = \; e^{v/4GM}$$

$$dW \; dZ \; = \; - \frac{WZ}{(4GM)^2} \; du \; dv \; \Rightarrow \; du \; dv \; = \; - \frac{(4GM)^2}{WZ} \; dW \; dZ \qquad (2.18a)$$

such that the metric in the form eq-(2.15, 2.16) can be rewritten in double null coordinates u, v as

$$ds^2 \; = \; - (1 - \frac{2GM}{r}) \; du \; dv \; + r^2 \; d\Omega^2 = \; - \frac{2GM}{r} \; e^{(v-u)/4GM} \; e^{-r/2GM} \; du \; dv \; + r^2 \; d\Omega^2 =$$

$$- \; \frac{4(2GM)^3}{r} \; e^{-r/2GM} \; dW \; dZ \; + \; r^2 \; d\Omega^2. \qquad (2.18b)$$

By defining $W = V + U$ and $Z = V - U$ the Fronsdal-Kruskal-Szekeres expression for the metric is then given by

$$ds^2 \; = \; - \; \frac{4(2GM)^3}{r(U,V)} \; e^{-r(U,V)/2GM} \; (\; dV^2 \; - dU^2 \;) \; + \; r^2(U,V) \; (d\Omega)^2. \qquad (2.18c)$$

with $d\Omega^2 = d\phi^2 + sin^2\phi \; d\theta^2$ and where $r = r(U,V)$ is a function of the *two* coordinates U, V which is *implicitly* given by the relations

$$U = (\frac{r}{2GM} - 1)^{1/2} \; e^{r/4GM} \; cosh \; (t/4GM); \quad V = (\frac{r}{2GM} - 1)^{1/2} \; e^{r/4GM} \; sinh \; (t/4GM). \qquad (2.19)$$

$$U^2 - V^2 = (\frac{r}{2GM} - 1) \; e^{r/2GM}. \qquad (2.20)$$

Therefore, the solution in the interior region $r < 2GM$ is no longer *static*. At the horizon $r = 2GM$ one has the null lines $U = \pm V$ corresponding to $t = \pm\infty$ respectively. Notice that one does *not* know the *explicit* analytical expression for $r = r(U,V)$. One only knows the converse relations $U = U(r,t)$ and $V = V(r,t)$.

The metric (2.18c) in the r, t coordinates is

$$ds^2 \;=\; -\;\frac{4(2GM)^3}{r}\; e^{-r/2GM}\;[\;(V_r\;dr + V_t\;dt)^2\;-(U_r\;dr + U_t\;dt)^2\;]\;+\;r^2\;(d\Omega)^2\;=$$

$$-\;\frac{4(2GM)^3}{r}\; e^{-r/2GM}\;[\;(V_r\;dr)^2\;+(V_t\;dt)^2\;-(U_r\;dr)^2\;-(U_t\;dt)^2\;+(2V_rV_t - 2V_rV_t)\;dr\;dt\;]$$
$$+\;r^2\;(d\Omega)^2. \tag{2.21}$$

where the partial derivatives

$$U_r = \frac{\partial U}{\partial r},\quad U_t = \frac{\partial U}{\partial t},\quad V_r = \frac{\partial V}{\partial r},\quad V_t = \frac{\partial V}{\partial t}. \tag{2.22}$$

are *known* from the defining relations of eq-(2.19). The metric acquires the form

$$ds^2 \;=\; -\;\frac{4(2GM)^3}{r}\; e^{-r/2GM}\;[\;A(r,t)\;dt^2 + B(r,t)\;dr^2 + C(r,t)\;drdt\;] + r^2\;(d\Omega)^2. \tag{2.23}$$

with

$$A(r,t) \;=\; (V_t)^2 \;-\; (U_t)^2,\quad B(r,t) \;=\; (V_r)^2 \;-\; (U_r)^2$$
$$C(r,t) \;=\; 2V_rV_t \;-\; 2U_rU_t. \tag{2.24}$$

After straightforward algebra, due to the identity $cosh^2(x) - sinh^2(x) = 1$, one recovers from the above eqs-(2.23, 2.24) the original Schwarzschild metric as expected

$$ds^2 \;=\; -\;(1 - \frac{2GM}{r})\;dt^2\;+\;\frac{dr^2}{(1 - \frac{2GM}{r})}\;+\;r^2(d\Omega)^2. \tag{2.25}$$

The reason why there is no *physical* singularity at $r = 2GM$ is because *all* the world lines trajectories upon reaching the horizon $r = 2GM$ at $t = \infty$ have the property that $dr^2 = 0$. Therefore, $(1 - 2GM/r)^{-1}\;(dr)^2 = \frac{0}{0}$ = undetermined at $r = 2GM$. There is no singularity at $r = 2GM$ in the Fronsdal-Kruskal-Szekeres expression for the metric (2.18c); the singularity is *diverted* into the Jacobian (it is singular at $r = 2GM$) resulting from the coordinate transformations from the (U, V) to the (r, t) variables, and for this reason the singularity at $r = 2GM$ is just a coordinate singularity. The true physical singularity lies at $r = 0$.

To sum up, we have revised the work of [1], [2], [3], [4] and discussed how their "vacuum" solutions $A^a_+ = 0, A^a_- = 0$ yield the same functional form of the Schwarzschild solution in the advanced and retarded temporal Eddington-Finkelstein coordinates form, and which in turn, leads to the Fronsdal-Kruskal-Szekeres expression of the metric in the region $r < 2GM$. In the next sections we shall find static and time-dependent vacuum solutions directly by setting the terms $A^a_\mu = 0$ in eq-(2.1) simplifying *enormously* the earlier calculations by [1], [2], [3], [4] .

3 d=p+q dimensions from D=d+n dimensions

Let us now consider a D-dimensional manifold M^D, with associated metric $\gamma(x,y)$ given by (2.1). We shall assume that γ_{AB} can be reduced to

$$\gamma_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{ab} \end{pmatrix}, \tag{3.1}$$

where $\mu,\nu = \{0,1,2,d-1\}$, $a,b = \{d,d+1,...,D-1\}$, $g_{\mu\nu}$ only depends on x^μ and $g_{ab} = \varphi^2(x^\mu)\tilde{g}_{ab}$, with $\tilde{g}_{ab} = \tilde{g}_{ab}(y^c)$.

The non vanishing Christoffel symbols,

$$\Gamma^A_{BC} = \frac{1}{2}\gamma^{AD}(\gamma_{DB,C} + \gamma_{DC,B} - \gamma_{BC,D}), \tag{3.2}$$

are

$$\begin{aligned} \Gamma^\mu_{\nu\alpha} &= \hat{\Gamma}^\mu_{\nu\alpha}, \\ \Gamma^\mu_{ab} &= -\varphi\varphi'^\mu\tilde{g}_{ab}, \\ \Gamma^a_{\mu b} &= \varphi^{-1}\varphi_{,\mu}\delta^a_b, \\ \Gamma^a_{bc} &= \tilde{\Gamma}^a_{bc}, \end{aligned} \tag{3.3}$$

where $\hat{\Gamma}^\mu_{\nu\alpha}$ and $\tilde{\Gamma}^a_{bc}$ are the Christoffel symbols associated with $g_{\mu\nu}$ and $\tilde{g}_{ab}$, respectively.

The nonvanishing components of the Riemann tensor are

$$\begin{aligned} R^\mu_{\nu\alpha\beta} &= \hat{R}^\mu_{\nu\alpha\beta}, \\ R^\mu_{a\nu b} &= -\varphi\mathcal{D}_\nu\varphi'^\mu\tilde{g}_{ab}, \\ R^a_{bcd} &= \hat{R}^a_{bcd} - \varphi'^\mu\varphi_{,\mu}(\delta^a_c\tilde{g}_{bd} - \delta^a_d\tilde{g}_{bc}). \end{aligned} \tag{3.4}$$

where $\mathcal{D}_\nu$ is the covariant derivative w.r.t the connection $\hat{\Gamma}^\mu_{\nu\alpha}$ associated with $g_{\mu\nu}$. From here, we get the nonvanishing components of the Ricci tensor

$$\begin{aligned} R_{\mu\nu} &= \hat{R}_{\mu\nu} - (D-d)\varphi^{-1}\mathcal{D}_\nu\varphi_{,\mu}, \\ R_{ab} &= \tilde{R}_{ab} - \varphi\mathcal{D}_\mu\varphi'^\mu\tilde{g}_{ab} - (D-d-1)\varphi'^\mu\varphi_{,\mu}\tilde{g}_{ab}. \end{aligned} \tag{3.5}$$

So, we get that the Ricci scalar is

$$R = \hat{R} - 2(D-d)\varphi^{-1}\mathcal{D}_\mu\varphi'^\mu + \varphi^{-2}\tilde{R} - (D-d)(D-d-1)\varphi^{-2}\varphi'^\mu\varphi_{,\mu}, \tag{3.6}$$

where $\hat{R} = g^{\mu\nu}\hat{R}_{\mu\nu}$ and $\tilde{R} = \tilde{g}^{ab}\tilde{R}_{ab}$.

Therefore, up to a total derivative we have

$$S = \int d^D x \sqrt{-\gamma} R = V_{D-d} \int d^d x \sqrt{-\hat{g}} \{\varphi^{D-d} \hat{R} + \kappa (D-d)(D-d-1) \varphi^{D-d-2} \\ + (D-d)(D-d-1) \varphi^{D-d-2} \varphi'^{\mu} \varphi_{,\mu}\}. \tag{3.7}$$

Here,

$$V_{D-d} = \int d^{D-d} x \sqrt{\tilde{g}}. \tag{3.8}$$

We also assume a homogeneous space for the metric $\tilde{g}_{ab}$, with $\tilde{R} = \kappa(D-d)(D-d-1)$ and $\kappa = \{-1, 1\}$. It is remarkable that the final result (3.7) does not depend either of the signature $p + q$ nor the signature $d + n$.

The field equations derived from (3.7) are

$$\varphi^{D-d} (\hat{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \hat{R}) - (D-d) \varphi^{D-d-1} \mathcal{D}_{\mu} \varphi_{,\nu} \\ + \tfrac{1}{2} g_{\mu\nu} [(D-d)(D-d-1) \varphi^{D-d-2} \varphi'^{\alpha} \varphi_{,\alpha} + 2(D-d) \varphi^{D-d-1} \mathcal{D}_{\alpha} \varphi'^{\alpha} \\ - k(D-d)(D-d-1) \varphi^{D-d-2}] = 0 \tag{3.9}$$

and

$$\varphi^{D-d-1} \hat{R} + \kappa (D-d)(D-d-1)(D-d-2) \varphi^{D-d-3} \\ - (D-d)(D-d-1)(D-d-2) \varphi^{D-d-3} \varphi'^{\mu} \varphi_{,\mu} \\ - 2(D-d)(D-d-1) \varphi^{D-d-2} \mathcal{D}_{\mu} \varphi'^{\mu}) = 0. \tag{3.10}$$

It is interesting that a considerable simplification of these field equations can be obtained when $D = d+1$. In fact from (3.10) we see that when $D = d+1$ one obtains the intriguing result $\hat{R} = 0$.

Eliminating the factor φ^{D-d} we can simplify (3.9) in the form

$$\hat{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \hat{R} - (D-d) \varphi^{-1} \mathcal{D}_{\mu} \varphi_{,\nu} \\ + \tfrac{1}{2} g_{\mu\nu} [(D-d)(D-d-1) \varphi^{-2} \varphi'^{\alpha} \varphi_{,\alpha} + 2(D-d) \varphi^{-1} \mathcal{D}_{\alpha} \varphi'^{\alpha} \\ - k(D-d)(D-d-1) \varphi^{-2}] = 0, \tag{3.11}$$

while eliminating the factor φ^{D-d-1} (3.10) becomes

$$\hat{R} + \kappa (D-d)(D-d-1)(D-d-2) \varphi^{-2} \\ - (D-d)(D-d-1)(D-d-2) \varphi^{-2} \varphi'^{\mu} \varphi_{,\mu} \\ - 2(D-d)(D-d-1) \varphi^{-1} \mathcal{D}_{\mu} \varphi'^{\mu} = 0. \tag{3.12}$$

4 1+1 dimensions from D-dimensional theory

Let us now consider the case $d = 2$, with $p = 1$ and $q = 1$. We still assume a D-dimensional manifold M^D, with associated metric $\gamma(x, y)$ given by (2.1). In fact we shall consider that γ_{AB} can be reduced to

$$\gamma_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{ab} \end{pmatrix}, \tag{4.1}$$

where $\mu, \nu = \{0, 1\}$, $a, b = \{2, 3, ..., D-1\}$, $g_{\mu\nu}$ only depends on x^μ and $g_{ab} = \varphi^2(x^\mu)\tilde{g}_{ab}$, with $\tilde{g}_{ab} = \tilde{g}_{ab}(y^c)$.

From (3.5) we get the nonvanishing components of the Ricci tensor

$$\begin{aligned} R_{\mu\nu} &= \hat{R}_{\mu\nu} - (D-2)\varphi^{-1}\mathcal{D}_\nu\varphi_{,\mu}, \\ R_{ab} &= \tilde{R}_{ab} - \varphi\mathcal{D}_\mu\varphi'^\mu\tilde{g}_{ab} - (D-3)\varphi'^\mu\varphi_{,\mu}\tilde{g}_{ab}. \end{aligned} \tag{4.2}$$

So, we obtain the Ricci scalar

$$R = \hat{R} - 2(D-2)\varphi^{-1}\mathcal{D}_\mu\varphi'^\mu + \varphi^{-2}\tilde{R} - (D-2)(D-3)\varphi^{-2}\varphi'^\mu\varphi_{,\mu}, \tag{4.3}$$

where $\hat{R} = g^{\mu\nu}\hat{R}_{\mu\nu}$ and $\tilde{R} = \tilde{g}^{ab}\tilde{R}_{ab}$.

Therefore, up to a total derivative we have

$$\begin{aligned} S = \textstyle\int d^2x\sqrt{-g}R = V_{D-2}\int d^2x\sqrt{-\hat{g}}\{&\varphi^{D-2}\hat{R} + \kappa(D-2)(D-3)\varphi^{D-4} \\ &+(D-2)(D-3)\varphi^{D-4}\varphi'^\mu\varphi_{,\mu}\}. \end{aligned} \tag{4.4}$$

Here,

$$V_{D-2} = \int d^{D-2}x\sqrt{\tilde{g}}. \tag{4.5}$$

We also assume a homogeneous space for the metric $\tilde{g}_{ab}$, with $\tilde{R} = \kappa(D-2)(D-3)$ and $\kappa = \{-1, 1\}$.

We observe that (4.4) can also be written as

$$S = \int dtL, \tag{4.6}$$

where the Lagrangian L is given by

$$L = V_{D-2}\int dx^1\sqrt{-\hat{g}}\{\varphi^{D-2}\hat{R} + \kappa(D-2)(D-3)\varphi^{D-4} + (D-2)(D-3)\varphi^{D-4}\varphi'^\mu\varphi_{,\mu}\}. \tag{4.7}$$

In the particular case of four dimensions, $D = 4$, eqs (4.4) and (4.7) are reduced to

$$S = \int d^2x\sqrt{-g}R = V_2\int d^2x\sqrt{-\hat{g}}(\varphi^2\hat{R} + +2\varphi'^\mu\varphi_{,\mu} + 2\kappa) \tag{4.8}$$

and

$$L = \int dx^1 \sqrt{-\hat{g}} \{\varphi^2 \hat{R} + 2\varphi'^{\mu} \varphi_{,\mu} + 2\kappa\}, \tag{4.9}$$

respectively.

5 Static Black Holes and the Euclidean Action-Entropy Relation

5.1 The Hilbert-Schwarzschild solution and Black Hole Entropy

To find static black hole solutions, for this purpose we assume

$$g_{\mu\nu} = \begin{pmatrix} -e^{\mu} & 0 \\ 0 & e^{\nu} \end{pmatrix}, \tag{5.1}$$

with $\mu = \mu(r)$ and $\nu = \nu(r)$, with $r = x^1$. We shall also assume that $\varphi = \varphi(r)$.

It is not difficult to see that

$$\hat{R}_{00} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\mu'\nu'), \tag{5.2}$$

$$\mathcal{D}_0 \varphi_{,0} = -\frac{1}{2} e^{\mu-\nu} \mu' \varphi', \tag{5.3}$$

$$\hat{R}_{11} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\nu', \tag{5.4}$$

and

$$\mathcal{D}_1 \varphi_{,1} = \varphi'' - \frac{1}{2}\nu'\varphi'. \tag{5.5}$$

Thus, the field equations obtained from (4.2) yield

$$R_{00} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\mu'\nu' + \frac{(D-2)}{2}\mu'\frac{\varphi'}{\varphi}) = 0, \tag{5.6}$$

$$R_{11} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\nu' + (D-2)(\frac{1}{2}\nu'\frac{\varphi'}{\varphi} - \frac{\varphi''}{\varphi}) = 0 \tag{5.7}$$

and

$$R_{ab} = \{\kappa(D-3) + e^{-\nu}[\frac{1}{2}(\nu' - \mu')\varphi\varphi' - \varphi\varphi'' - (D-3)\varphi'^2]\}\tilde{g}_{ab} = 0. \tag{5.8}$$

We recognize in (5.6), (5.7) and (5.8) the field equations for a static black hole . In fact, by using the combination

$$e^{-\mu}R_{00} + e^{-\nu}R_{11} = 0 \tag{5.9}$$

we get

$$\mu' + \nu' = \frac{2\varphi''}{\varphi'}. \tag{5.10}$$

The solution of this equation is

$$\mu + \nu = \ln \varphi'^2 + a, \tag{5.11}$$

where a is a constant.

Substituting (5.10) into the equation (5.8) we find

$$e^{-\nu}(\nu'\varphi\varphi' - 2\varphi\varphi'' - (D-3)\varphi'^2) = -k(D-3) \tag{5.12}$$

or

$$\gamma'\varphi\varphi' + 2\gamma\varphi\varphi'' + (D-3)\gamma\varphi'^2 = k(D-3), \tag{5.13}$$

where

$$\gamma = e^{-\nu}. \tag{5.14}$$

The solution of (5.13) for an ordinary D-dim spacetime (one temporal dimension) corresponding to a $(D-2)$-dim sphere for the homogeneous space can be written as

$$\gamma = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2}\varphi^{D-3}})\ (\frac{d\varphi}{dr})^{-2} \Rightarrow$$

$$g_{11} = e^{\nu} = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2}\varphi^{D-3}})^{-1}\ (\frac{d\varphi}{dr})^2, \tag{5.15}$$

where Ω_{D-2} is the appropriate solid angle in $(D-2)$-dim and G_D is the D-dim gravitational constant whose units are $(length)^{D-2}$. Thus $G_D M$ has units of $(length)^{D-3}$ as it should. When $D = 4$ as a result that the 2-dim solid angle is $\Omega_2 = 4\pi$ one recovers from eq-(5.15) the 4-dim Schwarzchild solution. The solution in eq-(5.15) is consistent with Gauss law and Poisson's equation in $D-1$ spatial dimensions obtained in the Newtonian limit.

For the most general case of the $(D-2)$-dim homogeneous space we should write

$$-\nu = \ln(k - \frac{\beta_D G_D M}{\varphi^{D-3}}) - 2\ln\varphi'. \tag{5.16}$$

where β_D is a constant and G_D is the gravitational constant in D-dim. Thus, according to (5.11) we get

$$\mu = \ln(k - \frac{\beta_D G_D M}{\varphi^{D-3}})\ +\ constant. \tag{5.17}$$

We can set the constant to zero, and this means the line element can be written as

$$ds^2 = -(k - \frac{\beta_D G_D M}{\varphi^{D-3}})(dt)^2\ +\ \frac{(d\varphi/dr)^2}{(k - \frac{\beta_D G_D M}{\varphi^{D-3}})}(dr)^2\ +\ \varphi^2(r)\ \tilde{g}_{ab}\ d\xi^a\ d\xi^b. \tag{5.18}$$

An important aspect is that, by taking for instance (5.7), the equations (5.6) and (5.8) do *not* determine the form of $\varphi(r)$. [13], [14], [15], [19]. It is also interesting to observe that the only effect of the homogeneous metric $\tilde{g}_{ab}$ is reflected in the $k = \pm 1$ parameter, associated with a positive (negative) constant scalar curvature of the homogeneous $(D-2)$-dim space. There are two interesting cases to study based on the boundary conditions obeyed by $\varphi(r)$: (i) the Hilbert textbook (black hole) solution when $\varphi(r) = r$ obeying $\varphi(r = 0) = 0$, $\varphi(r \to \infty) \to r$. And : (ii) the Abrams-Brillouin-Schwarzschild radial gauge [14], [15], [19] based on choosing the cutoff $\varphi(r = 0) = 2GM$ such that $g_{tt}(r = 0) = 0$ which apparently seems to "eliminate" the horizon and $\varphi(r \to \infty) \to r$. The original solution of 1916 found by Schwarzschild was based on the choice $\varphi(r) = [r^3 + (2GM)^3]^{1/3}$.

However, the choice $\varphi(r = 0) = 2GM$ has a serious $flaw$ and is : How is it *possible* for a point-mass at $r = 0$ to have a non-zero area $4\pi(2GM)^2$ and a *zero* volume *simultaneously* ? so it seems that one is forced to choose the Hilbert gauge [16] $\varphi(r) = r$ such $\varphi(r = 0) = 0$. Nevertheless it was shown [20], [21] that by choosing a *judicious* choice of $\varphi(r)$ one can cure this flaw and have the correct boundary condition $\varphi(r = 0) = 0$ while displacing the horizon from $r = 2GM$ to a location *arbitrarily* close to $r = 0$ as one desires, $r_h \to 0$, and where stringy geometry and Quantum Gravitational effects begin to take place.

A very straightforward solution to this cut-off problem was found in [20], [21] by choosing a radial gauge function like $\varphi(r) = r + 2GM\Theta(r)$, where the Heaviside Step function [1] is defined $\Theta(r) = 1$ when $r > 0$, $\Theta(r) = -1$ when $r < 0$ and $\Theta(r = 0) = 0$ (the arithmetic mean of the values at $r > 0$ and $r < 0$). When $\varphi \sim r$ for $r >> 2GM$ and one recovers the correct Newtonian limit in the asymptotic regime. It is now, via the Heaviside step function, that we may maintain the correct behaviour $\varphi(r = 0) = 0$, when $r = 0$, consistent with our intuitive notion that the spatial area and spatial volume of a point $r = 0$ *has* to be *zero*.

Since the notion of distance and separation of spacetime points only has meaning when it is referred to a gravitational field (metric) [30], [31], [32] when one has a metric of the form $g_{\mu\nu}[(\varphi(r)]$, it means that after performing the mapping from r to $\varphi(r)$ in the spacetime manifold $\mathcal{M}$, a *void* (hole) surrounding $\varphi = r = 0$ forms; i.e. a void in the region $0 < \varphi < 2GM$ with the singularity remaining at the center $r = 0 = \varphi(r = 0) = 0$ and a ring extending from $\varphi = 2GM$ to $\varphi = r = \infty$ (when $M = finite$). In the r-coordinates picture there is a *discontinuity* of the metric (and scalar curvature) at $r = 0$, the location of the point mass source. Because this is an infinitely compact source there is nothing wrong with having a discontinuity of the metric at $r = 0$. In the φ-coordinate picture, due to the correct condition $\varphi(r = 0) = 0$ consistent with the fact that a point must have zero area (since $\Theta(r = 0) = 0$), one can interpret the discontinuity of the metric as if the region of $0 < \varphi < 2GM$ were eliminated from the spacetime manifold to make the surface at $\varphi = 2GM$ a boundary of the spacetime while leaving the singularity at $r = 0$ behind.

The solutions [20], [21] had these salient features

- One is not gluing solutions with $M < 0$ into the region $r > 2G|M|$. The mass

[1] We thank Michael Ibison for pointing out the importance of the Heaviside step function and the use of the modulus $|r|$ to account for point mass sources at $r = 0$.

parameter $M > 0$. If one wishes to be strictly rigorous one may write the radial function as $\varphi(r) = r + 2G|M|\Theta(r)$ to ensure that $\varphi(r < 0) = -\varphi(r > 0) < 0$ and that solutions with $r < 0,\ M > 0$ have a one-to-one correspondence to the solutions with $r > 0,\ M < 0$ because $|-M| = |M|$. The latter $M < 0$ repulsive gravity regime is what it is called a "white" hole.

• It is shown explicitly that when one plugs eq-(5.15) directly into eqs-(5.13, 5.14) and despite that the derivatives $\frac{d\varphi}{dr} = 1 + 2G|M|\delta(r)$ and $(d^2\varphi/dr^2) = 2G|M|\delta'(r)$ are singular at $r = 0$, there is an exact and precise *cancellation* of these singular derivatives (and the ordinary derivatives of any radial function $\varphi(r)$) in eq-(5.13) ; i.e. the latter eq-(5.13) is satisfied for *any* radial function, irrespective if it has singular derivatives at $r = 0$ or not, for the solutions given by eq-(5.14, 5.15). Speaking of singular derivatives, it is well known that the Jacobian from the Fronsdal-Kruskal-Szekeres coordinates U, V to the r, t coordinates is singular at the horizon $r = 2GM$.

• There is a discontinuity of the metric at $r = 0$ where the magnitude of the g_{tt} component jumps from 0 to ∞ at $r = 0$ in the same fashion that the scalar curvature jumps from 0 to ∞ at $r = 0$ due to the presence of the point mass at $r = 0$. Such discontinuity of the metric at $r = 0$ is due to the discontinuity of the radial function given by $\varphi(r = 0) = 0,\ \varphi(r = 0^+) = 2GM$.

• Having $\varphi(r = 0) = 0$ and $\varphi(r = 0^+) = 2GM$, our solutions near the singularity can be represented by the right and left regions (quadrants) of the Rindler-wedge formed by the straight (null) lines $r = 0^+,\ t = +\infty$ and $r = 0^+,\ t = -\infty$ at $+45, -45$ degrees respectively. These (null) lines should be compared with the (null) lines $r = 2GM, t = \pm\infty$ corresponding to the text-book solution after performing the Fronsdal-Kruskal-Szekeres change of coordinates [16], [17], [18]

$$U = (\frac{r}{2GM} - 1)^{\frac{1}{2}}\ e^{r/4GM}\ cosh\ (\frac{t}{4GM}), \quad V = (\frac{r}{2GM} - 1)^{\frac{1}{2}}\ e^{r/4GM}\ sinh\ (\frac{t}{4GM})\ \Rightarrow$$

$$U^2 - V^2\ =\ (\frac{r}{2GM} - 1)\ e^{r/2GM}. \tag{5.19}$$

leading to a metric

$$ds^2\ =\ -\ \frac{4(2GM)^3}{r}\ e^{-r/2GM}\ (dV^2\ - dU^2\ )\ +\ r^2(d\Omega)^2. \tag{5.20}$$

In our case we must replace $r \to \varphi$ in eqs-(5.19, 5.20)

$$U = (\frac{\varphi}{2GM} - 1)^{\frac{1}{2}}\ e^{\varphi/4GM}\ cosh\ (\frac{t}{4GM}), \quad V = (\frac{\varphi}{2GM} - 1)^{\frac{1}{2}}\ e^{\varphi/4GM}\ sinh\ (\frac{t}{4GM})\ \Rightarrow$$

$$U^2 - V^2\ =\ (\frac{\varphi}{2GM} - 1)\ e^{\varphi/2GM}. \tag{5.21}$$

leading to a metric

$$ds^2\ =\ -\ \frac{4(2GM)^3}{\varphi(U,V)}\ e^{-\varphi(U,V)/2GM}\ (dV^2\ - dU^2\ )\ +\ \varphi^2(U,V)\ (d\Omega)^2. \tag{5.22}$$

such that an incoming photon, starting at point P in the right region (quadrant) of the Rindler wedge, moves upwards parallel to the -45 degrees null-line and reaches the null-line branch given by $r = 0^+, \varphi(r = 0^+) = 2GM$ at $t = \infty$, as measured by an asymptotic observer, at point P'. Then it tunnels through the spacetime void and reaches the spacelike singularity $\varphi(r = 0) = 0$ at point P''. This tunneling behaviour from P' to P'' is a direct *consequence* of the *discontinuity* of the metric at $r = 0$. Similar behaviour occurs for an infalling timelike path starting at Q : it reaches the null-line branch given by $r = 0^+, \varphi(r = 0^+) = 2GM$, $t = \infty$ at point Q'. Then it tunnels through the spacetime void reaching the spacelike singularity $\varphi(r = 0) = 0$ at point Q''.

In essence, the singularity $r = 0$ has been spliced-off from the rest of spacetime by *carving out* the future and past regions (quadrants) of the Rindler wedge (creating a spacetime void) leaving only the right and left regions (quadrants) bounded by the null lines $r = 0^+, \varphi(r = 0^+) = 2GM$ at $t = \pm\infty$. The fact that we end up only with the left and right regions of the Rindler wedge might have some relationship to the factor of *two* discrepancy of the Hawking radiation temperature which appears when working with the left-right versus the future-right regions of the Rindler wedge [33].

• Due to the *discontinuity* of the metric eq-(5.18) at $r = 0$, the location $r = 0, \varphi(r = 0) = 0$ corresponds to a *spacelike* singularity since $g_{tt}(r = 0) = \infty > 0$: it changes sign. Whereas $g_{rr}(r = 0) = 0$ because the quantity $r(1 + 2GM\delta(r))^2 = 0$ when $r = 0$, due to the fact that it is an *odd* function of r so the latter expression vanishes at $r = 0$. Therefore, since $g_{tt}(r = 0) = \infty > 0$ has changed sign in eq-(5.18), it is now *spacelike*, we must emphasize that *no violation* of the cosmic censorship conjecture occurs ! (that rules out *timelike* singularities).

A rigorous correct treatment of point mass distributions has been provided based on Colombeau's [22], [23], [24], [25], [26], [27], [28] theory of nonlinear distributions, generalized functions and nonlinear calculus. This permits the proper multiplication of distributions since the old Schwarz theory of linear distributions is invalid in nonlinear theories like General Relativity. Colombeau's nonlinear distributional geometry supersedes the no-go results of Geroch and Traschen [29] stating that there is no proper framework to study distributions of matter of co-dimensions higher than two (neither points nor strings in $D = 4$) in General Relativity. Colombeau's theory of Nonlinear Distributions (and Nonstandard Analysis) is the proper way to deal with point-mass sources in nonlinear theories like Gravity and where one may rigorously solve the problem without having to introduce a boundary of spacetime at $r = 0$.

Due to the essential technical subtlety in order to generate $\delta(r)$ terms in the right hand side of Einstein's equations, one must replace everywhere $r \to |r|$ as required when point-mass sources are inserted. The Newtonian gravitational potential due to a point-mass source at $r = 0$ is given by $-GM/|r|$ and is consistent with Poisson's law which states that the Laplacian of the Newtonian potential $-GM/|r| = 4\pi G\rho$ corresponding to a mass distribution $\rho = (M/4\pi r^2)\delta(r)$. However, the Laplacian in spherical coordinates of $(1/r)$ is *zero*. For this reason, there is a *fundamental* difference in dealing with expressions involving absolute values $|r|$ like $1/|r|$ from those which depend on r like $1/r$. Had one *not* use the modulus $|r|$ in the expression for the metric components $g_{tt} = 1 - 2GM/|r|$ one will not generate the desired $\delta(r)$ terms in the right hand side of Einstein's equations

$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G \ T_{\mu\nu} \neq 0$. Instead, one would get an expression *identically* equal to *zero* (consistent with the vacuum solutions in the absence of matter) instead of the $\delta(r)$ terms [20], [21].

Writing the metric components for the signature $(+,-,-,-)$

$$g_{00} \ = \ 1 - \frac{2GM}{|r|} \ = \ 1 - \frac{2GM}{r} \ \frac{r}{|r|} \ = \ 1 - \frac{2GM}{r} \ f(r); \quad f(r) \ \equiv \ \frac{r}{|r|}. \tag{5.23a}$$

$$g_{rr} \ = \ - \ \frac{1}{g_{00}}. \tag{5.23b}$$

such that the derivatives

$$f'(r) \ = \ \frac{df(r)}{dr} \ = \ \delta(r); \quad f''(r) \ = \ \frac{d^2 f(r)}{dr^2} \ = \ \delta'(r). \tag{5.24}$$

reveals that the *nonvanishing* $\mathcal{R}$ is given by :

$$\mathcal{R} \ = \ - 2GM \ [\ \frac{f''(r)}{r} \ + \ 2 \ \frac{f'(r)}{r^2} \ ] \ =$$

$$- \ 2GM \ [\ \frac{\delta'(r)}{r} \ + \ 2 \ \frac{\delta(r)}{r^2} \ ]. \tag{5.25}$$

where the signature chosen is $(+,-,-,-)$.

Therefore, the overall Einstein-Hilbert action involving both density and anisotropic pressure terms (the terms corresponding to derivatives of the delta function in eq-(5.25)) is exactly equal to an overall integral involving $2GM\delta(r)/r^2$:

$$S = \ - \frac{1}{16\pi G} \int \mathcal{R} \ 4\pi r^2 \ dr \ dt = \ \frac{1}{16\pi G} \int \ 2GM \ [\ \frac{\delta'(r)}{r} \ + \ 2 \ \frac{\delta(r)}{r^2} \ ] \ 4\pi r^2 \ dr \ dt. \tag{5.26}$$

Integrating by parts yields

$$\frac{1}{16\pi G} \int \ 8\pi GM \ [\ 2\delta(r) \ - \ \delta(r) \ ] \ dr \ dt \ = \frac{1}{16\pi G} \int \ 8\pi G \ (\frac{M \ \delta(r)}{4\pi r^2}) \ 4\pi r^2 \ dr \ dt \ =$$

$$\frac{1}{16\pi G} \int \ 8\pi G \ \rho(r) \ 4\pi r^2 \ dr \ dt \ = \ \frac{1}{2} \int \ M \ dt \ \Rightarrow \ \rho(r) \ \equiv \ \frac{M \ \delta(r)}{4\pi r^2}. \tag{5.27}$$

The Euclideanized Einstein-Hilbert action associated with the scalar curvature involving the delta functions terms (5.25), stemming from density and anisotropic pressure terms [20], [21], is obtained after a compactification of the temporal direction along a circle S^1 giving an Euclidean time coordinate interval of $2\pi t_E$ and which is defined in terms of the Hawking temperature T_H and Boltzman constant k_B as $2\pi t_E = (1/k_B T_H) = 8\pi GM$. The Euclidean action becomes

$$S_E \ = \ \frac{1}{2} \int_0^{2\pi t_E} \ M \ dt \ = \ (\frac{M}{2}) \ (2\pi t_E) \ = \ (\frac{M}{2}) \ (8\pi GM) \ =$$

$$4\pi \; G \; M^2 = \; \frac{1}{4} \; \frac{4\pi(2GM)^2}{G} \;\; = \; \frac{Area}{4 \; L_P^2}. \tag{5.28}$$

which is the Bekenstein-Hawking Black Hole Entropy given in terms of the horizon area $4\pi(2GM)^2$ by $S = Area/4$ in Planck area units $G = L_P^2$ ($\hbar = c = 1$). This Euclidean action = Entropy result was one of the most salient features in [20], [21].

5.2 The Euclidean Action-Entropy Relation in the Reissner-Nordstrom and Kerr-Newman Black holes case

In this subsection we shall provide the Euclidean action-Black Hole entropy relations [21] in the case of the Reissner-Nordstrom and Kerr-Newman solutions. Let us begin with the Einstein-Maxwell action

$$S \;=\; - \; \frac{1}{16\pi G} \int d^4x \; \sqrt{g} \; \mathcal{R} \; + \; \frac{1}{4e^2} \int d^4x \; \sqrt{g} \; F_{\mu\nu} \; F^{\mu\nu} \;\; =$$

$$- \; \frac{1}{16\pi G} \int d^4x \; \sqrt{g} \; [\; \mathcal{R} \; - \; \frac{4\pi G}{e^2} \; F_{\mu\nu} \; F^{\mu\nu} \;]. \tag{5.29}$$

We will calculate the entropy in the special case when the charge e satisfies the condition $4\pi G = e^2$, the charge e has length units. In this particular case the Euclidean action matches the entropy. In the case that $4\pi G \neq e^2$ the Euclidean action is proportional to the black hole entropy. The constant of proportionality is $4\pi G/e^2$.

The charged massive Reissner-Nordstrom solution has for metric components

$$g_{tt} \;=\; 1 - \frac{2G_N M}{r} \; + \; \frac{e^2}{r^2}, \quad g_{rr} = -\frac{1}{g_{tt}}. \tag{5.30}$$

the angular part is the same $r^2(d\Omega)^2$. In the point mass and point charge case, we should replace $r \rightarrow |r|$ in order to recover delta function point mass and point charge singularities at $r = 0$. In the region $r > 0$ the only contribution to the field equations is from the EM field stress-energy tensor. Einstein's equations in the case that $4\pi G = e^2$ are

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \; \Rightarrow \; \mathcal{R}_{\mu\nu} \; - \frac{1}{2} \; g_{\mu\nu} \; \mathcal{R} \;=\; 8\pi G \; T_{\mu\nu} =$$

$$- \; [\; g^{\alpha\beta} \; (F_{\mu\alpha} \; F_{\nu\beta} + \; F_{\nu\alpha} \; F_{\mu\beta}) \; - \; \frac{1}{2} \; g_{\mu\nu} \; F^{\alpha\beta} F_{\alpha\beta} \;], \; r > 0. \tag{5.31}$$

In $D = 4$ the trace of the stress EM energy tensor is zero consistent with the conformal invariance of the Maxwell action in $D = 4$. This results simply follows from a variation under conformal scalings

$$if \quad 2 \; \delta S = -\sqrt{g} \; T_{\mu\nu} \; \delta g^{\mu\nu} \;\; = \;\; - \; \lambda \; \sqrt{g} \; w(g^{\mu\nu}) \; T_{\mu\nu} \; g^{\mu\nu} = 0 \; \Rightarrow \; T = T_{\mu\nu} \; g^{\mu\nu} = 0. \tag{5.32a}$$

since the Weyl weight $w(g^{\mu\nu}) \neq 0$. The minus sign of the second term in the r.h.s of (5.31) is due to the variation of the determinant of the metric $g_{\mu\nu}$ resulting from the identities

$$\sqrt{det\ g_{\mu\nu}} \ = \ e^{\frac{1}{2} trace\ ln\ (g_{\mu\nu})} \ = \ e^{-\frac{1}{2} trace\ ln\ (g^{\mu\nu})}, \quad g^{\mu\nu} = (g_{\mu\nu})^{-1}. \tag{5.32b}$$

$$\delta\ \sqrt{g} \ = \ - \ \frac{1}{2}\ \sqrt{g}\ g_{\mu\nu}\delta g^{\mu\nu} \ = \ \frac{1}{2}\ \sqrt{g}\ g^{\mu\nu}\ \delta g_{\mu\nu},. \tag{5.33}$$

Therefore, when $r > 0$ the point mass terms don't contribute to the stress energy tensor and the relevant term is then the EM part of the action density :

$$\frac{1}{4e^2}\ F_{\alpha\beta}F^{\alpha\beta} \ = \ \frac{E^2(r)}{4e^2} \ = \ \frac{1}{4e^2}\ (\frac{e}{r^2})^2 \ = \ \frac{1}{16\pi G}\ \frac{e^2}{r^4},\ when\ 4\pi G = e^2. \tag{5.34}$$

the outer and inner horizons of the Reissner-Nordstrom massive charged black hole in the natural units is given by the solutions of the algebraic equation

$$1 - \frac{2GM}{r} + \frac{e^2}{r^2} \ = \ 0 \ \Rightarrow\ r_\pm \ = \ GM \ \pm\ \sqrt{(GM)^2 - e^2}. \tag{5.35}$$

From eq-(5.34) we can evaluate the EM part of the action bounded by the outer and inner horizons of the Reissner-Nordstrom massive charged black hole

$$\frac{1}{16\pi G}\ \int\int_{r_-}^{r_+}\ F_{\mu\nu}\ F^{\mu\nu}\ (4\pi r^2\ dr\ dt) \tag{5.36}$$

The spatial integral yields

$$\frac{1}{16\pi G}\ \int_{r_-}^{r_+}\ \frac{e^2}{r^4}\ 4\pi r^2\ dr \ =$$

$$\frac{e^2}{4G}\ [\ \frac{1}{r_-}\ -\ \frac{1}{r_+}\ ] \ = \ \frac{e^2}{4G}\ \frac{r_+\ - r_-}{r_+\ r_-} \ = \ \frac{e^2}{4G}\ \frac{2\sqrt{(GM)^2 - e^2}}{e^2}. \tag{5.37}$$

Upon a compactification the Euclidean thermal-time interval is $2\pi t = 1/k_B T$ (we will set $k_B = 1$). The temperature of the Reissner-Nordstrom Black Hole [7] is

$$T_H \ = \ \frac{1}{2\pi}\ \frac{\sqrt{(GM)^2 - e^2}}{2(GM)^2\ +\ 2GM\sqrt{(GM)^2 - e^2}\ \ -\ e^2} \ = \ \frac{1}{4\pi}\ \frac{r_+\ - r_-}{r_+^2}. \tag{5.38}$$

The full spatial-temporal integration corresponding to the EM part of the action bounded by the outer and inner horizons of the Reissner-Nordstrom Black Hole is then :

$$\frac{1}{16\pi G}\ \int_{r_-}^{r_+}\ \frac{e^2}{r^4}\ (4\pi r^2\ dr)\ \int_0^{1/k_B T}\ dt_E \ =$$

$$[\ \frac{2\pi\ e^2}{4G}\ ]\ [\ \frac{2\sqrt{(GM)^2 - e^2}}{e^2}\ ]\ \ [\ \frac{2(GM)^2 + 2GM\sqrt{(GM)^2 - e^2}\ \ - e^2}{\sqrt{(GM)^2 - e^2}}\ ] \ =$$

$$[\ \frac{\pi}{G}\ ]\ [\ GM\ +\ \sqrt{(GM)^2\ -\ e^2}\ ]^2\ =\ \frac{\pi\ r_+^2}{G}\ =\ \frac{4\pi r_+^2}{4G} = \frac{Area}{4L^2_{Planck}}. \qquad (5.39)$$

Therefore, we have shown that when $4\pi G\ =\ e^2$, the (Euclideanized) EM part of the action associated with the bulk region bounded by the outer and inner horizons of the Reissner-Nordstrom massive charged black hole is precisely equal to the Black Hole Entropy $4\pi r_+^2/4G$. The relationship between G and e is reminiscent of what occurs in Kaluza-Klein compactifications from $5D$ to $4D$. In Loop Quantum Gravity there is an undetermined Immirizi parameter in the Entropy-Area relation based on the $SU(2)$ spin-networks calculation.

The Reissner-Nordstrom black hole entropy can be recast as

$$S_E^{RN}\ =\ \frac{e^2}{4G}\ [\ \frac{1}{r_-}\ -\frac{1}{r_+}\ ]\ \frac{1}{T_H}\ =\ \frac{e^2}{4G}\ \frac{r_+\ -\ r_-}{r_+\ r_-}\ \frac{4\pi r_+^2}{r_+\ -\ r_-}\ =\ \frac{4\pi r_+^2}{4G}. \qquad (5.40)$$

since $e^2\ =\ r_+ r_-$ resulting from eq-(5.35) and after using eq-(5.38). In the extremal Reissner-Nordstrom black hole case, $GM\ =\ e$, the outer and inner horizons coincide $r_+ = r_- = GM$ so the spatial integral of the scalar curvature bounded by a domain of size zero is zero. However since the temperature in this extremal case is also *zero*, when one computes the Entropy in this case one will get $\frac{0}{0}$ which is undetermined, however due to the exact cancellation of the terms $r_+ - r_-$ in the numerator and denominator of eq-(5.40) the Entropy value becomes precisely equal to $(Area/4G)$ where $Area = 4\pi r_+^2 = 4\pi r_-^2 = 4\pi\ (GM)^2$ for the extremal Reissner-Nordstrom Black hole.

The charged rotating ring solution given by the Kerr-Newman Black Hole has an angular momentum per unit mass $a = \frac{J}{M}$ and the Hawking temperature corresponding to the outer horizon is [7]

$$T_H(M, e, a = \frac{J}{M})\ =\ \frac{1}{2\pi}\ \frac{\sqrt{(GM)^2 - a^2 - e^2}}{2(GM)^2\ +\ 2GM\sqrt{(GM)^2 - a^2 - e^2)}\ \ -\ e^2}\ =$$
$$\frac{1}{4\pi}\ \frac{r_+\ -\ r_-}{r_+^2\ +\ a^2}. \qquad (5.41)$$

The outer and inner horizons are solutions of the equation :

$$\Delta\ =\ r^2\ - 2GM\ r\ +\ a^2\ +\ e^2\ = 0\ \Rightarrow\ r_\pm\ =\ GM\ \pm\sqrt{(GM)^2\ - a^2\ - e^2}. \qquad (5.42)$$

The Entropy is

$$S_E\ =\ \frac{1}{4G}\ 4\pi\ [r_+^2\ +\ a^2]\ =\ [\ \frac{\pi}{G}\ ]\ [\ GM\ +\ \sqrt{(GM)^2\ - a^2 -\ e^2}\ ]^2\ =\ \frac{Area}{4L^2_{Planck}}. \qquad (5.43)$$

As a direct consequence of the temperature relation (5.41) and the rotational-energy density $(1/16\pi G)\ (J/Mr^2)^2\ =\ (1/16\pi G)\ (a/r^2)^2$ contribution to the energy, the Kerr-Newman back-hole entropy can still be re-written as

$$S_E^{KN}\ =\ \frac{e^2 + a^2}{4G}\ [\ \frac{1}{r_-}\ -\frac{1}{r_+}\ ]\ \frac{1}{T_H}\ =\ \frac{e^2 + a^2}{4G}\ \frac{r_+\ -\ r_-}{r_+\ r_-}\ \frac{4\pi(r_+^2\ +\ a^2)}{r_+\ -\ r_-}\ =\ \frac{4\pi(r_+^2\ + a^2)}{4G}. \qquad (5.44)$$

since $e^2 + a^2 = r_+ r_-$ resulting from (5.42). Based on the entropy functional forms given by eqs-(5.40, 5.44) (in the form energy $\times$ time) of the Reissner-Nordstrom and Kerr-Newman back-hole entropies, If one evaluates the bulk integral of the EM part of the action $\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}$ in the Kerr-Newman stationary solution expressed in Boyer-Lindquist coordinates one gets

$$S_E = \frac{1}{4e^2} \int dt \int d\phi \int_{r_-}^{r_+} dr \int_{-1}^{+1} d(cos\phi)\ (r^2 + a^2 cos^2\phi)\ \frac{e^2}{(r^2 + a^2 cos^2\phi)^2} =$$

$$\pi\ [\ \frac{iLi_2\ [-\frac{ia}{r}\ ]\ - iLi_2\ [\ \frac{ia}{r}\ ]}{2a}\ ]_{r_-}^{r_+}\ \frac{1}{T_H}. \tag{5.45}$$

where Li_2 is the di-logarithm. The poly-logarithm is defined by $Li_n(z) = \sum_{k=1} \frac{z^k}{k^n}$. Hence, it is only *to leading order* in powers of $\frac{a}{r}$ that one recovers from the integral of eq-(5.45) the Kerr-Newman black hole entropy

$$S_E \sim \pi\ [\ \frac{1}{r_-} - \frac{1}{r_+}\ ]\ \frac{1}{T_H} = \frac{\pi(r_+ - r_-)}{r_+\ r_-}\ \frac{4\pi(r_+^2 + a^2)}{r_+ - r_-} =$$

$$\frac{\pi}{e^2 + a^2}\ 4\pi(r_+^2 + a^2) = \frac{4\pi(r_+^2 + a^2)}{4G},\ when\ e^2 + a^2 = 4\pi G. \tag{5.46}$$

with the provision that the condition $e^2 + a^2 = 4\pi G$ is obeyed. The Reissner-Nordstrom entropy was recovered in eqs-(5.39, 5.40) when the condition $e^2 = 4\pi G$ was satisfied. To sum up, the expression

$$S_E = \frac{e^2 + a^2}{4G}\ [\ \frac{1}{r_-} - \frac{1}{r_+}\ ]\ \frac{1}{T_H}. \tag{5.47}$$

recaptures the Kerr-Newman entropy as well as the Reissner-Nordstrom, Kerr and Schwarzschild entropy when $a^2 = 0,\ e^2 = 0,\ e^2 = a^2 = 0$, respectively. The first law of thermodynamics relating the change of the internal energy dU with the change of entropy TdS and work dW is :

$$TdS_E - dW = dU \Rightarrow TdS_E - Jd\Omega - \Phi dQ = dU = d\ (M - Q\Phi - J\Omega). \tag{5.48}$$

where $\Phi = e/r_+$ is the electrostatic potential at the outer-horizon, Ω is the angular velocity of the outer horizon; J is the angular momentum and M is the ADM mass. The first law can be interpreted as the relationship among the global charges, parameters (M, e, J) and T, S which is obtained by performing a variation of the Euclidean action resulting from perturbing the location of the inner and outer horizons. Viewing the Hawking radiation and emission of particles as a quantum tunneling that shrinks the size of the horizons [35] is another way of perturbing the value of the Euclidean action. For a thorough discussion of interpreting Einstein's field equations as just a thermo-dynamical equation of state see [36] and Wald's entropy expression related to the global Noether charge of diffs under the Killing vector field which generates the event horizon in the stationary black hole background and which is given by a local geometric density integrated over a space-like section of the horizon [34].

6 Spatial-Temporal Vacuum Solutions

In this section we shall find novel spatial-temporal vacuum solutions based on our results in sections **3** and **4**. By taking now $g_{00} = -e^{\mu}$, $g_{11} = e^{\nu}$ and $g_{10} = g_{01} = 0$, with $\mu = \mu(t,r)$, $\nu = \nu(t,r)$ and $\varphi = \varphi(t,r)$, in this section we shall look for a more general solution of the field equations obtained from (4.2). For this purpose let us first observe the identity

$$\hat{R}_{00} = \hat{R}^1_{010} = g_{00}g^{11}\hat{R}^0_{101} = g_{00}g^{11}\hat{R}_{11} \tag{6.1}$$

or

$$g^{00}\hat{R}_{00} = g^{11}\hat{R}_{11}, \tag{6.2}$$

since $g^{00}g_{00} = g^{11}g_{11} = 1$. Observe that (6.2) is in fact what we used in (5.9) for the case of static black-holes.

Using eq- (4.2) one discovers that the identity (6.2) leads to

$$g^{00}\mathcal{D}_0\varphi_{,0} = g^{11}\mathcal{D}_1\varphi_{,1}. \tag{6.3a}$$

which explicitly becomes

$$-e^{-\mu}\,\frac{\partial^2\varphi}{\partial t^2} \;+\frac{1}{2}e^{-\mu}\,\frac{\partial\varphi}{\partial t}\,\frac{\partial\mu}{\partial t} \;+\frac{1}{2}e^{-\nu}\,\frac{\partial\varphi}{\partial r}\,\frac{\partial\mu}{\partial r} \;=$$

$$e^{-\nu}\,\frac{\partial^2\varphi}{\partial r^2} \;-\frac{1}{2}e^{-\nu}\,\frac{\partial\varphi}{\partial r}\,\frac{\partial\nu}{\partial r} \;-\frac{1}{2}e^{-\mu}\,\frac{\partial\varphi}{\partial t}\,\frac{\partial\nu}{\partial t}. \tag{6.3b}$$

Therefore the second expresion in (4.2) yields

$$k(D-3) - 2\varphi g^{11}\mathcal{D}_1\varphi_{,1} - (D-3)g^{00}\dot{\varphi}^2 - (D-3)g^{11}\varphi'^2 = 0. \tag{6.4}$$

Here, $\dot{\varphi} = \varphi_{,0}$. This is equivalent to substitute (5.10) into (5.8). After writing the explicit expression of the covariant derivative $\mathcal{D}_1\varphi_{,1}$ eq-(6.4) becomes in terms of μ and ν :

$$k(D-3) + e^{-\nu}\{\nu'\varphi\varphi' - 2\varphi\varphi'' - (D-3)\varphi'^2\} + e^{-\mu}\{\dot{\nu}\varphi\dot{\varphi} + (D-3)\dot{\varphi}^2\} = 0 \tag{6.5}$$

or

$$\gamma'\varphi\varphi' + 2\gamma\varphi\varphi'' + (D-3)\gamma\varphi'^2 = k(D-3) + e^{-\mu}\dot{\varphi}\{\dot{\nu}\varphi + (D-3)\dot{\varphi}\}, \tag{6.6}$$

where $\gamma = e^{-\nu}$.

On the other hand we also need to consider the field equation

$$\mathcal{D}_1\varphi_{,0} = 0, \tag{6.7}$$

which leads to

$$\dot{\varphi}' - \frac{1}{2}\mu'\dot{\varphi} - \frac{1}{2}\dot{\nu}\varphi' = 0. \tag{6.8}$$

We are going to find solutions to eqs–(6.3, 6.6, 6.8) by choosing the ansatz

$$e^{\mu(\varphi(t))} \;=\; e^{-\nu(\varphi(t))} \;=\; \gamma(t). \tag{6.9}$$

from eq-(6.8) one gets

$$\dot{\varphi}' \;-\; \frac{1}{2}\;\frac{\varphi'\;\dot{\varphi}\;(d\gamma/d\varphi)}{\gamma} \;+\; \frac{1}{2}\;\frac{\varphi'\;\dot{\varphi}\;(d\gamma/d\varphi)}{\gamma} \;=\; 0 \;\Rightarrow\; \dot{\varphi}' \;=\; 0. \tag{6.10}$$

a solution to (6.10) is clearly given by $\varphi(t) \;=\; t$. Plugging the solutions (6.9) into (6.6) when $\varphi(t) = t$ and $D = 4,\; k = 1$ gives

$$1 \;+\; e^{-\mu}\;[\;1 - t\frac{d\mu}{dt}\;] \;=\; 0 \tag{6.11}$$

since $(\dot{\varphi})^2 = 1$ and $\varphi' = \varphi'' = 0$. One deduces from (6.11) that

$$1 \;+\; e^{\mu} \;=\; t\;\frac{d\mu}{dt} \;\Rightarrow\; \int \frac{dt}{t} \;=\; \int \frac{d\mu}{1+e^{\mu}}. \tag{6.12}$$

Integrating (6.12) yields

$$ln\;(\frac{t}{t_o}) \;=\; ln\;(\frac{e^{\mu}}{e^{\mu}+1}) \;\Rightarrow\; \frac{t_o}{t} - 1 \;=\; e^{-\mu} \;\Rightarrow e^{\mu} \;=\; \frac{1}{\frac{t_o}{t} - 1}. \tag{6.13}$$

By inspection one can infer that $\mu + \nu = 0 \Rightarrow \mu = -\nu$ is a solution of eq-(6.3a, 6.3b), when $\varphi = t$ and $\mu = \mu(t),\; \nu = \nu(t)$. Therefore, the time dependent vacuum solution is given by

$$ds^2 \;=\; -\;\frac{dt^2}{(\frac{t_o}{t} - 1)} \;+\; (\frac{t_o}{t} - 1)\;dr^2 \;+\; t^2\;(d\Omega)^2. \tag{6.14a}$$

which is nothing but the Kantowski-Sachs cosmological vacuum solution obtained from the Schwarzschild solution after the exchange $r \leftrightarrow t$ if one sets the constant $t_o = 2GM$. At $t \to \infty$ it is asymptotically flat. It has a horizon at $t = 2GM$ and a curvature singularity at $t = 0$. By exchanging $t \leftrightarrow r$ in eq-(5.18) one gets the higher dimensional generalization of the Kantowski-Sachs metric eq-(6.14) for different values of D, k

$$ds^2 \;= -\;\frac{(dt)^2}{(-k + \frac{\beta_D G_D M}{t^{D-3}})} \;+\; (-k + \frac{\beta_D G_D M}{t^{D-3}})\;(dr)^2 \;+\; t^2\;\tilde{g}_{ab}\;d\xi^a\;d\xi^b. \tag{6.14b}$$

Inspired from the Fronsdal-Kruskal-Szekeres form of the metric in natural units $c = 1$, with a signature $(-, +, +, +, +)$, and working with coordinates r, t of length dimension, one can choose the metric of the form

$$ds^2 \ = \ \frac{4(2GM)}{\varphi(r,t)} \ e^{-\varphi(r,t)/2GM} \ (\ -dt^2 \ + \ dr^2 \ ) \ + \ \varphi^2(r,t) \ (d\Omega)^2, \quad d\Omega^2 = d\phi^2 + sin^2\phi \ d\theta^2. \tag{6.15}$$

as solutions to the vacuum field equations which are dependent on both r, t coordinates. The metric (6.15) correspond now to the case

$$e^{\mu(\varphi(r,t))} \ = \ e^{\nu(\varphi(r,t))} \ = \ \frac{4(2GM)}{\varphi(r,t)} \ e^{-\varphi(r,t)/2GM}. \tag{6.16}$$

where now $\mu(r,t) = \nu(r,t)$, instead of $\mu(r,t) = -\nu(r,t)$, and the radial function $\varphi(r,t)$ must obey the system of differential eqs-(6.3, 6.6, 6.8). As stated earlier, the Fronsdal-Kruskal-Szekeres coordinates U, V are explicitly given in terms of r, t by eq-(2.19).

The difficult problem is to *invert* these relations in order to obtain the functional relations $r = r(U,V), \ t = t(U,V)$. Whereas, by choosing the ansatz in eqs-(6.15, 6.16) one is able to derive a set of differential equations (6.3b, 6.6, 6.8) obeyed by $\mu(r,t) = \nu(r,t)$ and the radial function $\varphi(r,t)$, and whose solutions for the radial function $\varphi(r,t)$ must comprise a solution with the same functional form as the radial function $r = r(U,V)$ (whose functional form is unknown because we cannot invert the functional relations in eq-(2.16)). For instance, given $y(x) = xe^x$ the inverse function $x = W(y)$ is the celebrated Lambert W-function whose exact analytical expression is unknown.

We will now find explicit new solutions If one chooses

$$e^{\mu(\varphi(r,t))} \ = \ e^{-\nu(\varphi(r,t))} \ = \ \gamma(\varphi(r,t)). \tag{6.17}$$

One can see that eq-(6.3b) and eq-(6.8) are automatically satisfied for a radial function function of the form $\varphi(r,t) \ = \ t \pm r$ which is just the advanced and retarded time, respectively. Inserting these solutions into eq-(6.6) when $D = 4$ and $k = 1$ gives after some straightforward algebra :

$$(\gamma^2 + 1) \ \varphi \ (\frac{d\gamma}{d\varphi}) \ = \ -\gamma^3 \ + \gamma^2 \ + \gamma \ \Rightarrow \ \int \ d\gamma \ (\frac{\gamma^2 + 1}{-\gamma^3 \ + \gamma^2 \ + \gamma}) \ = \ \int \ \frac{d\varphi}{\varphi}. \tag{6.18}$$

upon integration it gives

$$ln \ [\frac{\gamma}{\gamma(\gamma - 1) \ - 1} \ ] \ = \ ln \ \frac{\varphi}{\varphi_o} \ \Rightarrow \ 1 \ + \frac{\varphi_o}{\varphi} \ = \ \gamma \ - \frac{1}{\gamma}. \tag{6.19}$$

where the integration constant is φ_o and has dimensions of length. Solving for γ in terms of φ leads to

$$\gamma(\varphi) \ = \ \frac{(1 \ + \frac{\varphi_o}{\varphi}) \ \pm \ \sqrt{(1 \ + \frac{\varphi_o}{\varphi})^2 + 4}}{2}. \tag{6.20}$$

In particular, when $\varphi(r,t) = t + r = u$, $\varphi(r,t) = (t - r) = v$, the solutions to the function $\gamma(r,t)$ are :

$$\gamma(r,t) \;=\; \gamma(t \pm r) \;=\; \frac{(1 \;+\frac{\varphi_o}{t\pm r}) \;\pm\; \sqrt{(1 \;+\frac{\varphi_o}{t\pm r})^2 + 4}}{2}. \qquad (6.21)$$

Therefore, the dynamical metric in terms of the advanced and retarded temporal coordinates $t \pm r$ is then given by

$$ds^2 \;=\; -\;\gamma(t \pm r)\;(dt)^2 \;+\; \frac{(dr)^2}{\gamma(t\pm r)} \;+\; (t\pm r)^2\;(d\Omega)^2. \qquad (6.22)$$

where the explicit functional form $\gamma(r,t) = \gamma(t \pm r)$ is given by eq-(6.20). Choosing a $+$ sign in eq-(6.21), one can perform a Taylor expansion of the expression inside the square root leading to

$$\gamma(r,t) \;=\; \gamma(t\pm r) \;\sim\; 1 \;+\; \frac{\varphi_o}{t\pm r} \;+\; \frac{t\pm r}{\varphi_o} + ... \qquad (6.23a)$$

Taylor expansion which is valid in the regime when

$$0 \;<\; |t \pm r| \;\; < \; |\varphi_o| \qquad (6.23b)$$

Since $\gamma = e^{-\nu(r,t)} \geq 0$ one must choose a $+$ sign in eq-(6.21), such that when one sets the integration constant $\varphi_o > 0$, one can see that the metric (6.22) is singular at $\varphi = t \pm r = 0$ because at those locations one has $\gamma(t\pm r = 0^+) = \infty$ and $\gamma(t \pm r = 0^-) = 0$. There is a *discontinuity* of γ at $t \pm r = 0$. If $\varphi_o < 0$ one would have the converse behaviour, $\gamma = \infty$ at $t \pm r = 0^-$ and $\gamma = 0$ at $t \pm r = 0^+$. We have not equated φ_o to $2GM$ in this dynamical case. At the end of this subsection we will explore further the physical interpretation to the metric solution (6.21, 6.22). A similar singularity along a null plane (wave-front) occurs also in the Aichelburg-Sexl metric

$$ds^2 \;=\; 8\; p\; log\; \rho\; \delta(u)\; du^2 \;+ du\; dw \;\; - dy^2 - dz^2. \qquad (6.24)$$

where p is the momentum and ρ is the mass density. The metric describes a plane-polarized *pp*-wave and is flat everywhere except on the location of the null plane $u = t - x = 0$ where it is singular. The variable w is defined by $w = t + x$. Such metric (6.24) is the ultra-relativistic limit of the Schwarzschild metric and represents the shock-wave geometry when the energy-momentum has a delta-like support on a null plane $u = t - x = 0$ and is generated by a particle moving at the speed of light along the x-direction.

When $|t \pm r| \to \infty$ the value of (6.21) for the $+$ sign tends to $\gamma \to (1 + \sqrt{5})/2 =$ Golden Mean $= 1.618...$, which is a curious numerical result. Since $\gamma(\pm\infty) \neq 1$, one may notice that the metric components g_{tt}, g_{rr} of eq-(6.22) do not become unity when $t \pm r \to \infty$, nevertheless the curvature tensor vanishes at $|t \pm r| = \infty$.

Time dependent black hole solutions with a nontrivial *dilaton* field background in $3D$ and higher dimensions were found by [37] based on the application of the $3D$ Janus geometry. The metric was of the form

$$ds^2 \;=\; f(r)\;[-dt^2 \;+dr^2 \;+r_o^2\;cos^2(\frac{t}{t_o})\;(d\Omega)^2\;]. \tag{6.25}$$

The spacetime is locally isomorphic to the $4D$ Anti de Sitter space and the curvature singularity at $t=0$ is of the orbifold type [37].

Inspired from the above Janus geometry we shall seek *vacuum* solutions (there is no dilaton background in this case) of the form

$$ds^2 \;=\; f(r)\;[\;-dt^2 \;+dr^2 \;+r_o^2\;cosh^2(\frac{t}{t_o})\;(d\Omega)^2\;]. \tag{6.26}$$

by replacing the oscillatory radial function $r_o cos(\frac{t}{t_o})$ for the non-oscillating one $r_o cosh(\frac{t}{t_o})$. The solutions for the conformal factor $f(r)$ are obtained by inserting the metric ansatz (6.26) into the eqs-(6.3b, 6.6, 6.8). Eq-(6.8) is automatically satisfied and after some algebra eqs-(6.3b, 6.6) lead respectively to :

$$-\frac{1}{t_o^2}\;+\;\frac{3}{4}\;(\frac{f'}{f})^2\;-\;\frac{1}{2}\;\frac{f''}{f}\;=\;0. \tag{6.27a}$$

$$1\;+\;[\;\frac{3}{4}\;(\frac{f'}{f})^2\;-\;\frac{f''}{f}\;]\;r_o^2\;cosh^2(\frac{t}{t_o})\;\;+\;\;\frac{r_o^2}{t_o^2}\;sinh^2(\frac{t}{t_o})\;=\;0. \tag{6.27b}$$

Using the identity $cosh^2x - sinh^2x = 1$ one can verify by simple inspection that the solution to eqs-(6.27a, 6,27b) (when $c=1$) is

$$r_o\;=\;t_o,\quad f(r)\;=\;e^{\pm\frac{2r}{r_o}}. \tag{6.28}$$

Consequently the metric (6.26) is

$$ds^2\;=\;e^{\pm\frac{2r}{r_o}}\;[\;-dt^2 \;+dr^2 \;+r_o^2\;cosh^2(\frac{t}{r_o})\;(d\Omega)^2\;]. \tag{6.29}$$

At $t=\infty$ the angular part of the metric diverges for finite r. Whether it is just a coordinate singularity or a physical one needs to be investigated. Solutions to eqs-(6.3, 6.6, 6.8) when $k=-1,0; D\geq 4$ and axially symmetric and stationary solutions of the form [7]

$$ds^2\;=\;-e^{A(\rho,z)}\;[\;dt - B(\rho,z)d\phi\;]^2\;+\;e^{-A(\rho,z)}\;\rho^2\;d\phi^2\;+\;C(\rho,z)\;d\rho^2\;+\;D(\rho,z)\;dz^2. \tag{6.30}$$

can also be found by similar means.

The most general expression for γ in eq-(6.21) for different values of D, k is

$$\gamma(r,t)\;=\;\gamma(t\pm r)\;=\;\frac{k\;+(\frac{\varphi_o}{t\pm r})^{D-3}\;\pm\;\sqrt{[\;k\;+(\frac{\varphi_o}{t\pm r})^{D-3}\;]^2+4}}{2}. \tag{6.31}$$

and the dynamical metric in terms of the advanced and retarded temporal coordinates $t\pm r$ in $d\geq 4$ is

$$ds^2 \;=\; -\;\gamma(t \pm r)\;(dt)^2 \;+\; \frac{(dr)^2}{\gamma(t \pm r)} \;+\; (t \pm r)^2\;\tilde{g}_{ab}\;d\xi^a\;d\xi^b. \tag{6.32}$$

where the explicit functional form $\gamma(r,t) = \gamma(t \pm r)$ is given by eq-(6.31).

A careful analysis of the two eqs-(6.5,6.8) reveals that the relation in the r.h.s of eq-(6.19) is a very special case of the most general relation in $D \geq 4$ and for $k = 1, 0, -1$

$$k \;+\; (\frac{\varphi_o}{\varphi(r,t)})^{D-3} \;=\; e^{-\nu(\varphi)}(\frac{\partial \varphi}{\partial r})^2 \;-\; e^{-\mu(\varphi)}(\frac{\partial \varphi}{\partial t})^2. \tag{6.33}$$

which can be derived directly from the two eqs-(6.3, 6.5,6.8) after defining

$$F(r,t) \;\equiv\; e^{-\nu(\varphi)}(\frac{\partial \varphi}{\partial r})^2 \;-\; e^{-\mu(\varphi)}(\frac{\partial \varphi}{\partial t})^2. \tag{6.34}$$

and performing some laborious but straightforward algebra in eqs-(6.3, 6.5, 6.8) leads to the two equations

$$\frac{(\partial F(r,t)/\partial t)}{(\partial F(r,t)/\partial r)} \;=\; \frac{(\partial \varphi(r,t)/\partial t)}{(\partial \varphi(r,t)/\partial r)}. \tag{6.35}$$

and

$$[\;\frac{\varphi}{(\partial\varphi/\partial r)}\;]\;[\;\frac{\partial F(r,t)}{\partial r}\;] \;=\; (D-3)\;[\;k - F(r,t)\;] \Rightarrow$$

$$\int \frac{dF}{(D-3)\;(k-F)} \;=\; \int \frac{d\varphi}{\varphi} \;\Rightarrow$$

$$F(r,t) \;=\; k \;-\; \frac{f(t)}{(\varphi(r,t))^{D-3}} \tag{6.36a}$$

where $f(t)$ is an arbitrary integration function depending solely on t. In order to satisfy eq-(6.35) one must have that $f(t) = constant$, therefore one arrives at

$$F(r,t) \;=\; k \;+\; (\frac{\varphi_o}{\varphi(r,t)})^{D-3}. \tag{6.36b}$$

after setting $-f(t) = constant = (\varphi_o)^{D-3}$. The parameter φ_o has length dimensions.

Namely, by defining $F(r,t)$ in eq-(6.34) one is able to *rewrite* the two eqs-(6.5, 6.8) in the form given by eqs-(6.35, 6.36). The geometrical meaning behind the definition of $F(r,t)$ in eq-(6.34) is that it represents the norm squared of the velocity associated with the gradient of the area-radius function $\varphi(r,t)$

$$g^{\mu\nu}\;\partial_\mu\varphi\;\partial_\nu\varphi \;=\; g^{tt}\;(\partial_t\varphi)^2 \;+\; g^{rr}\;(\partial_r\varphi)^2 \;\equiv\; F(r,t). \tag{6.37}$$

corresponding to the D-dim metric

$$ds^2 \;=\; -\;e^{\mu(r,t)}\;(dt)^2 \;+\; e^{\nu(r,t)}\;(dr)^2 \;+\; \varphi(r,t)^2\;(d\Omega_{D-2})^2 \tag{6.38}$$

when $\mu(r,t) = \mu(\varphi(r,t))$ and $\nu(r,t) = \nu(\varphi(r,t))$. In the *static* case one recovers from eq-(6.33) the solution in eq-(5.18)

$$k + (\frac{\varphi_o}{\varphi(r)})^{D-3} = e^{-\nu(\varphi)} (\frac{\partial \varphi}{\partial r})^2 \Rightarrow$$

$$g_{rr} = e^{\nu(\varphi)} = \frac{(\frac{\partial \varphi}{\partial r})^2}{k + (\frac{\varphi_o}{\varphi(r)})^{D-3}}. \tag{6.39}$$

One may notice that when $F(r,t) = 0$ the zero norm in eq-(6.37) corresponds geometrically to a horizon. For example, in the case $D = 4, k = 1$ and $\varphi(r,t) = t \pm r$ the location of a horizon occurs when

$$F(r,t) = 0 \Rightarrow 1 + \frac{\varphi_o}{t \pm r} = 0 \Rightarrow t \pm r = -\varphi_o \Rightarrow dt \pm dr = 0 \tag{6.40}$$

such that the value of γ in eq-(6.21) becomes $\gamma(t \pm r = -\varphi_o) = 1$. Hence, when the value $\gamma = 1$ and $dt \pm dr = 0$ one arrives at the expression for a particular null *radial* geodesic (there are others) associated with the metric in eq-(6.22) and given by $ds^2 = -dt^2 + dr^2 = 0$ which is compatible with the null surface $F(r,t) = 0$ condition of a horizon.

From the arguments described after eq-(6.23), related to the discontinuity of γ at $t \pm r = 0$, with values of $0, \infty$, one finds that *horizonless* solutions occur when $\varphi_o > 0,\ t \pm r \geq 0$ or when $\varphi_o < 0,\ t \pm r \leq 0$. The singularity at $t \pm r = 0$ is *timelike* and is visible to an observer. The existence of *horizons* occurs when $\varphi_o > 0,\ t \pm r \leq 0$ or when $\varphi_o < 0,\ t \pm r \geq 0$. The singularity at $t \pm r = 0$ is *spacelike* and would be hidden behind the horizons at $t \pm r = -\varphi_o$ (such that $\gamma(t \pm r = -\varphi_o) = 1$) from those observers in the regions $|t \pm r| > |\varphi_o|$. Notice that a *null* like singularity would require both that $ds^2 = 0$ and the curvature tensor $\mathcal{R}_{\mu\nu\rho\sigma}$ (and/or other curvature scalars) to *diverge* along the corresponding null like curve, which is not the case here.

Long ago Penrose [44] proposed the Cosmic Censorship Conjecture (CCC) stating that singularities which form in a gravitational collapse and consistent with Einstein's [13] field equations should never be visible to an outside observer or they should be hidden inside a horizon. Deshingkar [45] studied singularities which can form in a spherically symmetric gravitational collapse of a general matter field obeying the weak energy condition. He showed that no energy can reach an outside observer from a null naked singularity. That means they will not be a serious threat to the Cosmic Censorship Conjecture (CCC). For timelike naked singularities, where only the central shell gets singular, the redshift is always finite and they can in principle, carry energy to a faraway observer. Hence for proving or disproving CCC the study of timelike naked singularities is more important. The results of [45] were very general and independent of the initial data and the form of the matter.

Recently [46] we have shown the existence of *timelike* naked singularities which are *not* hidden by a horizon and which are associated to spherically symmetric (noncompact) matter sources extending from $r = 0$ to $r = \infty$ and obeying the weak energy conditions . These asymptotically flat solutions do represent observable *timelike* naked singularities where the scalar curvature $\mathcal{R}$ and volume mass density $\rho(r)$ are both *singular* at $r = 0$. Therefore, the *horizonless* solutions we just found above when $\varphi_o > 0,\ t \pm r \geq 0$ or $\varphi_o < 0,\ t \pm r \leq 0$, associated with a *timelike* singularity at $t \pm r = 0$ should not be discarded.

One could interpret the singularity from the perspective of radial ingoing/outgoing spherical waves whose shock wave front is singular at $t \pm r = 0$ due to infinite pressures along the wave front and which can be realized in terms of a delta-function field configuration depending on r, t; i.e when the energy-momentum has a delta-like support on the regions $t \pm r = 0$. In the Schwarzschild solution, the integration constant $2GM$ of length dimensions is defined in terms of the Kepler mass M as measured by an observer in the asymptotically flat spacetime region. The length parameter φ_o in our above solutions has a similar interpretation in terms of the global mass-energy content carried by the shock wave front as measured by an observer in the asymptotically-flat future (past) null infinity region. As stated above, a similar singularity along a null plane (shock wave-front) occurs also in the Aichelburg-Sexl metric (6.24) (the ultra-relativistic limit of the Schwarzschild metric) describing a plane-polarized *pp*-wave that is flat everywhere except on the location of the null plane $u = t - x = 0$ where it is singular.

To conclude, we have found (to our knowledge) new spatial-temporal vacuum solutions to Einstein's field equations, given by eqs-(6.21, 6.22, 6.31, 6.32) in $D \geq 4$, and for $k = 1, 0, -1$ in a straightforward fashion from the results of sections **3, 4**. In the particular case for $D = 4; k = 1$ one found solutions (with and without horizons) which display a singularity at $t \pm r = 0$. Furthermore, we have provided a physical interpretation to these solutions. The $4D$ metric solution (6.29) was inspired from the $3D$ Janus geometry one.

7 Cosmology based on the Interior Geometry of a Black Hole and a Modified Kantowski-Sachs metric

It has been known for some time that the external spatial and temporal coordinates *exchange* their character when the event horizon is crossed and the interior solution representing a *non−static* spacetime with a time-dependent metric is given by a *modified* Kantowski-Sachs metric (no longer a vacuum solution) for the interior spacetime region :

$$ds^2 \ = \ - \ \frac{dt^2}{[\frac{2Gm(t)}{t} - 1]} \ + \ [\frac{2Gm(t)}{t} - 1] \ dz^2 \ + \ t^2 \ d\Omega^2, \tag{7.1}$$

where now z is the spatial coordinate inside the black hole and *no* longer represents a radial coordinate inside. The range of values of z is $-\infty \leq \ z \ \leq +\infty$, whereas the ordinary radial variable $r \geq 0$. The components of the Einstein tensor in an ortho-normal reference frame are [38] (and references therein)

$$G_{tt} = \mathcal{R}_{tt} \ - \frac{1}{2} g_{tt} \ \mathcal{R} \ = \ \frac{2G}{t^2} \ \frac{dm(t)}{dt}, \tag{7.2}$$

$$G_{zz} = \mathcal{R}_{zz} \ - \frac{1}{2} g_{zz} \ \mathcal{R} \ = \ - \ \frac{2G}{t^2} \ \frac{dm(t)}{dt}, \tag{7.3}$$

$$G_{\theta\theta} = \mathcal{R}_{\theta\theta} \ - \frac{1}{2} g_{\theta\theta} \ \mathcal{R} \ = \ - \ \frac{G}{t} \ \frac{d^2 m(t)}{dt^2}, \tag{7.4}$$

$$G_{\phi\phi} = \mathcal{R}_{\phi\phi} \ - \frac{1}{2} g_{\phi\phi} \ \mathcal{R} \ = \ - \ \frac{G}{t} \ \frac{d^2 m(t)}{dt^2}. \tag{7.5}$$

This solution implies that $\rho = -p_z$, meaning that the pressure $p_z < 0$ when $\rho > 0$. The isotropic pressure condition requires equating the components of the Einstein tensor

$$G_{zz} = G_{\theta\theta} = G_{\phi\phi} \ \Rightarrow \ \frac{d^2 m(t)}{dt^2} \ - \ \frac{2}{t} \frac{dm(t)}{dt} = 0, \tag{7.6}$$

whose solution is

$$m(t) = C_1 + C_2 \ t^3 \tag{7.7}$$

and yields

$$ds^2 \ = \ - \ \frac{dt^2}{[\frac{2C_1}{t} + 2C_2 \ t^2 - 1]} \ + \ [\frac{2C_1}{t} + 2C_2 \ t^2 - 1] \ dz^2 \ + \ t^2 \ d\Omega^2 \tag{7.8}$$

which is the t-relative of the static form of the Anti-de Sitter-Schwarzshild solution

$$ds^2 \ = \ - \ [\ 1 \ - \frac{2GM}{r} - \frac{\Lambda}{3} \ r^2 \ ] \ dt^2 \ + \ \frac{dr^2}{[\ 1 \ - \frac{2GM}{r} - \frac{\Lambda}{3} \ r^2 \ ]} \ r^2 \ + \ d\Omega^2, \tag{7.9}$$

where $\Lambda < 0$ is the Cosmological constant in the Anti de Sitter-Schwarzshild case and $\Lambda > 0$ in the de Sitter-Schwarzshild case.

The *anisotropic* pressure of a self-gravitating *anisotropic* fluid is obtained when the the radial pressure is *not* equal to the tangential pressure. The pressure free case leads to the Lemaitre-Tolman metric [38]. For example, inspired from the static case problem studied by [39], [40], [41] we shall go ahead and smooth out the singularities at $t = 0$ by choosing a density given by the Gaussian

$$\rho(t) \ = \ \frac{M_o}{(4\pi\sigma^2)^{3/2}} \ e^{-t^2/4\sigma^2}. \tag{7.10}$$

from eqs-(7.2,7.3) one can infer that $\rho = -p_z$ and the tangential pressures are

$$p_\theta \ = \ p_\phi \ = \ - \rho(t) \ - \frac{t}{2} \ \frac{\partial \rho}{\partial t} \ = \ - \rho_o \ e^{-t^2/4\sigma^2} \ (\ 1 - \ \frac{t^2}{4\sigma^2} \ ). \tag{7.11}$$

The mass content at any instant t is

$$M(t,\sigma) \ = \ \int_0^t \ \frac{M_o}{(4\pi\sigma^2)^{3/2}} \ e^{-t^2/4\sigma^2} \ (4\pi \ t^2) \ dt \ = \ \frac{2M_o}{\sqrt{\pi}} \ \gamma \ [\ \frac{3}{2}, \ \frac{t^2}{4\sigma^2} \ ]. \tag{7.12a}$$

and is given in terms of the lower incomplete gamma function with parameter $a = \frac{3}{2}$.

Therefore, the metric is given explicitly in terms of the above mass function $M(t,\sigma)$ (7.12) as

$$ds^2 \;=\; -\;\frac{dt^2}{[\frac{2GM(t,\sigma)}{t} - 1]} \;+\; [\frac{2GM(t,\sigma)}{t} - 1]\; dz^2 \;+\; t^2\; d\Omega^2. \qquad (7.12b)$$

is a solution of Einstein's equations in the presence of matter whose density $\rho = -p_z$ and pressure configurations p_z, p_θ, p_ϕ are given by eqs-(7.10, 7.11).

The temporal horizon t_h is obtained by solving the *transcendental* equation obtained by setting in eq-(7.12b)

$$g_{zz}(t = t_h) \;=\; \frac{2GM(t = t_h, \sigma)}{t_h} \;-1 \;=\; 0 \;\Rightarrow\; t_h \;=\; (2GM_o)\;(\frac{2}{\sqrt{\pi}})\;\gamma\;[\;\frac{3}{2},\;\frac{t_h^2}{4\sigma^2}\;]. \qquad (7.13)$$

the *extremal* case corresponds when there is *one* horizon and it occurs when $t_h \sim 3.0\;\sigma$ and $GM_o \sim 1.9\;\sigma$. These results were found earlier for the *static* case solutions depending solely on r (instead of t) by [39], [40], [41] . For masses less than the critical value M_o there is *no* horizon. For masses larger than M_o there are *two* horizons as shown by [39]. The *no* horizon case is also a very interesting case.

Furthermore, the metric component $g_{zz}(t)$ is *not* singular at $t = 0$, due to the properties of the incomplete gamma function. For very small t, the behaviour of $2GM(t)/t$ is proportional to t^2 and hence $|g_{zz}| \rightarrow 1$ as $t \rightarrow 0$. And when $t \rightarrow \infty$, $|g_{zz}| \rightarrow 1$ also compatible with (temporal) asymptotic flatness. The scalar curvature is given by the expression

$$\mathcal{R}(t,\sigma) \;=\; -\;(8\pi G)\;\frac{2M_o\;e^{-t^2/4\sigma^2}}{(4\pi\sigma^2)^{3/2}}\;(2 - \frac{t^2}{4\sigma^2}). \qquad (7.14)$$

it is zero at $t = \infty$ due to the rapidly decaying behaviour of the exponential and the most important feature is that $\mathcal{R}(t = 0, \sigma) = -4GM_o/\sqrt{\pi}\sigma^3$ is *not* singular at $t = 0$ since the Gaussian distribution smears out the singularity at $t = 0$. When $\sigma = 0$ the scalar curvature blows up consistent with the fact that the zero width limit $\sigma \rightarrow 0$ of the Gaussian distribution $\rho(t, \sigma = 0) \rightarrow \delta(t)/4\pi t^2$ yields the delta function.

Let us focus in the extremal case when there is one horizon only. The relative amount of mass enclosed by the universe from $t = 0$ and $t = t_h \sim 3.0\;\sigma$ is

$$\frac{M(t = t_h,\;\sigma)}{M_o} \;=\; \frac{2}{\sqrt{\pi}}\;\gamma\;[\;\frac{3}{2},\;\frac{t_h^2}{4\sigma^2}\;] \;=\; \frac{2}{\sqrt{\pi}}\;\gamma\;[\;\frac{3}{2},\;\frac{9}{4}\;] \;=\; \frac{2}{\sqrt{\pi}} \times 0.69809 \;=\; 0.7877. \qquad (7.15)$$

The mass content between t_h and $t = \infty$ is given in terms of the upper incomplete gamma function

$$M(t_h, \sigma) \;=\; \int_{t_h}^{\infty}\;\frac{M_o}{(4\pi\sigma^2)^{3/2}}\;e^{-t^2/4\sigma^2}\;(4\pi\;t^2)\;dt \;=\; \frac{2M_o}{\sqrt{\pi}}\;\Gamma\;[\;\frac{3}{2},\;\frac{t_h^2}{4\sigma^2}\;]. \qquad (7.16)$$

The value of the upper incomplete gamma function when $t_h \sim 3.0\;\sigma$ is $\Gamma[\frac{3}{2}, \frac{9}{4}] = 0.188137$. Thus, the relative mass content in the interval between t_h and $t = \infty$ is then given by

$$\frac{M(t_h, \sigma)}{M_o} \ = \ \frac{2}{\sqrt{\pi}} \, \Gamma \, [\, \frac{3}{2}, \, \frac{9}{4} \,] \ = \ \frac{2}{\sqrt{\pi}} \ \times \ 0.188137 \ = \ 0.2129. \tag{7.17}$$

such that $0.2129 + 0.7877 \sim 1.0006$ which is very close to unity as expected since we rounded off the numbers. The sum of the upper and lower incomplete gammas yields the ordinary Euler gamma function

$$\Gamma \, [\, \frac{3}{2}, \, \frac{9}{4} \,] \ + \ \gamma \, [\, \frac{3}{2}, \, \frac{9}{4} \,] \ = \ \Gamma \, [\frac{3}{2}] \ = \ \frac{\sqrt{\pi}}{2}. \tag{7.18}$$

as expected. One can conclude that for measurements made between t_h and $t = \infty$ the fraction of the mass observed is 0.2129, and for measurements made between $t = 0$ and t_h the fraction of the mass observed is 0.7877. It is interesting that these numbers are close to the *reciprocal* values of what is observed in our universe; i.e 0.75 is dark energy and dark matter while 0.25 is what is observed.

If one sets

$$G M_o \sim 2 \, \sigma \ = \ R_{Hubble} \ \Rightarrow \ \frac{M_o}{m_{Planck}} \ = \ \frac{R_{Hubble}}{L_{Planck}} \ \sim \ 10^{61} \quad since \ G = L^2_{Planck}. \tag{7.19}$$

one arrives at a number consistent with the Dirac-Eddington-Weyl large number coincidences $M_o \sim 10^{61} \ m_{Planck} = 10^{80} \ m_{proton}$ since 10^{80} is of the same order of magnitude as the *square* of the ratio (10^{40}) of the Hubble scale and classical electron radius and the square of the ratio (10^{40}) of the electrostatic force between an electron and a proton versus their corresponding gravitational force. In this case, one finds (in natural units $\hbar = c = 1$) that the temporal horizon is given by $t_h = 3 \ \sigma = 1.5 \ t_{Hubble}$ such that the observed universe lies *inside* the temporal horizon because $t_{Hubble} < t_h$.

One may view the cosmological solution described in terms of the Gaussian density (7.10), the mass function (7.12) and the *modified* (non-vacuum) Kantowski-Sachs metric (7.9) as if the mass M_o of the entire universe were smeared over all of its temporal evolution (during its expansion) by means of a Gaussian distribution of width σ. In the extreme case, $\sigma \rightarrow 0$ scenario the density would have been just proportional to a delta function $\frac{\delta(t)}{4\pi t^2}$ as if the entire mass M_o of the universe were concentrated at a point in time : $t = 0$, the moment of the "Big Bang" singularity. When $\sigma \neq 0$ one no longer has a temporal delta function centered at $t = 0$, but instead a delta function which has been *smeared* into a temporal Gaussian distribution of width σ. Such width $\sigma \neq 0$ is related to the temporal horizon and the Hubble scale (today) by the relation $t_h \sim 3.0 \ \sigma = 1.5 \ t_{Hubble}$.

A more general interior spacetime metric is [38]

$$ds^2 \ = \ - \, B(z,t) \ dt^2 \ + \ A(z,t) \ dz^2 \ + \ F(z,t)^2 \ d\Omega^2. \tag{7.20}$$

In the very particular case that $M(t) = M_o = constant$ and $F(z,t) = t^2$, by following the results of [38] and after some algebra one recovers *once again* the Kantowski-Sachs metric that we found earlier in eq-(6.14) by *simpler* means.

To finalize let us discuss the t-relative version of Vaidya's metric. The Vaidya metric is a solution of Einstein's equations with spherical symmetry in the eikonal approximation to a radial flow of unpolarized radiation [42] that describes the temporal evolution of the evaporation process of black holes via the Hawking"s radiation mechanism. It is given by

$$ds^2 \;=\; -\,(1-\frac{2m(v)}{r})\;dv^2 \;+\; 2\;dr\;dv \;+r^2\;d\Omega^2 \tag{7.21}$$

and expressed in terms of the advanced time parameter $v = t + r_*$ associated with ingoing radial flow of radiation, and

$$ds^2 \;=\; -\,(1-\frac{2m(u)}{r})\;du^2 \;-\; 2\;dr\;du \;+r^2\;d\Omega^2, \tag{7.22}$$

expressed in terms of the retarded time parameter $u = t - r_*$ associated with outgoing radial flow of radiation. The Vaidya metrics are time-dependent solutions of Einstein's equations when $m(v)$ is an arbitrary increasing mass function of the advanced time $v = t + r_*$ and $m(u)$ is an arbitrary decreasing mass function of the retarded time $u = t - r_*$ and r_* is the tortoise radial coordinate $r_* = r + 2GM\;ln|\frac{r}{2GM} - 1|$. The location of the horizons associated to the metrics in eqs-(7.21, 7.22) are now *dynamical* and given by $r_h = 2m(v)$ and $r_h = 2m(u)$ respectively.

In the same fashion that the Kantowski-Sachs metric can be obtained from the Schwarzschild metric after the exchange of variables $t \leftrightarrow r$, it is warranted to explore the cosmological implications of seeing the universe as a dynamical black hole by starting with the metric obtained from the Vaidya metric after the exchange of variables $t \leftrightarrow r$

$$ds^2 \;=\; -\,(1-\frac{2m(\tilde v)}{t})\;d\tilde v^2 \;+\; 2\;dt\;d\tilde v \;+t^2\;d\Omega^2 \tag{7.23}$$

and

$$ds^2 \;=\; -\,(1-\frac{2m(\tilde u)}{t})\;d\tilde u^2 \;-\; 2\;dt\;d\tilde u \;+t^2\;d\Omega^2, \tag{7.24}$$

where the t-relatives of the advanced/retarded temporal coordinates and the temporal tortoise coordinate are now given by :

$$\tilde v \;=\; r + t_*, \quad \tilde u \;=\; r - t_*, \quad t_* \;=\; t \;+\; 2GM\;ln\;|\frac{t}{2GM} - 1|. \tag{7.25}$$

For recent work on the role of Entropy and the Universe as a black hole see [43]. The physical implications of the novel cosmological solution eqs-(7.12a, 7.12b) that is well behaved at $t = 0$ and the metric configurations (7.23. 7.24) deserve further investigation.

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