The Dirac Equation in a Gravitational Field

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Einstein's equivalence principle implies that Newton's gravity force has no local objective meaning. It is an inertial force, i.e. a contingent artifact of the covariantly tensor accelerating (*non-zero g's*) Local Non-Inertial Frame (LNIF[i]) detector.[ii] Indeed, Newton's gravity force disappears in a locally coincident non-accelerating (*zero g*) Local Inertial Frame (LIF[iii]). The presence or absence of tensor spacetime curvature is completely irrelevant to this fact. In the case of an extended test body, these remarks apply only to the Center of Mass (COM). Stresses across separated parts of the test body caused by the local objective tensor curvature are a logically independent separate issue. Garbling this distinction has generated not-even-wrong critiques of the equivalence principle among "philosophers of physics" and even among some venerable confused theoretical physicists. Non-standard terms coupling the spin-connection to the commutator of the Dirac matrices and to the Lorentz group Lie algebra generators are conjectured.

The 16 tetrad $e^{\mu(LNIF)}_{I(LIF)}(x)$ & $e^{I(LIF)}_{\mu(LNIF)}(x)$ and 24 spin connection $\omega^{IJ(LIF)}_{\mu(LNIF)}(x) = -\omega^{JI(LIF)}_{\mu(LNIF)}$ *coefficient* fields describe the transformations between locally coincident LIFs and LNIFs. The tetrads describe translational covariant tensor accelerations on massive test particles pushed off the zero g-force time-like geodesics of the symmetric torsion-free Levi-Civita connection. The spin-connections describe ordinary 3D space rotations of the LNIF about its center of mass as well as the space-time rotations (Lorentz boosts).[iv]

Neglecting the electro-weak-strong internal symmetry local gauge forces at first, the transformation of partial derivatives proceeds as

$$\partial_\mu = e^I_\mu(x)\partial_I + \omega^{IJ}_\mu L_{IJ} \tag{1.1}$$

where ∂_I & $L_{IJ} = -L_{JI}$ are the 10 elements of the Lie algebra of the Poincare group that generate the 4 space-time translations and the 6 space-time rotations. The inverse is

$$e^\mu_I(x)\left(\partial_\mu - \omega^{JK}_\mu(x)L_{JK}\right) = \partial_I \tag{1.2}$$

Imagine an LIF with a small displacement 4-vector δx_J from its origin. "Small" means relative to any local tensor curvature radii fields $\sqrt{R_{IJKL}(x)}^{-1}$ that may or may not be there and then. Note that "x" is the "origin" i.e. event at which the small detector is located. Then

$$L_{JK} \equiv \delta x_J \partial_K - \delta x_K \partial_J \equiv \delta x_{[J} \partial_{K]} \tag{1.3}$$

The Lagrangian for a neutral Dirac spinor second quantized free field on a spacelike slice of 4D globally flat Minkowski spacetime *in the absence of gravity* is

$$\begin{aligned} L(t) &= \iiint d^3 x L(x) \\ L(x) &= -\bar{\psi}(x)\left(m + \gamma^I \partial_I\right)\psi(x) \end{aligned} \tag{1.4}$$

The global dynamical action in a finite flat spacetime region is

$$S = \int dt L(t) \tag{1.5}$$

The Feynman quantum amplitude for that particular classical field configuration history is symbolically

$$A \approx e^{i\frac{S}{\hbar}} \tag{1.6}$$

We cannot do these global integrals in curved spacetime so easily because of the path-dependent anholonomy analogous to history dependent irreversibility in non-equilibrium thermodynamics where we no longer have state functions (e.g. exact Cartan 1-forms). Similarly, as in classical mechanics of particles when we no longer have a simple static potential, but also have velocity-dependent non-central forces. However, we can look at the local terms.

$$\gamma^\mu(x) = e_I^\mu(x)\gamma^I + \omega_{IJ}^\mu\left[\gamma^I, \gamma^J\right] \tag{1.7}$$

The second spin-connection term on the RHS is non-standard and is only an empirical conjecture that is Popper falsifiable. It is also a new torsion field coupling if torsion fields are present.

The Ansatz is then

$$\begin{aligned} &\gamma^I \partial_I \to \gamma^\mu \partial_\mu \\ &= \left(e_I^\mu(x)\gamma^I + \omega_{IJ}^\mu\left[\gamma^I, \gamma^J\right]\right)\left(e_\mu^{I'}(x)\partial_{I'} + \omega_\mu^{I'J'} L_{I'J'}\right) \\ &\neq \gamma^I \partial_I \end{aligned} \tag{1.8}$$

However, Einstein's local frame differential space-time interval $ds^2(x)$ is invariant under $LIF(x) \Leftrightarrow LNIF(x)$

$$
\begin{aligned}
&ds^2(x) = g_{\mu\nu(LNIF)}(x)e^{\mu}(x)e^{\nu}(x) = \eta_{IJ(LIF)}e^{I}(x)e^{J}(x) \\
&e^{\nu}(x)_{(LNIF)} = e_I^{\nu}(x)e^{(I)}(x)_{(LIF)} \\
&e_I^{\nu}(x) \equiv \frac{\partial x^{\nu}}{\partial x^{I}}(x)
\end{aligned}
\tag{1.9}
$$

where, again, "x" is the *local coincidence* of two detectors close to each other compared to the local radii of curvature if there is any. The $e's$ are local frame (co) vectors forming a "basis" in the sense of the algebra of vector spaces.

The important gravitational "acceleration" local T4(x) gauge potential is $A_{\mu}^{I}(x)$ where

$$
e_{\mu}^{I}(x) = \delta_{\mu}^{I} + A_{\mu}^{I}(x) \tag{1.10}
$$

For a simple example of a non-relativistic uniformly *slowly* rotating non-inertial frame about the z axis with ignorable time dilation and tangential length contraction, i.e., $\omega x'/c \ll 1$ etc.

$$
\begin{aligned}
&z' \equiv z_{(LIF)} = z_{(LNIF)} \equiv z \\
&t' \equiv t_{(LIF)} = t_{(LNIF)} \equiv t \\
&etc. \\
&x' = x\cos\omega t + y\sin\omega t \\
&y' = -x\sin\omega t + y\cos\omega t \\
&e_x^{x'} \equiv \frac{\partial x'}{\partial x} = \cos\omega t \to A_x^{x'} = \cos\omega t - 1 \\
&\underset{\omega\to 0}{Lim}\, A_x^{x'} = 0 \\
&e_{ct}^{x'} \equiv \frac{1}{c}\frac{\partial x'}{\partial t} = A_{ct}^{x'} = \frac{\omega}{c}(-x\sin\omega t + y\cos\omega t) \\
&\underset{\omega\to 0}{Lim}\, A_{ct}^{x'} = 0 \\
&etc.
\end{aligned}
\tag{1.11}
$$

$$
\begin{aligned}
&ds^2 = \eta_{IJ(LIF)}dx^{I}dx^{J} = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2 \\
&= g_{\mu\nu(LNIF)}dx^{\mu}dx^{\nu}
\end{aligned}
\tag{1.12}
$$

Each *locally coincident* frame carries its own non-intrinsic *historically contingent* representation $g_{\mu\nu(LNIF)}(x)$ & $\eta_{IJ(LIF)}$ of the metric tensor field. Only the locally variable spacetime interval $ds^2(x,x+dx) \equiv ds^2(x)$ has objective invariant physical meaning.

$$
g_{\mu\nu(LNIF)} \approx \eta_{\mu\nu} + h_{\mu\nu} \tag{1.13}
$$

$$h_{\mu\nu(rotatingLNIF)} = \begin{matrix} -\frac{\omega^2\rho^2}{c^2} & \frac{\omega\rho\sin\omega t}{c} & -\frac{\omega\rho\cos\omega t}{c} & 0 \\ \frac{\omega y}{c} & 0 & 0 & 0 \\ -\frac{\omega x}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \tag{1.14}$$

$$\rho = \sqrt{x^2 + y^2}$$
$$x = \rho\cos\omega t$$
$$y = \rho\sin\omega t$$

Einstein's *local equivalence principle* is mathematically expressed in terms of the non-tensor Levi-Civita connection that describes the contingent inertial g-forces felt on massive structures that are pushed off timelike[v] geodesics by non-gravity forces.

$$\begin{aligned} \Gamma_{IJK(LIF)} &= \frac{1}{2}\left(\partial_J g_{IK} + \partial_K g_{IJ} - \partial_I g_{JK}\right) = 0 \\ \Gamma_{\mu\nu\lambda(LNIF)} &= \frac{1}{2}\left(\partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\mu g_{\nu\lambda}\right) \neq 0 \end{aligned} \tag{1.15}$$

The presence or absence of curvature is completely irrelevant to the validity of the equivalence principle and some physicists are very confused on this issue. The curvature is measured in a completely different way from the measurement of the connection field even though the curvature is the *covariant* "curl" of the connection field. One uses pairs of closely spaced freely falling test particles each on timelike geodesics to measure the components of the curvature tensor. All inertial g-forces are eliminated in the curvature measurement, whereas the connection field measurement is that of inertial g-forces!

For example, in the above slowly rotating frame

$$\begin{aligned} \Gamma_{\mu\nu\lambda(LNIF)} &= \frac{1}{2}\left(\partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\mu g_{\nu\lambda}\right) \\ \Gamma_{00}^{i=x,y,z} &\approx \frac{1}{2}\eta^{i\lambda}\left(2\partial_0 h_{0\lambda} - \partial_\lambda h_{00}\right) \\ \Gamma_{00}^{x} &= \frac{1}{2}\eta^{xx}\left(2\partial_0 h_{0x} - \partial_x h_{00}\right) = -\frac{1}{2}\left(2\frac{\omega}{c^2}\partial_t \rho\sin\omega t + \partial_x \frac{\omega^2\left(x^2+y^2\right)}{c^2}\right) \\ &= -\left(\frac{2\omega^2 x}{c^2}\right) \end{aligned} \tag{1.16}$$

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Similarly, $\Gamma_{00}^{y} = -\frac{2\omega^2 y}{c^2}$. Therefore transforming this Cartesian representation to cylindrical coordinates gives $c^2\Gamma_{00}^{\rho}/2 = -\omega^2\rho$ the radially inward centripetal acceleration inertial g-force locally equivalent to Newton's gravity force. Imagine a rigid rotating cylinder. The electrical forces of the cylindrical wall provide the push off the timeline geodesic that you feel as "weight," e.g. artificial gravity on a rotating wheel space station.

$$
\begin{aligned}
&\left(e_I^{\mu}(x)\gamma^I + \omega_{IJ}^{\mu}\left[\gamma^I,\gamma^J\right]\right)\left(e_{\mu}^{I'}(x)\partial_{I'} + \omega_{\mu}^{I'J'}L_{I'J'}\right)\\
&= e_I^{\mu}(x)e_{\mu}^{I'}(x)\gamma^I\partial_{I'} + e_I^{\mu}(x)\gamma^I\omega_{\mu}^{I'J'}L_{I'J'} + \omega_{IJ}^{\mu}\left[\gamma^I,\gamma^J\right]e_{\mu}^{I'}(x)\partial_{I'} + \omega_{IJ}^{\mu}\left[\gamma^I,\gamma^J\right]\omega_{\mu}^{I'J'}L_{I'J'} \quad (1.17)\\
&= \gamma^I\partial_I + e_I^{\mu}(x)\gamma^I\omega_{\mu}^{I'J'}L_{I'J'} + \omega_{IJ}^{\mu}\left[\gamma^I,\gamma^J\right]e_{\mu}^{I'}(x)\partial_{I'} + \omega_{IJ}^{\mu}\left[\gamma^I,\gamma^J\right]\omega_{\mu}^{I'J'}L_{I'J'}
\end{aligned}
$$

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[i] LNIFs have indices $\mu,\nu,\lambda...$ raised and lowered with the locally variable curvilinear metric tensor field $g_{\mu\nu}(x)$ etc.

[ii] What is taught as Newton's gravity force in elementary physics is simply the universal covariant acceleration of a static LNIF detector in Spherically Symmetric Static curved spacetime.

[iii] LIFs have indices $I,J,K...$ that are raised and lowered with the SR constant Minkowski metric tensor $\eta_{IJ},\eta_I^J,\eta^{IJ}$

[iv] The Levi-Civita connection is mistakenly compared to a local gauge field by some physicists. This leads to the issue of the non-renormalizability of canonical top → down quantum gravity, e.g. starting with classical ADM formalism of lapse, shift and 3D geometry dynamical variables. The real gauge potentials are the tetrads and spin connections. The tetrads come from localizing the global 4-parameter translation sub-group T4 →T4(x) of the globally rigid 10-parameter Poincare universal spacetime symmetry group of Einstein's 1905 Special Relativity. If we stop there, we get precisely Einstein's 1915 General Relativity (GR) with curvature equivalent to disclination defects in a 4D Lorentzian world crystal lattice (Hagen Kleinert) without torsion gaps in the infinitesimal parallelograms where one parallel transports one edge by the other and compares the relative rotation of a third non-planar vector around the parallelogram closed loop. The angle of rotation (disclination) is the area of the parallelogram *multiplied by the sectional curvature in the limit that the area shrinks to zero*. In that case, the spin connections are dynamically redundant determined completely by the tetrads and their first partial derivatives in complicated antisymmetric combinations shown as Rovelli's eq. 2.89 in his book "Quantum Gravity".

$$\omega[e]_\mu^{IJ} = 2\,e^{\nu[I}\partial_{[\mu}e_{\nu]}{}^{J]} + e_{\mu K}e^{\nu I}e^{\sigma J}\partial_{[\sigma}e_{\nu]}{}^K . \qquad (2.89)$$

If we localize the full Poincare group we also get a new dynamical torsion gap "dislocation" tensor field in addition to the curvature "disclination" tensor field.
"A **disclination** is a line defect in which rotational symmetry is violated. In analogy with dislocations in crystals, the term, *disinclination*, for liquid crystals first used by F. C. Frank and since then has been modified to its current usage, *disclination*. It is a defect in the orientation of director whereas a dislocation is a defect in positional order." http://en.wikipedia.org/wiki/Disclination
We can continue, localizing the dilatation group and the four conformal boosts to uniform hyperbolic motion in SR to get a non-metricity tensor field. The Levi-Civita non-tensor connection field then gets additional tensor pieces. The term "tensor" is always relative to a group of physical local frame transformations. Unless otherwise stated, "tensor" always means local LNIF → LNIF' among covariantly accelerating non-inertial frames. Formal coordinate transformations in a fixed local frame are an equivalence class and are gauged away. The LNIF is an equivalence class of formal curvilinear transformations that are simply different descriptions of the same material detector.

[v] Timelike means inside the local light cone. Lightlike means on the local light cone. Spacelike means outside the local light cone. Each light cone has a future and past piece. Advanced Wheeler-Feynman electromagnetic *spherical* Huygens waves from the point origin of the light cone propagate expanding fronts backward in time to the past with

positive energy along the past light cone with phase $\theta_{adv} = kr + \omega t$. Retarded spherical waves propagate expanding fronts forward in time with positive energy along the future light cone with phase $\theta_{ret} = kr - \omega t$. The spherical wave form in general is
$\psi_{spherical} \equiv Ae^{i\theta} / r$