Adv. Studies Theor. Phys., Vol. 2, 2008, no. 17, 825 - 835

Gauge Theories of Kac-Moody Extensions of W_∞ Algebras as Effective Field Theories of Colored W_∞ Strings

Carlos Castro

Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, GA 30314, USA
castro@ctps.cau.edu

Abstract

A novel invariant gauge field theory realization of Kac-Moody extensions of $w_\infty(w_{1+\infty})$ algebras and based on a Lie group G is constructed. The most relevant physical feature of this theory is that it describes an effective field theory of "colored" internal $w_\infty(w_{1+\infty})$ strings when $G = SU(3)$. We conclude with a discussion of how these theories might provide infinite higher conformal-spins extensions of Grand Unified Models and the Standard Model in four dimensions.

Keywords: W_∞ Gauge Theories, Kac-Moody Algebras, Grand Unification, Higher spins, Strings, Membranes, Matrix Models

Sometime ago, a gauge theory of the Virasoro-Kac-Moody symmetry associated with an arbitrary grand-unified gauge group G was constructed by [1] that could be interpreted as an effective field theory of a coloured internal string. Such theory was the Kac-Moody extension of the gauge theory of the Virasoro group constructed earlier by [2] and which could be seen as a gauge theory of an internal string. The former theory automatically (geometrically and without the ad-hoc introduction of Higgs fields) breaks the symmetry down to $H \otimes U(1)$, where H is a subgroup of G and $U(1)$ is the Cartan subgroup of the Virasoro group. The symmetry breaking is what guarantees the *unitarity* of the theory since the adjoint representation of the Virasoro group is not unitary. Later on, a gauge theory of the w_∞ algebra (a higher conformal spin extension of the Virasoro algebra) was constructed by [3] and it was followed by a gauge theory of a diffeomorphism *subgroup* of the torus membrane and whose adjoint representation was unitary [5] .

It was shown recently [6] how $w_\infty, w_{1+\infty}$ gauge field theory actions in $2D$ emerge directly from an Einstein-Hilbert $4D$ Gravitational action. Strings and Membranes actions in $2D$ and $3D$ originated as well from $4D$ Einstein Gravity after recurring to the *nonlinear* connection formalism of Lagrange-Finsler and Hamilton-Cartan spaces [21]. We argued why quantum gravity in $3D$ can be described by a W_∞ Matrix Model in $D = 1$ that can be solved *exactly* via the collective field theory method [7], and why a quantization of $4D$ Gravity could be attained via a $2D$ Quantum W_∞ gauge theory coupled to an infinite-component scalar field multiplet belonging to the infinite-dim representation $V_{\alpha,\beta}$ (for $\alpha = -1/2, \beta = 0$) of the w_∞ group constructed by Feigin-Fuks-Kaplansky (FFK) [4].

Our results [6] were based on [8] where it have shown that $m+n$-dimensional Einstein gravity can be identified with an m-dimensional generally invariant gauge theory of $Diffs\ N$ (where N is an n-dim internal manifold) and coupled to a non-linear sigma scalar field whose self interaction potential term is related to the gauged Ricci scalar curvature of the internal manifold. When the internal manifold $\mathcal{N}$ is a homogeneous compact space one can perform a harmonic expansion of the fields w.r.t the internal y coordinates, and after integrating w.r.t these internal y coordinates, one will generate an infinite-component field theory on the m-dimensional space. A reduction of the Diffs $\mathcal{N}$, via the inner automorphims of a subgroup G of the Diffs $\mathcal{N}$, yields the usual Einstein-Yang-Mills theory interacting with a nonlinear sigma field. But in general, the theory described in [8] is by far *richer* than the latter theory.

It was found in [18] that the $D = m + n$ dimensional gravitational action restricted to $AdS_m \times S^n$ backgrounds admits a *holographic* reduction to a lower $d = m$-dimensional Yang-Mills-like gauge theory of diffs of S^n, interacting with a charged/gauged nonlinear sigma model plus boundary terms, by a simple tuning of the radius of S^n and the size of the throat of the AdS_m space. Namely, in the case of $AdS_5 \times S^5$, the holographic reduction occurs if, and only if, the size of the AdS_5 throat *coincides* precisely with the radius of S^5 ensuring a *cancellation* of the scalar curvature $g^{\mu\nu} R^{(m)}_{\mu\nu}$ of AdS_5 with the scalar curvature $g^{ab} R^{(n)}_{ab}$ of the internal S^5 space.

Zamolodchikov [9] was the first to pioneer the theory of higher conformal spin algebras w_N, $N = 2, 3, 4, ...$, in $2D$ that are the higher conformal spin extensions of the Virasoro algebra (w_2) that arise in various physical systems as in $2D$ quantum gravity, the quantum Hall effect, the membrane, the large N QCD, gravitational instantons, topological QFT, etc.... see [11] for an extensive review and references. The $w_{1+\infty}$ algebra is isomorphic to the area-preserving diffs algebra of the cylinder $S^1 \times R^1$. The w_∞ algebra is the area-preserving diffs algebra of the two-dim plane and is comprised of higher spin generators whose conformal spin range is $s = 2, 3, 4, ...$ and it is a subalgebra of $w_{1+\infty}$. For an extensive list of references on w_∞ algebras, w_∞ gravity, extended conformal field theories and their vast applications in physics see [10], [12], [13], [17].

The Kac-Moody extension of the $w_\infty(w_{1+\infty})$-algebra is defined by the relations

$$[L_{\vec{m}}\ ,\ T_{a,\vec{n}}]\ =\ -[\ (m_2+1)n_1 - m_1(n_2+1)\ ]\ T_{a,\vec{m}+\vec{n}}. \tag{1}$$

$$[L_{\vec{m}}\ ,\ L_{\vec{n}}]\ =\ [\ (n_2+1)m_1-(m_2+1)n_1\ ]\ L_{\vec{m}+\vec{n}} + \frac{c}{12}\,(m_1^3-m_1)\delta^{m_2,0}\ \delta^{n_2,0}\ \delta_{m_1+n_1,0}. \tag{2}$$

$$[T_{a,\vec{m}}\ ,\ T_{b,\vec{n}}]\ =\ f_{ab}^c\ T_{c,\vec{m}+\vec{n}}\ +\ \frac{\kappa}{16}\ m_1\ \delta^{m_2+1,0}\ \delta^{n_2+1,0}\ \delta_{m_1+n_1,0}. \tag{3}$$

c is the central charge of the $w_\infty(w_{1+\infty})$ algebra and κ is the level of the Kac-Moody extension. For a $SU(N)$ Kac-Moody extension of the w_∞ algebra, the central charge c and the level κ of the Kac-Moody algebra are related as $c = N\kappa$ by virtue of the Jacobi identities [12]. For the time being we will focus in the case that $c = \kappa = 0$. The indices a, b, c of the Kac-Moody extension are the Lie algebra $\mathbf{g}$ indices ranging from $1, 2, 3, ..., dim\ \mathbf{g}$ where $\mathbf{g}$ is the corresponding Lie algebra associated with the group G . The coefficients f^{abc} are the structure constants of the Lie algebra $\mathbf{g}$. The generators of the $SU(N)$ Kac-Moody extension of the area-preserving diffs algebra of a cylinder $S^1 \times R$ (in the centerless $c = \kappa = 0$ case) can be represented as

$$V_m^l\ =\ -i\ e^{im\theta}\ y^l\ [-im\ y\ \partial_y\ + (l+1)\ \partial_\theta];\quad T_m^{l,a}\ =\ -i\tau^a\ y^{l+1}\ e^{im\theta}. \tag{4}$$

where τ^a are the $N^2 - 1$ generators of $SU(N)$.

The range of the 2-dim lattice *vector* indices in eqs-(1-3)

$$\vec{m} = (m_1, m_2), \quad \vec{n} = (n_1, n_2). \tag{5a}$$

is given by

$$-\infty\ \leq\ m_1\ \leq\ \infty;\ -\infty\ \leq\ n_1\ \leq\ \infty;\quad m_2\ \geq\ s-2;\quad n_2\ \geq\ s-2; \tag{5b}$$

The conformal (internal) $su(1, 1)$ spin s associated with the (internal) $2D$ higher conformal spin generators $v_{m_1}^{m_2}, v_{n_1}^{n_2}....$ of the $w_\infty(w_{1+\infty})$ gauge algebra is represented by the indices $m_2, n_2, ...$ such that the conformal spin s is given by $s = m_2+2,\ s = n_2+2, ...$. The range of conformal spin values associated with the w_∞ algebra is $s = 2, 3, ...\infty$. The $w_{1+\infty}$ algebra conformal spin ranges from $s = 1, 2, 3, 4, ...\infty$. Whereas the indices $m_1, n_1, ...$ label the infinite number of Fourier modes associated with each single one of the conformal spin-s generators. The Virasoro algebra corresponds to the conformal spin-2 generator and can be denoted as the w_2 algebra.

Let $\phi^{\vec{k}}$ be a Hermitian scalar field $(\phi^{\vec{k}})^* = \phi^{-\vec{k}}$ and belonging to the $(\alpha = -1, \beta = 0)$ FFK representation of the $w_\infty(w_{1+\infty})$-group. Let $A_\mu^{\vec{k}}$ be a Hermitian gauge potential $(A_\mu^{\vec{k}})^* = A_\mu^{-\vec{k}}$ belonging to the $(\alpha = 1, \beta = 0)$ adjoint FFK representation. An invariant Lagrangian in a $4D$ spacetime is

$$\mathcal{L}_1 \ = \ \sum_{\vec{j},\vec{k}} \ - \frac{1}{4\rho^3} \ (\phi^3)^{-\vec{j}-\vec{k}} \ F_{\mu\nu}^{\vec{j}} \ F_{\mu\nu}^{\vec{k}} \ - \ \frac{\rho}{2} \ (\frac{1}{\phi})^{-\vec{j}-\vec{k}} \ (D_\mu \phi^{\vec{j}}) \ (D_\mu \phi^{\vec{k}}). \tag{6}$$

where μ, ν are the $4D$ spacetime indices and $\vec{j}, \vec{k}$ are the internal $2D$ lattice indices associated with the infinite-dim $w_\infty(w_{1+\infty})$ algebra. A mass parameter ρ is required in eq-(6) to render the $4D$ action dimensionless. The inverse of the scalar field is defined in terms of the norm-squared as

$$(\phi^{-1})^{\vec{k}} \ \equiv \ \frac{\phi^{\vec{k}}}{||\phi||^2}; \quad ||\phi||^2 \ \equiv \ g_{\vec{m}\vec{n}} \ \phi^{\vec{m}} \phi^{\vec{n}} \ = \ \sum_{\vec{m}} \ \phi_{\vec{m}} \ \phi^{\vec{m}} \ = \ \sum_{\vec{m}} \ \phi^{-\vec{m}} \ \phi^{\vec{m}}. \tag{7}$$

the invariant tensor that allows to lower indices is $g_{\vec{m}\vec{n}} = \delta^0_{\vec{m}+\vec{n}}$. The n-th power of the scalar field $\phi^{\vec{k}}$ belonging to a (α, β) FFK representation of the $w_\infty(w_{1+\infty})$ group is given by

$$(\phi^n)^{\vec{k}} \ \equiv \ \sum_{\vec{k}_1+\vec{k}_2+.....+\vec{k}_n=\vec{k}} \ \phi^{\vec{k}_1} \ \phi^{\vec{k}_2} \ .... \ \phi^{\vec{k}_n}. \tag{8}$$

and its weight is $(n\alpha, n\beta)$. The $F_{\mu\nu}^{\vec{k}}$ field strength associated with the Hermitian gauge potential $A_\mu^{\vec{k}}$ belonging to the adjoint $(\alpha = 1, \beta = 0)$ FFK representation of the $w_\infty(w_{1+\infty})$ group is defined as

$$F_{\mu\nu}^{\vec{k}} \ = \ \partial_\mu A_\nu^{\vec{k}} - \partial_\nu A_\mu^{\vec{k}} + ie \, [\, (m_2+1)(2m_1-k_1) - m_1(2m_2-k_2) \,] \ A_\mu^{\vec{m}} A_\nu^{\vec{k}-\vec{m}} \, . \tag{9}$$

The covariant derivative of a hermitian scalar multiplet $\phi^{\vec{k}}$ belonging to the $(\alpha = -1, \beta = 0)$ FFK representation of the $w_\infty(w_{1+\infty})$ group is

$$D_\mu \phi^{\vec{k}} \ = \ \partial_\mu \phi^{\vec{k}} \ + ie \ [\ m_1 k_2 - (m_2+1) k_1 \ ] \ A_\mu^{\vec{m}} \ \phi^{\vec{k}-\vec{m}}. \tag{10}$$

The Kac-Moody extension of the invariant Lagrangian $\mathcal{L}_1$ is more subtle. The full-fledged $w_\infty(w_{1+\infty})$-Kac-Moody *covariantized* $\mathcal{F}_{\mu\nu}^{c\vec{k}}$ field strength is defined in terms of the $w_\infty(w_{1+\infty})$ field strength $F_{\mu\nu}^{\vec{k}}$ given above by eq-(9), and the $w_\infty(w_{1+\infty})$-Kac-Moody field strength $F_{\mu\nu}^{c\vec{k}}$ as follows

$$\mathcal{F}_{\mu\nu}^{c\vec{k}} \ = \ F_{\mu\nu}^{c\vec{k}} \ + \frac{e}{\rho} \ F_{\mu\nu}^{\vec{m}} \ \phi^{c\vec{k}-\vec{m}}. \tag{11}$$

where the Kac-Moody field strength $F_{\mu\nu}^{c\vec{k}}$ in the r.h.s of eq-(11) is given in terms of the gauge fields $A_\mu^{\vec{k}}, \ A_\nu^{c\vec{k}}$ as

$$F_{\mu\nu}^{c\vec{k}} \ = \ \partial_\mu A_\nu^{c\vec{k}} \ - \partial_\nu A_\mu^{c\vec{k}} \ + \ g f_{ab}^c \ A_\mu^{a\vec{m}} \ A_\nu^{b\vec{k}-\vec{m}} \ +$$

$$ie\ [\ m_1(k_2+1)-(m_2+1)k_1\ ]\ (\ A_\mu^{\vec{m}} A_\nu^{c\vec{k}-\vec{m}}\ - A_\nu^{\vec{m}} A_\mu^{c\vec{k}-\vec{m}}\ ). \qquad (12)$$

Notice that the presence of the $w_\infty(w_{1+\infty})$-Kac-Moody Hermitian scalar multiplet $\phi^{c\vec{k}}$ in eq- (11) is also required in the definition of $\mathcal{F}_{\mu\nu}^{c\vec{k}}$ and it ensures that $\mathcal{F}_{\mu\nu}^{c\vec{k}}$ belongs to a $(\alpha = 0, \beta = 0)$ FFK representation. It is the full-fledged $w_\infty(w_{1+\infty})$-Kac-Moody *covariantized* $\mathcal{F}_{\mu\nu}^{c\vec{j}}$ field strength which transforms *covariantly* under the action of the $w_\infty(w_{1+\infty})$ group as a $(\alpha = 0, \beta = 0)$ FFK representation

$$\delta_W\ \mathcal{F}_{\mu\nu}^{c\vec{k}}\ =\ i\ \{\ (m_2+1)\ [(\alpha+1)m_1+\beta-k_1]-m_1\ [(\alpha+1)m_2-k_2]\ \}\ \xi^{\vec{m}}\ \mathcal{F}_{\mu\nu}^{c\vec{k}-\vec{m}}\ =$$

$$i\ [\ m_1(k_2+1)-(m_2+1)k_1\ ]\ \ \xi^{\vec{m}}\ \mathcal{F}_{\mu\nu}^{c\vec{k}-\vec{m}} \qquad (13)$$

Therefore, an invariant Lagrangian (in a $4D$ spacetime) under the full-fledged action action of the $w_\infty(w_{1+\infty})$-Kac-Moody group is then given by

$$\mathcal{L}_2\ =\ \sum_{\vec{j},\vec{k}}\ -\frac{1}{4\rho}\ (\phi)^{-\vec{j}-\vec{k}}\ \mathcal{F}_{\mu\nu}^{c\vec{j}}\ \mathcal{F}_{\mu\nu}^{c\vec{k}}\ -\ \frac{\rho}{2}\ (\frac{1}{\phi})^{-\vec{j}-\vec{k}}\ (\mathcal{D}_\mu \phi^{c\vec{j}})\ (\mathcal{D}_\mu \phi^{c\vec{k}}). \qquad (14)$$

The full-flegded $w_\infty(w_{1+\infty})$-Kac-Moody *covariantized* derivative $\mathcal{D}_\mu \phi^{c\vec{k}}$ is defined

$$\mathcal{D}_\mu \phi^{c\vec{k}}\ =\ \partial_\mu \phi^{c\vec{k}}\ + g f_{ab}^c\ A_\mu^{a\vec{m}}\ \phi^{b\vec{k}-\vec{m}}\ + ie\ [\ m_1 k_2-(m_2+1)k_1\ ]\ A_\mu^{\vec{m}}\ \phi^{c\vec{k}-\vec{m}}. \quad (15)$$

Under infinitesimal gauge transformations of the Kac-Moody algebra associated with the infinitesimal parameter $\xi^{a\vec{m}}$ one has

$$\delta_{KM}\ \mathcal{F}_{\mu\nu}^{c\vec{k}}\ =\ f_{ab}^c\ \xi^{a\vec{m}}\ \mathcal{F}_{\mu\nu}^{b\vec{k}-\vec{m}};\quad \delta_{KM}\ \mathcal{D}_\mu \phi^{c\vec{k}}\ =\ f_{ab}^c\ \xi^{a\vec{m}}\ \mathcal{D}_\mu \phi^{b\vec{k}-\vec{m}}. \qquad (16)$$

Under the infinitesimal action of the $w_\infty(w_{1+\infty})$ algebra associated with the infinitesimal parameter $\xi^{\vec{m}}$ one has

$$\delta_W\ \mathcal{D}_\mu \phi^{c\vec{k}}\ =\ i\ [\ m_1 k_2-(m_2+1)k_1\ ]\ \xi^{\vec{m}}\ \mathcal{D}_\mu \phi^{c\vec{k}-\vec{m}}. \qquad (17a)$$

$$\delta_W\ \mathcal{F}_{\mu\nu}^{c\vec{k}}\ =\ i\ [\ m_1(k_2+1)\ \ -\ (m_2+1)k_1\ ]\ \xi^{\vec{m}}\ \mathcal{F}_{\mu\nu}^{c\vec{k}-\vec{m}}. \qquad (17b)$$

Under the combined action of the $w_\infty(w_{1+\infty})$-Kac-Moody algebra one has

$$\delta_{WKM}\ \mathcal{D}_\mu \phi^{c\vec{k}}\ =\ i\ [\ m_1 k_2\ -(m_2+1)k_1\ ]\ \xi^{\vec{m}}\ \mathcal{D}_\mu \phi^{c\vec{k}-\vec{m}} + f_{ab}^c\ \xi^{a\vec{m}}\ \mathcal{D}_\mu \phi^{b\vec{k}-\vec{m}}. \quad (18)$$

$$\delta_{WKM}\,\mathcal{F}^{c\vec{k}}_{\mu\nu} \;=\; i\,[\,m_1(k_2{+}1) \;-\; (m_2{+}1)k_1\,]\,\xi^{\vec{m}}\,\mathcal{F}^{c\vec{k}-\vec{m}}_{\mu\nu} \;+\; f^c_{ab}\,\xi^{a\vec{m}}\,\mathcal{F}^{b\vec{k}-\vec{m}}_{\mu\nu} \quad (19)$$

The gauge invariance of $\mathcal{L}_1, \mathcal{L}_2$ follows from the fact that each term in the Lagrangians forms a scalar product

$$< \chi \mid \phi > \;=\; (\chi^{\vec{k}})^*\,\phi^{\vec{k}} \;=\; (\chi^{-\vec{k}})\,\phi^{\vec{k}}. \quad (20)$$

which is invariant under the gauge transformations. In order to write invariant actions based on a scalar product the weights must obey $\alpha^* + \alpha + 1 = 0$ and $\beta^* - \beta = 0$ where α^*, β^* are the weights of the *dual* representation $V^*_{\alpha,\beta} = V_{-1-\alpha,-\beta}$. This is the case of each term in $\mathcal{L}_1, \mathcal{L}_2$. The Lagrangian is real-valued (invariant under charge conjugation) as a result of the Hermiticity of the field strengths and covariant derivatives, and does not contain any Higgs-type potential for the scalar fields. Nevertheless, despite the absence of an ad-hoc Higgs type potential as the authors [1] explained for the Virasoro (w_2) Kac-Moody algebra case, the vacuum expectation values $< \phi^k >= \rho\delta^k_0$ and $< \phi^{c,k} >= \rho^c\delta^k_0$ (the indices c span over the dimension of the Lie algebra $\mathbf{g}$ associated with the group G) lead to a symmetry breaking down to $H \otimes U(1)$, where H is a subgroup of G and $U(1)$ is the Cartan subgroup of the Virasoro group. This $w_\infty(w_{1+\infty})$-Kac-Moody gauge theory is a non-linearly realized gauge theory by virtue of the relation (11) which establishes a *nonlinear* relation among the *covariantized* field strength $\mathcal{F}^{c\vec{k}}_{\mu\nu}$ and $\mathcal{F}^{\vec{k}}_{\mu\nu}, \phi^{c\vec{k}}$. For instance, the Lagrangians of eqs-(6, 14) explicitly furnish *nonlinear* equations of motion for the scalar fields $\phi^{\vec{k}}, \phi^{c\vec{k}}$. As emphasized by [1] this type of gauge theories $differ$ from the standard Callan-Coleman-Wess-Zumino non-linear field realization. Despite that the theory in [1] is made of non-unitary FFK representations of the Virasoro group it, nevertheless, has a positive-definite Hamiltonian resulting from the fact that after the symmetry breaking down to the Cartan subgroup $U(1)$ all physical fields form unitary representations of the unbroken subgroup.

The symmetry breaking process in the $w_\infty(w_{1+\infty})$-Kac-Moody case is far more complex . It remains to be studied the unitarity of the theories associated with the unbroken subgroups. For instance, a symmetry breaking mechanism of $w_\infty(w_{1+\infty})$ down to the Virasoro algebra (w_2) should lead to an infinite collection of massive higher spin fields. It has ben speculated that the infinite number of massive Virasoro-string states lying along a Regge trajectory might follow from a symmetry breaking mechanism of the $w_\infty(w_{1+\infty})$ symmetry associated to the infinite number of *massless* higher spin states of $w_\infty(w_{1+\infty})$-strings living in a flat target spacetime background [10], [13]. Among the most salient features of this theory based on the novel Lagrangians $\mathcal{L}_1, \mathcal{L}_2$ is that it *is* a field theory realization of the $w_\infty(w_{1+\infty})$-Kac-Moody algebra which was seen as an unresolved problem a while ago [10]. To sum up, the most relevant physical feature of this work is that the Lagrangian $\mathcal{L}_1 + \mathcal{L}_2$ should describe an effective field theory of "coloured" internal $w_\infty(w_{1+\infty})$ strings when $G = SU(3)$. The large $N \to \infty$ limit of the $SU(N)$ extension of w_∞ algebras were studied by [12] and correspond to area-preserving (symplectic) diffs in $four$ dimensions.

$$[V_m^{l,\vec{k}} \ , \ V_n^{j,\vec{l}}] \ = \ [\ (j+1)m - (l+1)n\ ]\ V_{m+n}^{l+j,\vec{k}+\vec{l}} \ + \ [\ k_1 l_2 - k_2 l_1\ ]\ V_{m+n}^{l+j+1,\vec{k}+\vec{l}}. \quad (21)$$

To finalize we will discuss how to build Lagrangians corresponding to higher conformal-spin extensions of Grand Unified Models and the Standard Model. The $w_\infty, w_{1+\infty}$ gauge invariant Lagrangian density in $4D$ was constructed by [3]

$$\mathcal{L} \ = \ \sum_{\vec{i},\vec{j}} \ (\Phi^6)^{-\vec{i}-\vec{j}}\ \mathcal{F}^{\vec{i}}_{\mu\nu}\ \mathcal{F}^{\mu\nu,\vec{j}} + \sum_{\vec{k}}\ (\mathcal{D}_\mu \Phi^{-\vec{k}})\ (\mathcal{D}^\mu \Phi^{\vec{k}}) + V(\Phi^{\vec{k}}). \qquad (22)$$

where we have set the mass scale parameter $\rho = 1$. As usual, the gauge field $A^{\vec{k}}_\mu$ is Hemitian (w.r.t a well defined scalar product) $(A^{\vec{k}}_\mu)^* = A^{-\vec{k}}_\mu = A_{\mu,\vec{k}}$ and belongs to the adjoint representation $V_{\alpha,\beta}$ constructed by Feigin-Fuks-Kaplansky (FFK) [4] with $\alpha = 1, \beta = 0$ as before. However, the scalar field $\Phi^{\vec{k}}$ in (22) is now an infinite-component complex scalar multiplet belonging to the infinite-dim FFK vector representation $V_{\alpha,\beta}$ with $(\alpha = -1/2, \beta = 0)$, instead of belonging to a $(\alpha = -1, \beta = 0)$ FFK representation as before in eqs-(6,14). It is for this reason that a potential term is now *allowed* in (22) because any polynomial comprising powers of the *bilinear* combination given by the scalar product $\Phi^{-\vec{k}}(x)\Phi^{\vec{k}}(x)$ is gauge invariant, due to the fact that each bilinear factor obeys the condition $\alpha^* + \alpha + 1 = 0$ and $\beta^* - \beta = 0$.

The covariant derivative in (22) is now given by

$$(\mathcal{D}_\mu \Phi^{\vec{k}}) \ = \ \partial_\mu \Phi^{\vec{k}} \ + \ ie\ [\ (m_2+1)(\frac{m_1}{2} - k_1) \ - \ (\frac{m_2}{2} - k_2)m_1\ ]\ \mathcal{A}^{\vec{m}}_\mu\ \Phi^{\vec{k}-\vec{m}}. \quad (23)$$

The gauge invariant Lagrangian based on the Virasoro w_2 algebra involving only the conformal spin 2 current (stress energy tensor) was constructed by [2] and can be obtained from the w_∞ Lagrangian by a simple truncation. One can add a fermionic Lagrangian to the one in eq-(22)

$$\mathcal{L}_f \ = \ \sum_{\vec{k},\vec{m}} \ \bar{\Psi}^{-\vec{k}}\ \Gamma^\mu\ \{\ \partial_\mu \Psi^{\vec{k}} + ie\ [\ (m_2{+}1)(\frac{m_1}{2}{-}k_1) - (\frac{m_2}{2}{-}k_2)m_1\ ]\ \mathcal{A}^{\vec{m}}_\mu\ \Psi^{\vec{k}-\vec{m}}\ \}. \quad (24)$$

we have omitted the spinor indices. $\Psi^{\vec{k}}$ is a $4D$ space-time infinite-component spinor-multiplet belonging to a $(\alpha = -1/2, \beta = 0)$ FKK representation and Γ^μ are the $4D$ spacetime Clifford algebra 4×4 matrices. A Kac-Moody extension of the Lagrangians (22, 24) differs from the expressions in eqs-(6,14). For instance, by choosing G to be one of the grand unification groups $SU(5) \subset SO(10) \subset E_6 \subset E_7 \subset E_8$, these Kac-Moody extensions of the $w_\infty(w_{1+\infty})$ gauge field theories represented by (22, 24) yield the Lagrangians that describe the infinite higher conformal-spins extensions of the Grand Unified Models and the Standard Model in $4D$. More precisely, as discussed in [6], one may define the Lagrangian density of a Yang-Mills-like $w_\infty, w_{1+\infty}$ gauge field theory coupled to a scalar field $\mathbf{\Phi}$ valued in the *adjoint* representation of $w_\infty, w_{1+\infty}$ and subject to a self-interacting scalar potential $V(\mathbf{\Phi})$ by

$$\mathcal{L} \;=\; Trace \; [\; -\frac{1}{2}\; \mathbf{F}_{\mu\nu}\; \mathbf{F}^{\mu\nu} \;+\; D_\mu \mathbf{\Phi}\; D^\mu \mathbf{\Phi} \;+\; V(\mathbf{\Phi})\;]. \tag{25}$$

The trace operation given by an infinite sum over all the generators of the $w_\infty, w_{1+\infty}$ algebra can be replaced by an integration over the internal y^1, y^2 coordinates of the internal two-dim surface $\mathcal{N}$ of the form

$$\mathcal{L} \;=\; \int d^2y \; [\; -\frac{1}{2}\; F_{\mu\nu}\; F^{\mu\nu} \;+\; D_\mu \Phi\; D^\mu \Phi \;+\; V(\Phi)\;] \tag{26}$$

leading to to an effective $6D$ theory (2 internal dimensions and 4 spacetime dimensions) for the fields $A_\mu(x^\mu, y^a)$ and $\Phi(x^\mu, y^a)$ with $y^a = y^1, y^2$ representing the internal coordinates of the 2-dim internal manifold. The commutators $[\mathbf{A}_\mu, \mathbf{A}_\nu]$ are replaced by Poisson brackets $\{A_\mu, A_\nu\}$ w.r.t the internal y^1, y^2 coordinates. The main problem is to find irreducible *unitary* representations (different from the non-unitary FFK representations) of $w_\infty, w_{1+\infty}$ to carry over the program based on eqs-(25,26).

Finally, some concluding remarks are in order. W strings based on Exceptional algebras E_6, E_7, E_8 and other Lie algebras have been studied by [14]. Higher dimensional extensions of $2D$ w_∞ algebras were analyzed by [15], thus it remains an open problem how to construct gauge theories based on these higher-dim extensions of w_∞ algebras; i.e. how to construct gauge theories of p-volume preserving diffs and relate them to an effective field theory of p-branes . Higher spins theories in Anti de Sitter spaces were developed by [16] long ago and are currently studied vigorously.

Upon quantization, the classical $w_\infty(w_{1+\infty})$ algebras get *deformed* into $W_\infty(W_{1+\infty})$ algebras constructed by [13] and which coincide also with Moyal deformations of the classical $w_\infty(w_{1+\infty})$ algebras [12]. The Moyal deformation quantization of the Lagrangians in eqs-(6, 14) presented in this work deserve further investigation. Moyal deformations of gravity via $SU(\infty)$ gauge theories and holography were constructed in [19]. The W_∞ gravity formulation of [17] based on the $4D$ self-dual gravity associated to the geometry of the contangent space of 2-dim Riemann surfaces could also be interpreted from a Fedosov deformation quantization procedure of symplectic manifolds [20]. Recent work on Fedosov deformation quantization of gravity based on Lagrange-Finsler geometric methods has been carried out by [21]. Hawking radiation, W_∞ algebras and trace anomalies is an active field of research at the present, see [25] and references therein.

Non-critical W_∞ (super) strings were found to be devoid of BRST anomalies in dimensions $D = 27$ ($D = 11$), respectively [23], and which coincide with the alleged critical (super) membrane dimensions $D = 27$ ($D = 11$) [24]. A QCD membrane from the large N limit of the $SU(N)$ Yang-Mills theory *quenched* down to a line was found by [22]. Clearly, a lot remains to be done ahead in this fascinating field of W_∞ algebras. For example, the supersymmetrization program of this work.

Acknowledgements

We thank Y. M. Cho for providing important references; to S. Vacaru for many discussions on nonlinear connections and Lagrange-Finsler spaces and M. Bowers for assistance.

References

[1] Y. Cho, Phys. Rev. Letts **67** (1991) 3469. Y. Cho, Phys Rev **D 49** (1994) 1011. Y. Cho and S. Zoh, Phys. Letts **B 292** (1992) 55.

[2] Y. Cho, W. Zoh, Phys. Review **D 46**, 3483 (1992). Y. Cho and J. Yoon, Phys. Review **D 47**, 3465 (1992).

[3] W. Zhao, Jour. Math. Phys. **40** 4325 (1999). Phys. Letts **B 459** (1999) 134.

[4] B. Feigin and D. Fuks, Funct. Anal. Appl **16**, 47 (1982). I. Kaplansky, Comm. Math. Phys **86**, 49 (1982).

[5] I. Antoniadis, P. Ditsas, E. Floratos, and J. Iliopoulos, Nucl. Phys. **B 300** (1988) 549.

[6] C. Castro, "Strings and Membranes from Einstein Gravity, Matrix Models and W_∞ Gauge Theories as paths to Quantum Gravity" to appear in the IJMPA, 2008.

[7] A. Kavalov and B. Sakita, " W_∞ and w_∞ gauge theories and contraction" [arXiv : hep-th/9603024]. S. Iso, D. Karabali and B. Sakita, Phys. Lett **B 296**, 143 (1992).

[8] Y.Cho, K, Soh, Q. Park, and J. Yoon, Phys. Lets **B 286** (1992) 251. Q. Park, Phys. Letts **B 269** (1991) 271. J. Yoon, Phys. Letts **B 308**. 240 (1993) 240. J. Yoon, Phys. Lett **A 292**, 166 (2001). J. Yoon, Class. Quan. Grav **18**, 1863 (1999).

[9] A.B. Zamoldchikov, Teor. Fiz **65**, 347 (1985).

[10] E. Sezgin, " Area-preserving Diffs, w_∞ algebras , w_∞ gravity " hep-th/9202080]. E. Sezgin, " Aspects of W_∞ Symmetry " [arXiv : hep-th/9112025]. X. Shen, " W_∞ and String Theory" [arXiv : hep-th/9202072].

[11] P. Bouwknegt, K. Schouetens, Phys. Reports **223**, 183 (1993).

[12] I. Bakas, Phys. Letts **B 228**, 57 (1989). I. Bakas and E. Kiritsis, Nucl. Phys **B 343**, 185 (1990). I. Bakas, B. Khesin and E. Kiritsis, Comm. Math. Phys **151**, 233 (1993). I. Bakas, "W_∞ symmetry of the Nambu-Goto string in $4D$ " [arXiv : hep-th/9310121].

[13] C. Pope, L. Romans, X. Shen, Nuc. Phys, **B 339**, 191 (1990). C. Pope, L. Romans and X. Shen, Phys. Letts **B 245**, 72 (1990). E. Bergshoeff, P. Howe, C. Pope, L. Romans, E. Sezgin, X. Shen and K. Stelle, "Quantization Deforms w_∞ to W_∞ Gravity"

[14] H. Lu, C. Pope, S. Schrans and X. Wang, "On Sibling and Exceptional W Strings" hep-th/9202060.

[15] M. Calixto, " Generalized W_∞ Higher Spin Algebras and Symbolic Calculus on Flag manifolds" [arXiv : hep-th/0301200].

[16] M. Vasiliev, " Higher Spin Gauge Theories, Star Product and AdS spaces " [arXiv : hep-th/9910096] .

[17] C. Hull, Phys. Letts **B 269**, 257 (1991).

[18] C. Castro, Europhysics Letters **61**, 480 (2003)

[19] C. Castro, General Relativity and Gravitation, **36**, 2605 (2004).

[20] C. Castro, Journal of Geometry and Physics **33**, 173 (2000).

[21] S. Vacaru, "Generalized Lagrange Transforms: Finsler Geometry Methods and Deformation Quantization of Gravity" [arXiv : hep-th/0707.1526]. "Finsler-Lagrange Geometries and Standard Theories in Physics: New Methods in Einstein and String Gravity" [arXiv : hep-th/0707.1524].

[22] C. Castro, "Branes from Moyal Deformation quantization of Generalized Yang-Mills Theories" [arXiv : hep-th/9908115]. S. Ansoldi, C. Castro, E. Spallucci, Phys. Lett **B 504**,174 (2001). S. Ansoldi, C. Castro, E. Spallucci, Class. Quant. Grav. **18**, L23 (2001). S. Ansoldi, C. Castro, E. Spallucci, Class. Quan. Gravity **18**, L17 (2001). S. Ansoldi, C. Castro, E. Guendelman and E. Spallucci, "Nambu-Goto Strings from SU(N) Born-Infeld model", Class.Quant.Grav. **19** L135 (2002).

[23] C. Castro, Jour. Chaos, Solitons and Fractals **7**, 711 (1996).

[24] U. Marquard, R. Kaiser and M. Scholl, Phys. Lett **B 227**, 234 (1989). U. Marquard and M. Scholl, Phys. Lett **B 227**, 227 (1989).

[25] L. Bonora and M. Cvitan, "Hawking radiation, W_∞ algebra and trace anomalies" arXiv.org : 0804.0198.

Received: June 5, 2008