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THE EXCEPTIONAL E_8 GEOMETRY OF CLIFFORD (16) SUPERSPACE AND CONFORMAL GRAVITY YANG–MILLS GRAND UNIFICATION

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We continue to study the Chern–Simons E_8 Gauge theory of Gravity developed by the author which is a unified field theory (at the Planck scale) of a Lanczos–Lovelock Gravitational theory with a E_8 Generalized Yang–Mills (GYM) field theory, and is defined in the $15D$ boundary of a 16D bulk space. The Exceptional E_8 Geometry of the 256-dim slice of the 256×256-dimensional flat *Clifford* (16) space is explicitly constructed based on a spin connection Ω_M^{AB}, that gauges the generalized Lorentz transformations in the tangent space of the 256-dim curved slice, and the 256×256 components of the vielbein field E_M^A, that gauge the nonabelian translations. Thus, in one-scoop, the vielbein E_M^A encodes *all* of the 248 (nonabelian) E_8 generators and 8 additional (abelian) translations associated with the *vectorial* parts of the generators of the diagonal subalgebra $[Cl(8) \otimes Cl(8)]_{\text{diag}} \subset Cl(16)$. The generalized curvature, Ricci tensor, Ricci scalar, torsion, torsion vector and the Einstein–Hilbert–Cartan action is constructed. A preliminary analysis of how to construct a Clifford Superspace (that is far *richer* than ordinary superspace) based on orthogonal and symplectic Clifford algebras is presented. Finally, it is shown how an E_8 ordinary Yang–Mills in $8D$, after a sequence of symmetry breaking processes $E_8 \to E_7 \to E_6 \to SO(8,2)$, and performing a Kaluza–Klein–Batakis compactification on CP^2, involving a nontrivial *torsion*, leads to a (Conformal) Gravity and Yang–Mills theory based on the Standard Model in $4D$. The conclusion is devoted to explaining how Conformal (super) Gravity and (super) Yang–Mills theory in any dimension can be embedded into a (super) Clifford-algebra-valued gauge field theory.

Keywords: C-space gravity; Clifford algebras; grand unification; exceptional algebras; string theory.

1. Introduction

Grand-Unification models in $4D$ based on the exceptional E_8 Lie algebra have been known for sometime [29]. Both gauge bosons A_μ^a and left-handed (two-component) Weyl fermions are assigned to the adjoint 248-dim representation that coincides with the fundamental representation (a very special case for E_8). The Higgs bosons Φ are chosen from among the multiplets that couple to the symmetric product of two fermionic representations $\Psi_L^a C \Psi_L^b \Phi_{ab}$ (C is the charge conjugation matrix) such that $[\mathbf{248} \times \mathbf{248}]_S = \mathbf{1} + \mathbf{3875} + \mathbf{27000}$. Bars and Gunaydin [29] have argued that a physically relevant subspace in the symmetry *breaking* process of E_8 is $SO(16) \to SO(10) \times SU(4)$, where the 128 remaining massless fermions (after symmetry breaking) are assigned to the $(16, \bar{4})$ and $(\bar{16}, 4)$ representations. $SU(4)$ serves as the family unification group (four fermion families plus four mirror fermion families of opposite chirality) and $SO(10)$ is the Yang–Mills GUT group.

This symmetry breaking channel occurs in the **135**-dim representation of $SO(16)$ that appears in the $SO(16)$ decomposition of the **3875**-dim representation of E_8 : $\mathbf{3875} = \mathbf{135} + \mathbf{1820} + \mathbf{1920}$. By giving a large v.e.v (vacuum expectation value) to the Higgs Φ_{ab} in the **135**-dim representation of $SO(16)$, corresponding to a symmetric traceless tensor of rank 2, *all* fermions and gauge bosons become super-heavy except for the adjoint representations of gauge bosons given in terms of the $SO(10) \times SU(4)$ decomposition as $(45, 1) + (1, 15)$. The spinor representations of the massless fermions is $128 = (16, \bar{4}) + (\bar{16}, 4)$, leading to 4 fermion families plus their 4 mirror ones. In this process, only 120 fermions and 188 gauge bosons of the initial 248 have gained mass.

In $SO(10)$ GUT a right-handed massive neutrino (a $SU(5)$ singlet) is added to each Standard Model generation so that 16 (two-component) Weyl fermions can now be placed in the **16**-dim spinor representation of $SO(10)$ and, which in turn, can be decomposed in terms of $SU(5)$ representations as $\mathbf{16} = \mathbf{1} + \mathbf{5}^* + \mathbf{10}$ [32]. In the second stage of symmetry breaking, the fourth family of $\mathbf{5}^* + \mathbf{10}$; $\mathbf{5} + \mathbf{10}^*$ becomes heavy without affecting the remaining 3 families. Later on [30] found that a Peccei–Quinn symmetry could be used to protect light fermions from acquiring super large masses. If this protection is to be maintained without destroying perturbative unification, *three* light families of fermion generations are singled out which is what is observed. In addition to the other three mirror families, several exotic fermions also remain light.

The other physically relevant symmetry breaking channel is $E_8 \to E_6 \times SU(3)$ with 3 fermion families (and their mirrors) assigned to the 27 ($\bar{27}$) dim representation of E_6:

$$248 = (1, 8) + (78, 1) + (27, 3) + (\bar{27}, \bar{3}).$$

In this case, in addition to the 16 fermions assigned to the 16-dim dim spinor representation of $SO(10)$, there exist 11 exotic (two-component) Weyl fermions for each generation. The low energy phenomenology of Superstring-inspired E_6 models has been studied intensively. New particles including new gauge bosons, massive neutrinos, exotic fermions, Higgs bosons and their superpartners, are expected to exist. See [34] for an extensive review and references.

The supersymmetric E_8 model has more recently been studied as a fermion family and grand unification model [28] under the assumption that there is a vacuum gluino condensate but this condensate is *not* accompanied by a dynamical generation of a mass gap in the pure E_8 gauge sector. A study of the interplay among

Exceptional Groups, del Pezzo surfaces and the extra massless particles arising from rational double point singularities can be found in [10].

Clifford algebras and E_8 are key ingredients in Smith's $D_4 - D_5 - E_6 - E_7 - E_8$ grand unified model in $D = 8$ [7]. Exceptional, Jordan, Division and Clifford algebras are deeply related and essential tools in many aspects in Physics [12, 18–25, 44]. Ever since the discovery [1] that $11D$ supergravity, when dimensionally reduced to an n-dim torus led to maximal supergravity theories with hidden exceptional symmetries E_n for $n \leq 8$, it has prompted intensive research to explain the higher dimensional origins of these hidden exceptional E_n symmetries [2, 5]. More recently, there has been a lot of interest in the infinite-dim hyperbolic Kac–Moody E_{10} and nonlinearly realized E_{11} algebras arising in the asymptotic chaotic oscillatory solutions of Supergravity fields close to cosmological singularities [1, 2].

The classification of symmetric spaces associated with the scalars of N extended Supergravity theories, emerging from compactifications of $11D$ supergravity to lower dimensions, and the construction of the U-duality groups as spectrum-generating symmetries for four-dimensional BPS black-holes [5] also involved exceptional symmetries associated with the exceptional magic Jordan algebras $J_3[R, C, H, O]$. The discovery of the anomaly free 10-dim heterotic string for the algebra $E_8 \times E_8$ was another hallmark of the importance of Exceptional Lie groups in Physics.

Supersymmetric nonlinear σ models of Kahler coset spaces $\frac{E_8}{SO(10)\times SU(3)\times U(1)}$; $\frac{E_7}{SU(5)}$; $\frac{E_6}{SO(10)\times U(1)}$ are known to contain three generations of quarks and leptons as (quasi) Nambu–Goldstone *superfields* [26] (and references therein). The coset model based on $G = E_8$ gives rise to 3 left-handed generations assigned to the **16** multiplet of $SO(10)$, and 1 right-handed generation assigned to the $\mathbf{16}^*$ multiplet of $SO(10)$. The coset model based on $G = E_7$ gives rise to 3 generations of quarks and leptons assigned to the $\mathbf{5}^* + \mathbf{10}$ multiplets of $SU(5)$, and a Higgsino (the fermionic partner of the scalar Higgs) in the **5** representation of $SU(5)$.

An E_8 gauge bundle was instrumental in the understanding the topological part of the M-theory partition function [8, 9]. A mysterious E_8 bundle which restricts from 12-dim to the 11-dim bulk of M theory can be compatible with 11-dim supersymmetry. The nature of this 11-dim E_8 gauge theory remains unknown. We hope that the Chern–Simons E_8 gauge theory of gravity in $D = 15$ advanced in this work may shed some light into solving this question.

E_8 Yang–Mills theory can naturally be embedded into a $Cl(16)$ algebra Gauge Theory and the $11D$ Chern–Simons (Super) Gravity [4] is a very small sector of a more fundamental polyvector-valued gauge theory in Clifford spaces. Polyvector-valued Supersymmetries [15] in Clifford-spaces turned out to be more fundamental than the supersymmetries associated with M, F theory superalgebras [14]. For this reason we believe that Clifford structures may shed some light into the origins behind the hidden E_8 symmetry of $11D$ Supergravity and reveal more important features underlying M, F theory.

In [31] we constructed a Chern–Simons E_8 gauge theory of (*Euclideanized*) Gravity in $D = 15$ (the 15-dim boundary of a 16-dim space) based on an *octic* E_8 invariant expression in $D = 16$ constructed by [27], and proposed that a grand unification of gravity with all the other forces is possible after including *Supersymmetry* in order to incorporate spacetime fermions (which are the *gauginos* of the theory). The E_8 invariant action had 37 terms and contained: (i) the Lanczos–Lovelock Gravitational action associated with the 15-dim boundary $\partial\mathcal{M}^{16}$ of the 16-dim manifold; (ii) 5 terms with the same structure as the Pontryagin $p_4(F^{IJ})$ 16-form associated with the $SO(16)$ spin connection Ω_μ^{IJ} where the indices I, J run from $1, 2, \ldots, 16$; (iii) the fourth power of the standard quadratic E_8 invariant $[I_2]^4$; (iv) plus 30 additional terms involving powers of the E_8-valued $F_{\mu\nu}^{IJ}$ and $F_{\mu\nu}^{\alpha}$ field-strength (2-forms).

The main purpose of this work is to extend the above results on Chern–Simons E_8 Gauge theories of Gravity in order to build the Exceptional E_8 Geometry of $Cl(16)$ (super)space; to construct an E_8 gauge theory of gravitation and to show how to obtain unified field theories of Conformal (super) Gravity and (super) Yang–Mills by exploiting the algebraic structures of (super) Clifford algebras in higher dimensions, in particular $8D$ and $16D$. A candidate action for an Exceptional E_8 gauge theory of gravity in $8D$ was constructed recently [54]. It was obtained by recasting the E_8 group as the semi-direct product of $GL(8, R)$ with a deformed Weyl-Heisenberg group associated with canonical-conjugate pairs of vectorial and antisymmetric tensorial generators of rank two and three. This decomposition of the E_8 algebra generators in terms of $GL(8, R)$ is presented in the Appendix **A**.

2. The Exceptional E_8 Geometry of $Cl(16)$ Superspace and Unification

2.1. *A Chern–Simons E_8 gauge theory gravity and grand-unification in higher dimensions*

In this section we will begin by reviewing our work [31] by showing why the E_8 algebra is a subalgebra of $Cl(16) = Cl(8) \otimes Cl(8)$ and how E_8 admits a 7-grading decomposition in terms $Sl(8, R)$ [5, 7], and provide the action corresponding to the Chern–Simons E_8 gauge theory of (*Euclideanized*) Gravity [31] which naturally furnishes a Gravity-E_8 Generalized Yang–Mills unified field theory in $15D$ (the boundary of a $16D$ bulk space). We then proceed to discuss in detail why a E_8 singlet chiral spinor Ψ_α has enough degrees of freedom to accommodate 4 fermion families (plus 4 mirror ones) belonging to the **16**-dim representations of the $SO(10)$ GUT group in $4D$, after a dimensional reduction from $16D$ to $4D$ is performed.

It is well known among the experts that the E_8 algebra admits the $SO(16)$ decomposition $\mathbf{248} \rightarrow \mathbf{120} \oplus \mathbf{128}$. The E_8 admits also a $SL(8, R)$ decomposition [5]. Due to the triality property, the $SO(8)$ admits the vector $\mathbf{8}_v$ and spinor representations $\mathbf{8}_s, \mathbf{8}_c$. After a triality rotation, the $SO(16)$ vector and spinor representations decompose as [5]

$$\mathbf{16} \to \mathbf{8}_s \oplus \mathbf{8}_c. \tag{2.1a}$$

$$\mathbf{128}_s \to \mathbf{8}_v \oplus \mathbf{56}_v \oplus \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v. \tag{2.1b}$$

$$\mathbf{128}_c \to \mathbf{8}_s \oplus \mathbf{56}_s \oplus \mathbf{8}_c \oplus \mathbf{56}_c. \tag{2.1c}$$

To connect with (real) Clifford algebras [7], i.e. how to fit E_8 into a Clifford structure, start with the 248-dim fundamental representation E_8 that admits a $SO(16)$ decomposition given by the 120-dim bivector representation plus the 128-dim chiral-spinor representations of $SO(16)$. From the modulo 8 periodicity of Clifford algebras over the reals one has $Cl(16) = Cl(2 \times 8) = Cl(8) \otimes Cl(8)$, meaning, roughly, that the $2^{16} = 256 \times 256$ $Cl(16)$-algebra matrices can be obtained effectively by replacing each single one of the *entries* of the $2^8 = 256 = 16 \times 16$ $Cl(8)$-algebra matrices by the 16×16 matrices of the second copy of the $Cl(8)$ algebra. In particular, $120 = 1 \times 28 + 8 \times 8 + 28 \times 1$ and $128 = 8 + 56 + 8 + 56$, hence the 248-dim E_8 algebra decomposes into a $120 + 128$ dim structure such that E_8 can be represented indeed within a tensor product of $Cl(8)$ algebras.

At the E_8 Lie algebra level, the E_8 gauge connection decomposes into the $SO(16)$ vector $I, J = 1, 2, \ldots, 16$ and (chiral) spinor $A = 1, 2, \ldots, 128$ indices as follows

$$\begin{aligned} &\mathcal{A}_\mu = \mathcal{A}_\mu^{IJ} X_{IJ} + \mathcal{A}_\mu^A Y_A, \quad X_{IJ} = -X_{JI}, \\ &I, J = 1, 2, 3, \ldots, 16, \quad A = 1, 2, \ldots, 128, \end{aligned} \tag{2.2}$$

where X_{IJ}, Y_A are the E_8 generators. The Clifford algebra $(Cl(8) \otimes Cl(8))$ structure behind the $SO(16)$ decomposition of the E_8 gauge field $\mathcal{A}_\mu^{IJ} X_{IJ} + \mathcal{A}_\mu^A Y_A$ can be deduced from the expansion of the generators X_{IJ}, Y_A in terms of the $Cl(16)$ algebra generators. The $Cl(16)$ bivector basis admits the decomposition

$$X^{IJ} = a_{ij}^{IJ} (\gamma_{ij} \otimes \mathbf{1}) + b_{ij}^{IJ} (\mathbf{1} \otimes \gamma_{ij}) + c_{ij}^{IJ} (\gamma_i \otimes \gamma_j) \tag{2.3}$$

where γ_i are the Clifford algebra generators of the $Cl(8)$ algebra present in $Cl(16) = Cl(8) \otimes Cl(8)$; $\mathbf{1}$ is the unit $Cl(8)$ algebra element that can be represented by a unit 16×16 diagonal matrix. The tensor products $\otimes$ of the 16×16 $Cl(8)$-algebra matrices, like $\gamma_i \otimes \mathbf{1}$, $\gamma_i \otimes \gamma_j, \ldots$ furnish a 256×256 $Cl(16)$-algebra matrix, as expected. Therefore, the decomposition in (2.3) yields the $28 + 28 + 8 \times 8 = 56 + 64 = 120$-dim bivector representation of $SO(16)$; i.e. for each *fixed* values of IJ there are 120 terms in the r.h.s of (2.3), that match the number of *independent* components of the E_8 generators $X^{IJ} = -X^{JI}$, given by $\frac{1}{2}(16 \times 15) = 120$. The decomposition of Y_A is more subtle. A spinor Ψ in $16D$ has $2^8 = 256$ components and can be decomposed into a 128 component left-handed spinor Ψ^A and a 128 component right-handed spinor $\Psi^{\dot{A}}$. The 256 spinor indices are $\alpha = A, \dot{A}; \beta = B, \dot{B}, \ldots$ with $A, B = 1, 2, \ldots, 128$ and $\dot{A}, \dot{B} = 1, 2, \ldots, 128$, respectively.

Spinors are elements of right (left) ideals of the $Cl(16)$ algebra and admit the expansion $\Psi = \Psi_\alpha \xi^\alpha$ in a 256-dim spinor basis ξ^α which in turn can be expanded as sums of Clifford polyvectors of *mixed* grade; i.e. into a sum of scalars, vectors, bivectors, trivectors, ... Minimal left/right ideals elements of Clifford algebras may be systematically constructed by means of idempotents $e^2 = e$ such that the geometric product of $Cl(p,q)e$ generates the ideal [25].

The commutation relations of E_8 are [5]

$$[X^{IJ}, X^{KL}] = 4(\delta^{IK}X^{LJ} - \delta^{IL}X^{KJ} + \delta^{JK}X^{IL} - \delta^{JL}X^{IK}).$$

$$[X_{IJ}, Y^{\alpha}] = -\frac{1}{2}\Gamma^{\alpha\beta}_{IJ}Y_{\beta}; \quad [Y^{\alpha}, Y^{\beta}] = \frac{1}{4}\Gamma^{\alpha\beta}_{IJ}X^{IJ}, \quad \Gamma^{\alpha\beta}_{IJ} = [\Gamma_I, \Gamma_J]^{\alpha\beta}. \tag{2.4}$$

The combined E_8 indices are denoted by $\mathcal{A} \equiv [IJ]$, α $(120 + 128 = 248$ indices in total) that yield the Killing metric and the structure constants

$$\eta^{\mathcal{AB}} = \frac{1}{60}TrT^{\mathcal{A}}T^{\mathcal{B}} = -\frac{1}{60}f^{\mathcal{A}}_{\mathcal{CD}}f^{\mathcal{BCD}}. \tag{2.5a}$$

$$\begin{gathered} f^{IJ,KL,MN} = -8\delta^{IK}\delta^{LJ}_{MN} + \text{permutations}; \quad f^{IJ}_{\alpha\beta} = -\frac{1}{2}\Gamma^{IJ}_{\alpha\beta}; \\ \eta^{IJKL} = -\frac{1}{60}f^{IJ}_{\mathcal{CD}}f^{KL,\mathcal{CD}}. \end{gathered} \tag{2.5b}$$

The E_8 algebra as a sub-algebra of $Cl(8)\otimes Cl(8)$ is consistent with the $SL(8,R)$ 7-grading decomposition of $E_{8(8)}$ (with 128 noncompact and 120 compact generators) as shown by [5]. Such $SL(8,R)$ 7-grading is based on the diagonal part $[SO(8) \times SO(8)]_{\text{diag}} \subset SO(16)$ described in full detail by [5] and can be deduced from the $Cl(8) \otimes Cl(8)$ 7-grading decomposition of E_8 provided by Larsson [7] as follows,

$$[\gamma^{\mu}_{(1)} \oplus \gamma^{\mu\nu}_{(1)} \oplus \gamma^{\mu\nu\rho}_{(1)}] \otimes \mathbf{1}_{(2)} + \mathbf{1}_{(1)} \otimes [\gamma^{\mu}_{(2)} \oplus \gamma^{\mu\nu}_{(2)} \oplus \gamma^{\mu\nu\rho}_{(2)}] + \gamma^{\mu}_{(1)} \otimes \gamma^{\nu}_{(2)}. \tag{2.6}$$

These tensor products of elements of the two factor $Cl(8)$ algebras, described by the subscripts $(1), (2)$, furnishes the 7-grading of $E_{8(8)}$

$$8 + 28 + 56 + 64 + 56 + 28 + 8 = 248. \tag{2.7}$$

8 corresponds to the $8D$ vector γ^{μ}; 28 is the $8D$ bivector $\gamma^{\mu\nu}$; 56 is the $8D$ trivector $\gamma^{\mu\nu\rho}$, and $64 = 8 \times 8$ corresponds to the tensor product $\gamma^{\mu}_{(1)} \otimes \gamma^{\nu}_{(2)}$. In essence one can rewrite the E_8 algebra in terms of $8+8$ vectors Z^a, Z_a $(a = 1, 2, \ldots, 8)$; $28 + 28$ bivectors $Z^{[ab]}, Z_{[ab]}$; $56 + 56$ trivectors $E^{[abc]}, E_{[abc]}$, and the $SL(8,R)$ generators E^b_a which are expressed in terms of a $8 \times 8 = 64$-component tensor Y^{ab} that can be decomposed into a symmetric part $Y^{(ab)}$ with 36 independent components, and an anti-symmetric part $Y^{[ab]}$ with 28 independent components. Its trace $Y^{cc} = N$ yields an element N of the Cartan subalgebra such that the degrees $-3, -2, -1, 0, 3, 2, 1$ of the 7-grading of $E_{8(8)}$ can be read from [51]

$$[N, Z^a] = 3Z^a; \quad [N, Z_a] = -3Z_a; \quad [N, Z_{ab}] = 2Z_{ab}; \quad [N, Z^{ab}] = -2Z^{ab}. \tag{2.8a}$$

$$[N, E^{abc}] = E^{abc}; \quad [N, E_{abc}] = -E_{abc}; \quad [N, E^b_a] = 0 \tag{2.8b}$$

where the 63 generators E^b_a (after subtracting the trace)

$$E^b_a = \frac{1}{8}(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]} + \Gamma^{ab}_{\dot{\alpha}\dot{\beta}}X^{[\dot{\alpha}\dot{\beta}]}) + Y^{(ab)} - \frac{1}{8}\delta^{ab}N. \tag{2.8c}$$

for the (vector) indices $a, b = 1, 2, \ldots, 8$ span the $SL(8, R)$ subalgebra of $E_{8(8)}$. The spinorial indices $\alpha, \beta = 1, 2, \ldots 8$ and $\dot{\alpha}, \dot{\beta} = 1, 2, \ldots 8$ correspond to the chiral/antichiral $\mathbf{8}_s, \mathbf{8}_c$ spinor representations of $SO(8)$. The 64 components Y^{ab} are part of the 128 chiral $SO(16)$ spinorial components $Y^A = (Y^{\alpha\dot{\beta}}, Y^{ab})$ after performing the $SO(8)$ decomposition of the chiral spinorial $SO(16)$ indices into $64 + 64$ components $Y^{\alpha\dot{\beta}}, Y^{ab}$, respectively. Whereas the 120 $SO(16)$ bivectors X^{IJ} are decomposed in terms of $X^{[\alpha\beta]}$, $X^{[\dot{\alpha}\dot{\beta}]}$ and $X^{\alpha\dot{\beta}}$ with $28 + 28 + 64 = 120$ components, respectively. We refer to [51] for details.

The $Cl(16)$ gauge theory that encodes the E_8 gauge theory in D-dim is based on the E_8-valued field strengths

$$F^{IJ}_{\mu\nu} X_{IJ} = (\partial_\mu \mathcal{A}^{IJ}_\nu - \partial_\nu \mathcal{A}^{IJ}_\mu) X_{IJ} + \mathcal{A}^{KL}_\mu \mathcal{A}^{MN}_\nu [X_{KL}, X_{MN}] + \mathcal{A}^\alpha_\mu \mathcal{A}^\beta_\nu [Y_\alpha, Y_\beta]. \tag{2.9}$$

$$F^A_{\mu\nu} Y_\alpha = (\partial_\mu \mathcal{A}^\alpha_\nu - \partial_\nu \mathcal{A}^\alpha_\mu) Y_\alpha + \mathcal{A}^\alpha_\mu \mathcal{A}^{IJ}_\nu [Y_\alpha, X_{IJ}]. \tag{2.10}$$

The E_8 actions in $4D$ are

$$\begin{aligned} S_{\text{Topological}}[E_8] &= \int d^4x \frac{1}{60} Tr[F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} T_{\mathcal{A}} T_{\mathcal{B}}] \epsilon^{\mu\nu\rho\tau} = \int d^4x F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} \eta_{\mathcal{AB}} \epsilon^{\mu\nu\rho\tau} \\ &= \int d^4x [F^{IJ}_{\mu\nu} F^{KL}_{\rho\tau} \eta_{IJKL} + F^\alpha_{\mu\nu} F^\beta_{\rho\tau} \eta_{\alpha\beta} + 2 F^{IJ}_{\mu\nu} F^\beta_{\rho\tau} \eta_{IJ\beta}] \epsilon^{\mu\nu\rho\tau} \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} S_{YM}[E_8] &= \int d^4x \sqrt{g} \frac{1}{60} Tr[F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} T_{\mathcal{A}} T_{\mathcal{B}}] g^{\mu\rho} g^{\nu\tau} = \int d^4x \sqrt{g} F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} \eta_{\mathcal{AB}} g^{\mu\rho} g^{\nu\tau} \\ &= \int d^4x \sqrt{g} [F^{IJ}_{\mu\nu} F^{KL}_{\rho\tau} \eta_{IJKL} + F^\alpha_{\mu\nu} F^\beta_{\rho\tau} \eta_{\alpha\beta} + 2 F^{IJ}_{\mu\nu} F^\beta_{\rho\tau} \eta_{IJ\beta}] g^{\mu\rho} g^{\nu\tau}. \end{aligned} \tag{2.12}$$

The above E_8 actions can be embedded onto more general $Cl(16)$ actions with a much larger number of terms as shown in [31].

The action that defines a Chern–Simons E_8 gauge theory of (*Euclideanized*) Gravity in 15-dim (the boundary of a $16D$ space) was based on the *octic* E_8 invariant constructed by [27] and is defined [31]

$$\begin{aligned} S &= \int_{\mathcal{M}^{16}} \langle F F \cdots F \rangle_{E_8} \\ &= \int_{\mathcal{M}^{16}} (F^{M_1} \wedge F^{M_2} \wedge \cdots \wedge F^{M_8}) \Upsilon_{M_1 M_2 M_3 \cdots M_8} \\ &= \int_{\partial\mathcal{M}^{16}} \mathcal{L}^{(15)}_{CS} (\mathbf{A}, \mathbf{F}) \,. \end{aligned} \tag{2.13}$$

The E_8 Lie-algebra valued 16-form $\langle F^8 \rangle$ is *closed*: $d(\langle F^{M_1} T_{M_1} \wedge F^{M_2} T_{M_2} \wedge \cdots \wedge F^{M_8} T_{M_8} \rangle) = 0$ and *locally* can always be written as an exact form in terms of an E_8-valued Chern–Simons 15-form as $I_{16} = d\mathcal{L}^{(15)}_{CS}(\mathbf{A}, \mathbf{F})$. For instance, when $\mathcal{M}^{16} = S^{16}$ the 15-dim boundary integral (2.15) is evaluated in the two coordinate patches of the equator $S^{15} = \partial\mathcal{M}^{16}$ of S^{16} leading to the integral of $tr(\mathbf{g}^{-1}d\mathbf{g})^{15}$ (up to numerical factors) when the gauge potential $\mathbf{A}$ is written locally as $\mathbf{A} = \mathbf{g}^{-1}d\mathbf{g}$ and $\mathbf{g}$ belongs to the E_8 Lie-algebra. The integral is characterized by the elements of the homotopy group $\boldsymbol{\pi}_{15}(E_8)$. S^{16} can also be represented in terms of quaternionic and octonionic projectives spaces as HP^4, OP^2 respectively.

In order to evaluate the operation $\langle \cdots \rangle_{E_8}$ in the action (2.13) it involves the existence of an *octic* E_8 group invariant tensor $\Upsilon_{M_1, M_2, \ldots, M_8}$ that was recently constructed by Cederwall and Palmkvist [27] using the Mathematica package GAMMA based on the full machinery of the Fierz identities. The entire *octic* E_8 invariant contains powers of the $SO(16)$ bivector X^{IJ} and spinorial Y^α generators $X^8, X^6Y^2, X^4Y^4, X^2Y^6, Y^8$. The corresponding number of terms is $6, 11, 12, 5, 2$ respectively giving a total of **36** terms for the octic E_8 invariant involving **36** numerical coefficients multiplying the corresponding powers of the E_8 generators. There is an extra term (giving a total of **37** terms) with an *arbitrary* constant multiplying the fourth power of the E_8 quadratic invariant $I_2 = -\frac{1}{2}tr[(F^{IJ}_{\mu\nu}X_J)^2 + (F^\alpha_{\mu\nu}Y_\alpha)^2]$.

Thus, the E_8 invariant action has 37 terms containing: (i) the Lanczos–Lovelock Gravitational action associated with the 15-dim boundary $\partial\mathcal{M}^{16}$ of the 16-dim manifold; (ii) 5 terms with the same structure as the Pontryagin $p_4(F^{IJ})$ 16-form associated with the $SO(16)$ spin connection Ω^{IJ}_μ and where the indices I, J run from $1, 2, \ldots, 16$; (iii) the fourth power of the standard quadratic E_8 invariant $[I_2]^4$; (iv) plus 30 additional terms involving powers of the E_8-valued $F^{IJ}_{\mu\nu}$ and $F^\alpha_{\mu\nu}$ field-strength (2-forms).

Therefore, the essence of the action in Eq. (2.13) as explained in [31] comprises a Chern–Simons (Euclideanized) Gravity coupled to a E_8 Generalized Yang–Mills theory in the $15D$ boundary of a $16D$ manifold. Certainly, the bulk in $16D$ theory has large number of degrees of freedom. One can freeze-off a large number of them, such that upon a dimensional reduction to $4D$ and truncation of degrees of freedom, one will obtain $4D$ Gravity interacting with a E_8 Yang–Mills theory. A supersymmetrization program yields $4D$ SUGRA coupled to E_8 SUSY YM (after dimensional reduction and truncation). Why not start with the quartic invariant in $8D$ and/or a quadratic invariant in $4D$ instead of the *octic* invariant in $16D$? Because it *is* the $SO(16)$ maximal subgroup of $E_{8(8)}$ that requires us to build a theory in the $16D$ bulk space and its $15D$ boundary. To sum up, a Chern–Simons E_8 Gauge Theory of gravity requires a $16D$ bulk space which upon dimensional reduction to $4D$ and truncation of degrees of freedom leads to the desired $4D$ Gravitational *and* E_8 Yang–Mills theory. Thus we have a natural Gravity-E_8 Yang–Mills unification theory stemming from the Chern–Simons E_8 Gauge Theory of Gravity in higher dimensions.

A supersymmetric version of the octic E_8 invariant action (2.13) involves a vector supermultiplet A^m_μ, Ψ^m_α in $D = 16$, with 248 spacetime fermions Ψ^m_α in the fundamental 248-dim representation of E_8 ($m = 1, 2, \ldots, 248$), and 248 spacetime vectors (gluons) A_μ^m in the 248-dim adjoint representation, $\mu = 1, 2, 3, \ldots, 16$. The $16D$ spacetime chiral spinor index α runs over $1, 2, 3, \ldots, 128$. One should not confuse the spacetime spinorial indices with the internal space ones associated with the spinorial generators of E_8. The fermions are the *gluinos* in this very special case because the 248-dim fundamental and 248-dim adjoint representations of the exceptional E_8 group coincide. The exceptional group E_8 is unique in this respect. In ordinary supersymmetric Yang–Mills the superpartners of the fermions are scalars, however, in the supersymmetric E_8 Yang–Mills case, the fermions Ψ_α^m (gluinos) and the vectors A_μ^m (gluons) comprise the vector supermultiplet.

It is true that the chiral spinors Ψ_α^m in $D = 16$ have many degrees of freedom, since m ranges over the 248 generators of E_8 (120 vector and 128 spinorial generators) and $\alpha = 1, 2, 3, \ldots, 128$. It is interesting to notice that if one had a *singlet* spinor Ψ_α, belonging to the **1**-dim trivial "scalar" representation of E_8 and if, and only if, one could *mix* the *internal spinorial* indices of $SO(10)$ and $SU(4) \sim SO(6)$ with the $4D$ *spacetime spinorial* indices, *after* the dimensional reduction process from $16D \to 4D$, the initial 128 spacetime spinorial components in $16D$ ($\alpha = 1, 2, 3, \ldots, 128$) of the E_8 singlet chiral spinor Ψ_α will have enough room to accommodate 64 two-component chiral Weyl spinors in $4D$ (since $2 \times 64 = 128$). And, in turn, the 64 two-component Weyl spinors in $4D$ could assemble themselves into 4 copies of 16 two-component Weyl spinors in $4D$, where the 16 Weyl spinors can be assigned to the **16**-dim chiral spinorial representation of $SO(10)$.

Therefore, one could argue that the E_8 singlet chiral spinor Ψ_α in $16D$ has precisely the right number of degrees of freedom to accommodate the 4 families of fermions in the 16-dim chiral spinorial representation of the $SO(10)$ GUT group in $4D$. The 4 mirror fermion families would be assigned to the anti-chiral Weyl spinor $\Psi_{\dot\alpha}$. To sum up, starting from a E_8 *singlet* spinor in $16D$, one would have recovered upon dimensional reduction to $4D$ the same number of (four) fermion families as those in the E_8 GUT models of [29] in $4D$, with the provision that one could *mix* the *internal spinorial* indices of $SO(10), SU(4) \sim SO(6)$, with the $4D$ *spacetime spinorial* indices in the dimensional reduction process. In this fashion, one would have encoded *all* $4D$ fermions of *all* the families into a E_8 *singlet* fermion Ψ_α in $16D$. This possibility warrants further investigation.

To conclude, we must emphasize that the Chern–Simons E_8 Gauge theory of (Euclideanized) Gravity [31] in higher dimensions (the $15D$ boundary of a $16D$ space), *unifying* a Lanczos–Lovelock gravitational theory *with* a E_8 Generalized Yang–Mills theory, involves a gauge theory of E_8 comprised of *higher* powers of the field strengths $\mathbf{F}$ rather than the mere quadratic ones as in ordinary YM, *in addition* to a Lanczos–Lovelock gravitational theory which also involves *higher* powers of the curvature field strengths $\mathbf{R}$ rather than the linear power $\mathbf{R}$ as in ordinary Einstein Gravity. Despite the *higher* powers of field strengths and curvatures, the equations of motion for the graviton and YM field are *no* higher than *two* [4] avoiding the problem of ghosts in higher derivative theories. This is the *hallmark* of theories based on actions of the Chern–Simons form (Lanczos–Lovelock type for gravity). Since Witten [6] has shown that $3D$ Chern–Simons gravity is exactly solvable as a quantum theory, despite that ordinary $3D$ gravity is not perturbative renormalizable, one might have a (non)perturbative *finitely* renormalizable and *unitary* (free of ghosts) Quantum theory of Lanczos–Lovelock Gravity, *unified* with a E_8 Generalized Yang–Mills theory, based on the quantization of the Chern–Simons E_8 Gauge theory of Gravity [31] in higher dimensions. The issue of instabilities and anomalies has to be analyzed in full depth since anomalies may spoil the quantum consistency of the theory. For very deep connections between $3D$ Chern–Simons Gravity, Extremal Conformal Field Theories with central charges $c = 12k$ (k is an integer) and the Monster Group in classifying the physical states of a BTZ black-hole, see [6].

The underlying reason why one has a *unified* Lanczos–Lovelock gravitational theory with a E_8 Generalized Yang–Mills theory, in *one scoop*, is due to very *special* properties of the E_8 Yang–Mills algebra involving $SO(16)$ bivectorial generators $\mathcal{A}_\mu^{IJ}, I, J = 1, 2, 3, \ldots, 16$, and $SO(16)$ chiral *spinorial* ones $\mathcal{A}_\mu^A, A = 1, 2, 3, \ldots, 128$. Commutators of the bivector generators yields bivector ones; commutators of spinorial generators yields *also* bivector ones, and commutators of bivectors with spinorial generators yields spinorial ones. This *mixing* is what accounts for having a *unified* Lanczos–Lovelock gravitational theory with a E_8 Generalized Yang–Mills theory. The $SO(16)$ bivector pieces appearing in the E_8 commutators encode *both* the terms in the Lanczos–Lovelock gravitational sector theory, *as well as* the $SO(16)$ bivector parts of the E_8 Generalized Yang–Mills gauge theory (120 of them). The $SO(16)$ spinorial pieces appearing in the commutators of the spinorial generators encode the $SO(16)$ spinorial parts of the E_8 Generalized Yang–Mills (128 of them).

2.2. *The exceptional E_8 geometry of $Cl(16)$-superspaces*

In this section we will develop a novel theory (to our knowledge) by generalizing the construction of the Chern–Simons E_8 Gauge theory of (Euclideanized) Gravity to the Exceptional E_8 Geometry of the C-space associated to the Clifford algebra $C(16) = Cl(8) \otimes Cl(8)$ by taking the 256-dim *diagonal* slice of the 256×256-dim space associated with $Cl(16)$ algebra, such that we may decompose the symmetries of the *tangent* space of such 256-dim diagonal space, as comprised of $\frac{1}{2}\, 256 \times 255 = 32640$ "*rotations*" and 256 "translations". In the same fashion, one could have decomposed the 248 roots of E_8 as $240 + 8$, where the 8 roots correspond to the Cartan subalgebra of E_8 which has rank 8. The 240 roots generate "rotations" and the 8 roots generate "translations". We prefer however the *former* geometrical description directly in terms of the 256-dim slice of the C-space corresponding to the $C(16)$ algebra. This is what we call the Exceptional E_8 Geometry of $Cl(16)$ spaces. At the end of this section we explain how to extend this construction to Clifford Superspaces.

On the speculative side, one could also think of this 256-dim slice of $Cl(16)$-space, as if it were the world-manifold of an extended object moving in a flat 256×256-dim background space. Such extended object corresponds to a $Cl(8)$-space-valued extended object; namely, an extended object spanning the 256 dimensions associated with the C-space corresponding to the 256-dim $Cl(8)$ algebra ($2^8 = 256$). In [3] we described an action which unified extended objects of different intrinsic dimensions (strings, membranes ... p-branes) in one footing by embedding a Clifford-space valued world-manifold of dimensions 2^d into a target C-space background of dimensions 2^D. The Extended Relativity theory in *curved* C-spaces and Phase-Space Clifford spaces can be found in [3, 38].

This slicing of $Cl(16)$-space is compatible with the E_8 algebra being a subalgebra of $Cl(8) \otimes Cl(8)$ and consistent with the $Sl(8, R)$ 7-grading decomposition of $E_{8(8)}$ (with 128 noncompact and 120 compact generators) as shown by [5]. Such $Sl(8, R)$ 7-grading is based on the *diagonal* part $[SO(8) \times SO(8)]_{\text{diag}} \subset SO(16)$ described in full detail by [5] and can be deduced from the $Cl(8) \otimes Cl(8)$ 7-grading decomposition of E_8 provided by Larsson [7].

To construct the Exceptional E_8 Geometry (Gravity) of $Cl(16)$-spaces we take the 256-dim slice by choosing 256 gamma matrices of 16×16 components $(\Upsilon_A)^{\alpha\beta}$, where $A = 1, 2, 3, \ldots, 256$ spans over the 256-dim slice, and $\alpha, \beta = 1, 2, 3, \ldots, 16$ are $SO(8)$ spinorial indices. Such (Υ_A) matrices live in the *diagonal* part of $Cl(16)$: $[Cl(8) \otimes Cl(8)]_{\text{diag}} \subset Cl(16)$. For instance, in $4D$ one has 16 gamma 4×4 matrices spanning the $Cl(4)$ algebra which is $2^4 = 16$-dim. In $8D$ one has 256 gamma 16×16 matrices spanning the $2^8 = 256$-dim $Cl(8)$ algebra.

The generalized spin-connection and vielbein are Ω_M^{AB} and E_M^A, where M is a $Cl(8)$-algebra-valued *polyvector* index spanning 256 degrees of freedom corresponding to the scalar $\mathbf{1}$, vector Υ^μ, bivector $\Upsilon^{\mu\nu}$, trivector $\Upsilon^{\mu\nu\rho}$, ... of the $[Cl(8) \otimes Cl(8)]_{\text{diag}}$ diagonal-subalgebra generators of $Cl(16)$. The 256-dim slice of $Cl(16)$-space is associated with and underlying $8D$ spacetime which is a subspace of the $16D$ spacetime corresponding to the $Cl(16)$ algebra. Once again, one encounters $16D$ as we did in the Chern–Simons E_8 gauge theory of gravity [31]. A, B are the *tangent*-space $Cl(8)$-algebra-valued *polyvector* indices. The spin-connection gauges the *extended* Lorentz symmetries of the tangent space of the 256-dim slice of $Cl(16)$-space. The vielbein E_M^A gauges the (nonabelian) translations in the 256-dim slice of $Cl(16)$-space. The generalized gauge connection is decomposed into a spin-connection and a (nonabelian) translation part as follows

$$\mathcal{A}_M = \Omega_M^{AB}[\Upsilon_A, \Upsilon_B] + \mathcal{A}_M^A \mathcal{P}_A. \tag{2.14a}$$

The 256×256 components of $\mathcal{A}_M^A$ match the 256×256 components of the vielbein E_M^A, hence, by setting the correspondence $\mathcal{A}_M^A \leftrightarrow E_M^A$ it gives

$$\mathcal{A}_M = \Omega_M^{AB}[\Upsilon_A, \Upsilon_B] + E_M^A \mathcal{P}_A. \tag{2.14b}$$

The E_8 generators are *part* of the vielbein one-form $E_M^A \mathcal{P}_A dX^M$. The 256 indices of $A = 1, 2, 3, \ldots, 256$ are spanned by the $\mathcal{P}_A$ generators in the 256-dim tangent space of the 256-dim curved slice of $Cl(16)$-space. The latter indices can be broken into $256 = 248 + 8$. 248 of them are assigned to the nonabelian E_8 gauge symmetries and the remaining 8 indices correspond to the 8 abelian translation generators P_a associated with the 8 translations in $D = 8$. Hence we have

$$\mathcal{P}_A \equiv 248\ E_8 \quad \text{generators for } A = 1, 2, 3, \ldots, 248, \tag{2.15}$$

$$\mathcal{P}_A \equiv 8 P_{a=1,2,3,\ldots,8} \quad \text{generators when } A = 249, 250, \ldots, 256. \tag{2.16}$$

The 248 E_8 generators can be decomposed explicitly in terms of the $sl(8, R)$ 7-grading of $E_{8(8)}$ as shown in Eq. (2.8) [5], [51] : 8 + 8 vectors Z^a, Z_a; 28 + 28 bivectors $Z^{[ab]}, Z_{[ab]}$; 56 + 56 trivectors $E^{[abc]}, E_{[abc]}$; the tensor E_a^b generator of $SL(8, R)$ with 63 elements and the trace $\mathcal{N} = Y^{cc}$ of the tensor Y^{ab} as shown in Eqs. (2.8). The $GL(8, R)$ subalgebra with $64 = 63 + 1$ generators is comprised of E_a^b and $\mathcal{N}$. Therefore we may write the E_8 sector (with 248 generators) of the 256 $\mathcal{P}_A$ generators of the 256-dim slice as follows

$$\begin{aligned} \mathcal{A}_M^{(E_8)} &= \mathcal{E}_M^a Z_a + \mathcal{E}_{M,a} Z^a + \mathcal{E}_{M,[ab]} Z^{[ab]} + \mathcal{E}_M^{[ab]} Z_{[ab]} \\ &\quad + \mathcal{E}_{M,[abc]} E^{[abc]} + \mathcal{E}_M^{[abc]} E_{[abc]} + \mathcal{E}_{M,b}^a E_a^b + \mathcal{E}_M \mathcal{N}. \end{aligned} \tag{2.17}$$

The remaining contribution (to the 256 generators of the vielbein) from the 8 abelian translation generators P_a are written as $E_M^a P_a$. Hence, one has a *triad* of vector generators Z^a, Z_a and P_a compatible with the *triality* property of $SO(8)$. The vector generators Z^a, Z_a of $E_{8(8)}$ decompose into the following $SL(8, R)$ representations as follows [51]

$$Z^a = \frac{1}{4} \Gamma^a_{\alpha\dot{\alpha}} (X^{\alpha\dot{\alpha}} + Y^{\alpha\dot{\alpha}}); \quad Z_a = -\frac{1}{4} \Gamma^a_{\alpha\dot{\alpha}} (X^{\alpha\dot{\alpha}} - Y^{\alpha\dot{\alpha}}). \tag{2.18a}$$

where the 120 $SO(16)$ bivectors X^{IJ} are decomposed in terms of $X^{[\alpha\beta]}$, $X^{[\dot{\alpha}\dot{\beta}]}$ and $X^{\alpha\dot{\beta}}$ with $28 + 28 + 64 = 120$ components, respectively. The commutators are [51]

$$[Z^a, Z^b] = 0; \quad [Z_a, Z_b] = 0, \quad [Z_a, Z^b] = E_a^b - \frac{3}{8} \delta_a^b \mathcal{N}. \tag{2.18b}$$

where $[P_a, P_b] = 0$ and the commutators of the E_8 generators with the abelian translations P_a are all $[E_8, P_a] = 0$. Since the commutator of $[\Upsilon_i, \Upsilon_j] \neq 0$, and the E_8 group is also nonabelian, we have in general *nonabelian* generalized translations $[\mathcal{P}_A, \mathcal{P}_B] \neq 0$. A Nonabelian complex gravity in Phase spaces [41], involving symmetric and anti-symmetric metrics, was instrumental in the construction of a General Relativity theory based on Born's Reciprocity principle (of maximal speeds and maximal proper forces) and gauging Low's Quaplectic group [42].

The generalized curvature and torsion two-forms in C-space associated with the spin connection and vielbein one-forms

$$\mathbf{\Omega}^{AB} \equiv \Omega^{AB}_M dX^M, \quad \mathbf{E}^A \equiv E^A_M dX^M \tag{2.19}$$

are

$$\mathcal{R}^{AB}_{MN} dX^M \wedge dX^N = \mathbf{R}^{AB} = \mathbf{d}\mathbf{\Omega}^{AB} + \mathbf{\Omega}^A_C \wedge \mathbf{\Omega}^{CB}, \tag{2.20a}$$

$$\mathcal{T}^A_{MN} dX^M \wedge dX^N = \mathbf{T}^A = \mathbf{d}\mathbf{E}^A + \mathbf{\Omega}^A_B \wedge \mathbf{E}^B + \mathbf{E}^B \wedge \mathbf{E}^C f^A_{BC}. \tag{2.20b}$$

The structure constants f^A_{BC} associated with the $E_8 \times P_8$ algebra ($248 + 8 = 256$ generators) are given in Appendices **A**, **B**, as well as the structure constants of the $\mathcal{J}^{AB}$ algebra involving the $Cl(8)$ algebra generators $\mathcal{J}_{AB} = [\Upsilon_A, \Upsilon_B]$.

Below we shall define the Clifford-space **d** exterior derivative operator, the exterior product, the super-extension of **d**, the ordinary Dirac operator and its super-extension when orthogonal *and* symplectic Clifford algebras are introduced in order to construct a Clifford Superspace, super-connections, super-polyvectors, etc. such that we can assign all bosons and fermions into a single super-Clifford connection and correct the problems of [13]. A true grand unification *requires* a Clifford Superspace involving Supergravity and Super-Yang–Mills theory and which is consistent with superstring, M, F theory. However Clifford Superspaces yields a far richer plethora of tensorial gauge fields (higher spin theories), tensorial coordinates, etc.

The Ricci tensor, Ricci scalar, Torsion tensor, Torsion vector are defined as

$$\mathcal{R}_{MN} = E^P_A \mathcal{R}^{AB}_{MP} E_{BN}; \quad \mathcal{R} = G^{MN} \mathcal{R}_{MN} = \mathcal{R}^{AB}_{MN} E^M_B E^N_A,$$
$$\mathcal{T}_M = \mathcal{T}^A_{MN} E^N_A; \quad \mathcal{T}_{MNP} = \mathcal{T}^A_{MN} E_{AP}, \tag{2.21}$$

where the metric is defined in terms of the vielbein and the 256-dim tangent space metric η_{AB} as

$$G_{MN} \equiv E^A_M E^B_N \eta_{AB}; \quad \eta_{AB} = G_{MN} E^M_A E^N_B; \quad E_{BN} = \eta_{AB} E^A_N. \tag{2.22}$$

The tangent space polyvector-valued indices are $A, B, C, \ldots = 1, 2, \ldots, 256$. The base space polyvector-valued indices are $M, N, P, Q, \ldots = 1, 2, \ldots, 256$. The inverse vielbein E^M_A is defined as $E^M_A E^B_M = \delta^B_A$.

An important remark is in order before continuing. We *should not confuse* this metric G^{MN} with the C-space metric (comprising a line-metric, area-metric, volume-metric, ... hyper-volume metric) [3] that can be written as sums of antisymmetrized products of the underlying $8D$ spacetime metric $g^{\mu\nu}$ as $g^{\mu_1\nu_1} g^{\mu_2\nu_2} \cdots g^{\mu_n\nu_n}$ + signed permutations. It can also be expressed in terms of the determinant of the $n \times n$ *block* matrix $\mathbf{G}^{IJ}$ whose entries are the metric elements $g^{\mu_i\nu_j}$. Notice that the Exceptional geometry we are constructing involves *nonabelian* translations (it is "noncommutative") as opposed to the traditional abelian translations in the tangent spacetime.

In the traditional description of C-spaces [3] there is one component of the C-space metric $G^{\text{scalar,scalar}} = \Phi$ corresponding the scalar element of the Clifford algebra that must be included as well. Such scalar component is a dilaton-like Jordan–Brans–Dicke scalar field. In [40] we were able to show how Weyl-geometry solves the riddle of the cosmological constant within the context of a Robertson–Friedmann–Lemaitre–Walker cosmology by coupling the Weyl scalar curvature to the Jordan–Brans–Dicke scalar ϕ field with a self-interacting potential $V(\phi)$ and kinetic terms $(\mathcal{D}_\mu\phi)(\mathcal{D}^\mu\phi)$. Upon eliminating the Weyl gauge field of dilations A_μ from its algebraic (nonpropagating) equations of motion, and fixing the Weyl gauge scalings, by setting the scalar field to a constant ϕ_o such that $\phi_o^2 = \frac{1}{16}\pi G$, where G is the present day observed Newtonian constant, we were able to prove that $V(\phi_o) = \frac{3H_o^2}{8\pi G}$ was *precisely* equal to the observed vacuum energy density of the order of $10^{-122}M^4_{\text{Planck}}$. H_o is the present value of the Hubble scale.

After this detour, the Einstein–Hilbert–Cartan action is comprised of scalar curvature plus torsion squared terms

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}^{256}}[d^{(256)}\mathbf{X}]\sqrt{|\det G^{MN}|}[a_1\mathcal{R} + a_2\mathcal{T}_M\mathcal{T}^M + a_3\mathcal{T}_{MNP}\mathcal{T}^{MNP}] \quad (2.23a)$$

where the 256-dim measure of integration is defined by

$$[d^{(256)}\mathbf{X}] = d\sigma\prod dx_\mu\prod dx_{\mu_1\mu_2}\prod dx_{\mu_1\mu_2\mu_3}\cdots\prod dx_{\mu_1\mu_2\mu_3\cdots\mu_8} \quad (2.23b)$$

in terms of the 256 components of the polyvector $\mathbf{X}$ which can be expanded in the Υ basis corresponding to the diagonal subalgebra $[Cl(8)\otimes Cl(8)]_{\text{diag}} \subset Cl(16)$ as

$$\mathbf{X} = \sigma\mathbf{1} + x_\mu\Upsilon^\mu + x_{\mu_1\mu_2}\Upsilon^{\mu_1\mu_2} + \cdots x_{\mu_1\mu_2\mu_3\cdots\mu_8}\Upsilon^{\mu_1\mu_2\cdots\mu_8}. \quad (2.23c)$$

In order to match dimensions in the expansion (2.24) one requires to introduce powers of a length scale [3] which we could set equal to the Planck scale and set it to unity. In Clifford Phase Spaces [38] one needs two length scales parameters, a lower and an upper scale. Fermionic matter and scalar-field actions can be constructed in terms of Dirac–Barut–Hestenes spinors as in [3, 54]. A generalized Einstein-Hilbert gravity action in C-spaces was given in [3] where in very special cases the C-space scalar curvature $\mathcal{R}$ admits an expansion in terms of sums of powers of the ordinary scalar curvature R, Riemann curvature $R_{\mu\nu\rho\sigma}$ and Ricci $R_{\mu\nu}$ tensor of the underlying Riemannian spacetime manifold.

An alternative action to the one in Eq. (2.23a) is the one given by the Yang–Mills action associated with the gauge field described by Eq. (2.14a)

$$S \sim \int_{\mathcal{M}^{256}}[d^{(256)}\mathbf{X}]\sqrt{|\det G^{MN}|}\ \text{Trace}\ [(\mathcal{F}^{AB}_{MN}\mathcal{J}_{AB})^2 + (\mathcal{F}^{A}_{MN}P_A)^2]. \quad (2.24)$$

where the field strength $\mathcal{F}^{AB}_{MN} = \mathcal{R}^{AB}_{MN}$ is identified with the curvature, and $\mathcal{F}^{A}_{MN}$ is given by the 248 elements of the E_8 field strength and the 8 extra components associated with the translations.

Do we have an example of the 256-dim slice of $Cl(16)$-space? Let us choose the *ordinary* space (comprised of *vectorial* coordinates) with $2 \times 64 + 2 \times 64 = 128 + 128 = 256$-dimensions

$$\Sigma^{256} = (O \times O)\mathbf{P}^2 \times (O \times O)\mathbf{P}^2 \tag{2.25}$$

where $(O \times O)\mathbf{P}^2$ is the octo-octonionic two-dim projective space whose isometry group is E_8. Since the isometry group of each copy $(O \times O)$ $\mathbf{P}^2$ is E_8, then the isometry group of Σ^{256} is $E_8 \times E_8$ group, which is the familiar symmetry group of the anomaly free Heterotic string in $10D$. The number of generators is $248 + 248 = 496$ and the algebra has rank $8 + 8 = 16$. The $E_8 \times E_8 \subset Cl(16) \times Cl(16) = Cl(32)$. The bivector generators of $Cl(32)$ correspond to the $SO(32)$ group which is associated with the anomaly-free open superstring in $10D$ and also has 496 generators and rank 16. An open question is to find a realization of the space Σ^{256} in terns of *polyvector* coordinates to see if in fact it admits a reinterpretation as the $Cl(8)$-space associated to an underlying $8D$ manifold.

A different kind of Exceptional E_8 Geometry of $11D$ SUGRA (Supergravity) was investigated by [51] based on the formulation of $11D$ SUGRA with a local $SO(16)$ invariance, after enlarging the $SO(2,1) \times SO(8)$ symmetry of the tangent space, after a compactification from $11D$ to $3D$, to the group $SO(2,1) \times SO(16)$, by introducing new gauge degrees of freedom carried by the new field called the Kaluza–Klein vector B_μ^m, with $\mu, \nu = 0, 1, 2$, and $m, n = 3, 4, \ldots, 10$. The ordinary 8 x_m coordinates and additional 28 tensorial (bivectors) $x_{[mn]}$ coordinates were needed. This model is quite different than the one described in this work based on $Cl(16)$-spaces. It is interesting, however, that 28 tensorial $x_{[mn]}$ coordinates and tensorial gauge tranformations were essential features in the construction [5]. Since we have polyvector-valued coordinates comprised of anti-symmetric tensorial coordinates of rank $2, 3, \ldots, 8$ as well, it is warranted to explore further relations between the work of [5] and ours. Generalized Yang–Mills field theories based on tensorial gauge transformations in C-space and extensions of the Standard Model were investigated in [54]. This is the reason why a Clifford Superspace is needed to incorporate Supergravity into the picture as we shall see below.

For a recent E_8 algebraic interpretation in terms of the $Cl(8,8)$ algebra and $SO(8,8)$, see [43]. In [45] we discussed the relationship between an Octonionic string and Octonionic Gravity based on an Octonionic $(1,1)$ world sheet of real dimensions $8 + 8 = 16$, with 8 spatial and 8 temporal dimensions; i.e, the $16D$ space is two-dimensional from the octonionic point of view. The connection stems from the fact that the 16-dim C-space corresponding to the Clifford algebra $Cl(4)$ associated with an underlying a $4D$ space (let us our 4-dim spacetime), is comprised of a basis of $2^4 = 16$ elements given by 1 scalar, 4 vectors, 6 bivectors, 4 axial vectors, and 1 pseudoscalar, and have a one-to-one correspondence to the 16 vectors of $SO(8,8)$.

To finalize, we explain how to construct the exterior Clifford calculus and its supersymmetric extension in Clifford Superspaces. In order to achieve this one needs an orthogonal Clifford algebra $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}\mathbf{1}$, where $\mu, \nu = 1, 2, \ldots, m$, as well as a symplectic Clifford algebra [49] defined by $[\Sigma_{2i-1}, \Sigma_{2j}] = \delta_{ij}\mathbf{1}$, and other relations, where the components of the canonical symplectic two-form in a $2n$-space (say a Phase space), with indices running $i, i = 1, 2, \ldots, n$ can be written as a $2n \times 2n$ antisymmetric matrix with $1, -1$ off the main diagonal. One has now ordinary Dirac spinors as well as symplectic spinors [47, 48].

The bosonic differentials obey $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$, while the Grassmanian ones, defined in terms of anti-commuting coordinates $\{\theta^i, \theta^j\} = 0$, obey $d\theta^i \wedge d\theta^j = d\theta^j \wedge d\theta^i$.

The Cliford space differential is

$$\mathbf{d} = d\sigma \frac{\partial}{\partial \sigma} + dx^\mu \frac{\partial}{\partial x^\mu} + dx^{\mu_1\mu_2} \frac{\partial}{\partial x^{\mu_1\mu_2}} + \cdots dx^{\mu_1\mu_2\cdots\mu_m} \frac{\partial}{\partial x^{\mu_1\mu_2\cdots\mu_m}} \tag{2.26}$$

where the tensorial coordinates are fully antisymmetric in their indices.

The Clifford Superspace differential requires adding the Grassmanian contribution to the bosonic differential

$$d\theta \frac{\partial}{\partial \theta} + d\theta^i \frac{\partial}{\partial \theta^i} + d\theta^{i_1 i_2} \frac{\partial}{\partial \theta^{i_1 i_2}} + \cdots d\theta^{i_1 i_2 \cdots i_{2n}} \frac{\partial}{\partial \theta^{i_1 i_2 \cdots i_{2n}}} \tag{2.27}$$

where the tensorial Grassmanian coordinates are fully symmetric in their indices.

The super-Dirac operator is [49] obtained by adding the Grassmanian contribution to the ordinary Dirac operator

$$\sum_1^m \Gamma^\mu \frac{\partial}{\partial x^\mu} + 2 \sum_1^n \left(\sum^{2j} \frac{\partial}{\partial \theta^{2j-1}} - \sum^{2j-1} \frac{\partial}{\partial \theta^{2j}} \right). \tag{2.28}$$

The superdimension is defined by $m - 2n$. In ordinary $2D$ Conformal Field Theory bosons have a central charge $c = 1$, while fermions have $c = \frac{1}{2}$. If $m - 2n = 0$ this means that we have an equal number of bosonic and fermionic degrees of freedom, which is what occurs in supersymmetric field theories.

We will define the Clifford Superspace extension of the super-Dirac operator by including the tensorial bosonic and tensorial Grassmanian variables. Namely, by having

$$\begin{aligned} \mathbf{D} = \mathbf{1} \frac{\partial}{\partial \sigma} + \Gamma^\mu \frac{\partial}{\partial x^\mu} + \Gamma^{\mu_1\mu_2} \frac{\partial}{\partial x^{\mu_1\mu_2}} + \cdots + \Gamma^{\mu_1\mu_2\cdots\mu_m} \frac{\partial}{\partial x^{\mu_1\mu_2\cdots\mu_m}} \\ + \mathbf{1} \frac{\partial}{\partial \theta} + 2 \left(\sum^{2j} \frac{\partial}{\partial \theta^{2j-1}} - \sum^{2j-1} \frac{\partial}{\partial \theta^{2j}} \right) \\ + 2 \left(\sum^{2j_1, 2j_2} \frac{\partial}{\partial \theta^{2j_1 - 1, 2j_2 - 1}} - \sum^{2j_1 - 1, 2j_2 - 1} \frac{\partial}{\partial \theta^{2j_1, 2j_2}} \right) + \cdots \end{aligned}$$

$$
+2\left(\sum^{2j_1,2j_2,\ldots,2j_n} \frac{\partial}{\partial\theta^{2j_1-1,2j_2-1,\ldots,2j_n-1}}\right.
$$

$$
\left. - \sum^{2j_1-1,2j_2-1,\ldots,2j_n-1} \frac{\partial}{\partial\theta^{2j_1,2j_2,\ldots,2j_n}}\right). \tag{2.29}
$$

where $\Sigma^{i_1 i_2 \cdots i_k}$ are the sums of symmetrized products of the Σ's consistent with the tensorial Grassmanian coordinates being fully symmetric in their indices. This is the reciprocal of the tensorial bosonic coordinates being fully antisymmetric in their indices and the $\Gamma^{\mu_1\mu_2\cdots\mu_k}$ being the sums of antisymmetrized products of the Γ's with unit weight.

The contraction of super-differential operators, the Laplace–Beltrami super-differential operators, the solution to the super-harmonic oscillator, ... for ordinary superspace can be found in [49]. Their construction can be extended to Clifford superspaces by similar methods outlined here. The exterior products of the (Clifford-algebra-valued) spin-connection and vielbein one-forms in Clifford-space given by Eqs. (2.19, 2.20) is

$$
\mathbf{\Omega} \wedge \mathbf{E} = \mathbf{\Omega}^A_C \wedge \mathbf{E}^C \mathcal{P}_A = \Omega^{AB}_M E^C_N [[\Upsilon_A, \Upsilon_B], \mathcal{P}_C] dX^M \wedge dX^N. \tag{2.30a}
$$

$$
\begin{aligned}
\mathbf{\Omega} \wedge \mathbf{\Omega} &= \mathbf{\Omega}^A_C \wedge \mathbf{\Omega}^{CB} [\Upsilon_A, \Upsilon_B] \\
&= \Omega^{AC}_M \Omega^{CB}_N [[\Upsilon_A, \Upsilon_C], [\Upsilon_C, \Upsilon_B]] dX^M \wedge dX^N.
\end{aligned} \tag{2.30b}
$$

$$
\mathbf{d\Omega} = \frac{\partial \Omega^{AB}_N}{\partial X^M} [\Upsilon_A, \Upsilon_B] dX^M \wedge dX^N. \tag{2.30c}
$$

$$
\mathbf{dE} = \frac{\partial E^A_N}{\partial X^M} \mathcal{P}_A dX^M \wedge dX^N. \tag{2.30d}
$$

and can also be constructed in Clifford-superspace by including both orthogonal and symplectic Clifford algebras and generalizing the Clifford super-differential exterior calculus in ordinary superspace [49], to the full fledged Clifford-Superspace outlined here. Notice that the commutators in Eqs. (2.30) are just the polyvector valued extensions of the usual Poincare algebra commutators $[\mathcal{M}_{\mu\nu}, \mathcal{M}_{\rho\sigma}]$ and $[\mathcal{M}_{\mu\nu}, P_\rho]$, when the Lorentz algebra generators are realized in terms of Clifford bivectors as $\mathcal{M}_{\mu\nu} \sim [\gamma_\mu, \gamma_\nu]$. In Clifford spaces in order to evaluate the commutators involving *polyvector* generators requires the computation of all commutators of all of the Clifford-algebra generators $\Gamma^\mu, \Gamma^{\mu_1\mu_2}, \Gamma^{\mu_1\mu_2\mu_3}, \ldots$, see [50]. Generalized orthogonal Clifford algebras; symplectic Clifford algebras as subalgebras of super-Clifford algebras; symplectic Clifford algebraic field theory ... can be found in [47]. The full development of the Clifford Superspace exterior differential calculus will be the subject of further investigation. In particular the construction of generalized Supergravity and Super–Yang–Mills theory in Clifford Superspaces and the Clifford (super) space extensions of (super)twistors.

2.3. *A conformal gravity and standard model unification in 4D from E_8 Yang–Mills in $D = 8$*

Before embarking into this section we must say that the discussion, reasoning and most of the results in this section are different from [7, 13]. There are some overlaps with Smith's model of gravity and particle physics involving $Cl(8)$ algebras. In particular, we avoid the problems encountered in the model [13] by attempting to assign all (massless fields prior to symmetry breaking) gauge bosons, Higgs scalars and matter fermions as elements of a single E_8 gauge connection *without* invoking Supersymmetry nor a Quillen's BRST-like superconnection. Within the realm of the *super* Clifford algebras developed by Dixon [47, 49], based on orthogonal *and* symplectic Clifford algebras, one can accommodate bosonic and fermionic gauge degrees of freedom, as well as scalar and fermionic matter, into a single super-Clifford-algebra *polyvector*-valued connection, by choosing a sufficiently large algebra. An example of this assignment of many fields within a single Clifford-algebra-valued *polyvector*-connection has been discussed in detail by [54]. Polyvectors contain scalars, pseudoscalars, vectors, axial-vectors, bivectors (antisymmetric tensors of rank 2), . . . thus one can have scalar matter *and* vector gauge bosons within a polyvector. In Clifford-superspace one incorporates fermionic matter, gauginos, gravitinos, . . . as well.

As mentioned above, orthogonal *and* symplectic Clifford algebras have been widely used by [49] to develop a Clifford algebraic formulation of Superspaces present in Supersymmetry and Supergravity. In this fashion one can now accommodate bosons and fermions into a single super-Clifford-algebra polyvector-valued-connection avoiding the problems of [13]. The tetrad e_μ^m, gravitino Ψ_μ, photon, photino, Yang–Mills gauge fields, gauginos, the scalars Higgs, Higgsinos, . . . all can now be assembled into the super-Clifford-algebra polyvector-valued connection. Ordinary supersymmetry rotates elements within a given multiplet. In a scalar supermultiplet, scalars (spin 0) and fermions (spin $\frac{1}{2}$) are rotated into each other. In a vector multiplet, one has the gauge field (spin 1) rotated into a gaugino (spin $\frac{1}{2}$). In the graviton multiplet, the graviton (spin 2) is rotated into a gravitino (spin $\frac{3}{2}$). In Clifford Superspaces, however, *all* these fields from *all* these multiplets are encoded into a single super-Clifford *polyvector* multiplet and can be rotated into each other under polyvector-valued extensions of SUSY.

A Polyvector-valued extension of ordinary Poincare super-algebras in connection to the M, F theory super-algebras [14], involving tensorial antisymmetric charges and based on Clifford spaces was studied by [15]. Dixon [47] has also recurred to an algebraic design in Nature and unification based on the four Division algebras Real, Complex, Quaternions and Octonions. An Ashtekar formulation of Gravity in $8D$ using the octonionic structure constants has been attained by [39]. For complex, quaternionic and octonionic gravity see [46] and references therein.

After this preamble, as discussed in the introduction, the authors have analyzed in detail [29, 30] the symmetry breaking of $E_8 \to SO(10) \times SU(4)$, where $SO(10)$

is the GUT group and $SU(4)$ is the four families (plus four mirror families) unification group. The symmetry breaking leaves 128 ($i = 1, 2, 3, \ldots, 128$) *massless* (chiral) spinors Ψ^i_α (out of the initial 248 since 120 have acquired large masses). It leaves 60 gauge bosons *massless* and 188 *massive*. The remaining 60 unbroken gauge symmetries admit the $SO(10) \times SU(4)$ decomposition given by $(45, 1) + (1, 15)$ and, as expected, the 45 gauge bosons are assigned to the adjoint representation of the GUT group $SO(10)$, and 15 gauge bosons are associated with the gauging of the $SU(4)$ family group.

The analysis of [29, 30] was restricted to $4D$. If one begins with a E_8 Yang–Mills in $16D$ the number of degrees of freedom is far larger. By having 128 massless spinors, since each chiral spinor in $16D$ is comprised of 128 components, in the dimensional reduction process $16D \to 4D$, one would end up with a plethora of 4×64 fermion families, plus 4×64 mirror fermion families, of two-component (left-handed and right-handed) Weyl spinors in $4D$ assigned to the 16-dim chiral (anti-chiral) spinor representation of $SO(10)$. One would argue that this leaves us with too many families, a 64-fold increase ... For this reason one would have to freeze (truncate to zero) a large number of degrees of freedom of most of the fermions (and scalars emerging from the gauge bosons) in the dimensional reduction process, in order to end up with Conformal Gravity and a $SO(10) \times SU(4)$ Yang–Mills theory interacting with 4 fermion families (plus their 4 mirror families) in $4D$. Another possibility that one can envision is to find a mechanism, through the symmetry breaking process from $SO(10) \times SU(4)$ to the Standard Model group, that brings about large masses for *most* of the fermion families, except for 3 or 4 light ones at lower energies which is compatible with what is observed.

Despite that a large number of families might destroy perturbative unification, asymptotic freedom, ... there is nothing wrong, in principle, after the symmetry breaking process (there are other different symmetry breaking branches, we choose one in particular)

$$E_8 \to SO(10) \to SU(5) \to SU(3)_c \times SU(2)_L \times U(1)_Y \to SU(3)_c \times U(1)_{EM} \tag{2.30}$$

to end up with a large number of massive fermions, which in turn, can themselves be regrouped into 64 hierarchical layers of increasing mass, with 4 families in each layer, from ordinary energies *all* the way to the GUT scale M_{GUT} and Planck energy scale 10^{19} Gev. As we go from lower to higher energies, the 64 hierarchical layers of fermions begin to unfold. Hence, in this scenario there would not be a "desert" between lower energies and the GUT and Planck scale. Therefore, at first hand we should not disregard this possibility. Of course, to avoid the plethora of particles, and to simplify matters, it is more natural to start with an ordinary E_8 Yang–Mills in $4D$ which must *not be confused* with the E_8 Generalized Yang–Mills theory associated with the Chern–Simons E_8 Gauge theory of Gravity in higher dimensions involving polynomials in the E_8 field strength, Riemann curvature and torsion [31] and discussed in the previous section.

The GUT group $SO(10)$ is very physically appealing for many reasons as stated in [32]. In particular, it admits two physically relevant branchings [32]

$$E_8 \to SO(10) \times SU(4) \to \\ SO(10) \to SO(6) \times SO(4) \sim SU(4) \times SU(2)_L \times SU(2)_R \tag{2.31}$$

leading at the end of the chain to the Patti–Salam unification group. There is also the $SU(5)$ GUT model branch

$$E_8 \to SO(10) \times SU(4) \to SO(10) \to \\ SU(5) \times U(1) \to [SU(3)_c \times SU(2)_L \times U(1)_Y] \times U(1)_{B-L} \tag{2.32}$$

where the brackets include the Standard Model group and $B - L$ denotes baryon minus lepton number.

Starting now from an ordinary E_8 Yang–Mills gauge field theory in $8D$ we shall follow two different channels of symmetry breaking of the $E_{8(-24)}$ algebra with 112 noncompact and 136 compact generators such that the character of the real form is $122 - 136 = -24$. The first channel is obtained by finding the judicious subgroups H of $E_{8(-24)}, E_{7(-5)}, E_{6(-14)}$, respectively, from the table of the classification of real forms and cosets $\frac{G}{H}$ of exceptional groups in [33] and it leads to

$$E_{8(-24)} \to E_{7(-5)} \times SU(2) \to E_{6(-14)} \times U(1) \to \\ SO(8,2) \times U(1) \to SO(8,2). \tag{2.33}$$

After 4 successive symmetry breakings, and following closely the tables in [33], one finally has reached the sought after $8D$ Conformal group $SO(8,2)$ appearing in the last term of the sequence and which will furnish a Conformal Gravitational theory in $8D$ after *gauging* the Conformal group. At this stage of matters we shall not be concerned about the details of the 4 symmetry breaking mechanisms from the beginning of the chain of symmetries to the end. At the moment we are only concerned with the *algebraic* group structures present in the branching chain of group symmetries in (2.29).

The next step is to recur to the Kaluza–Klein compactification process of the $8D$ Conformal Gravity theory down to $4D$. Contrary to the lore that it is not possible to obtain the Standard Model group $SU(3) \times SU(2) \times U(1)$ in $4D$ directly from a Kaluza–Klein compactification of Gravity from $8D$ to $4D$, (higher dimensions than $D = 8$ were thought to be needed to attain this goal) Batakis [11] uncovered an extra $SU(2) \times U(1)$ gauge field structure to the $SU(3)$ gauge field structure, from a Kaluza–Klein compactification process of the form $\mathcal{M}^8 \to \mathcal{M}^4 \times CP^2$, provided a nontrivial *torsion* in the total space is incorporated. Such *torsion* creates a new possibility for the construction of a unified theory in $8D$ not envisioned before. In particular, C-spaces have torsion [3]. Therefore, a compactification of the $8D$ theory down to $4D$ along the internal space CP^2 will lead to a (Conformal)

Gravitational *and* Yang–Mills theory in $4D$ based on the Standard Model group $SU(3)_c \times SU(2)_L \times U(1)_Y$. Therefore, starting from an E_8 ordinary Yang–Mills in $8D$, after a sequence of symmetry breaking processes, and performing a Kaluza–Klein–Batakis compactification on CP^2, we are able to recover a Gravity–Yang–Mills theory based on the Standard Model.

Another channel of symmetry breaking processes is

$$E_{8(-24)} \to E_{6(-14)} \times SU(3) \to SO(8,2) \times U(1) \to SO(8,2). \tag{2.34}$$

As stated in the introduction, the $SU(3)$ in the first stage of the symmetry breaking process is the 3 fermion family unification group as it occurs in the Heterotic string theory in $10D$, with 3 fermion families (and their mirrors) assigned to the 27 ($\bar{2}7$)-dim representation of E_6:

$$248 = (1,8) + (78,1) + (27,3) + (\bar{2}7, \bar{3}). \tag{2.35}$$

Once again, at the end of the chain (2.34) we recover the conformal group $SO(8,2)$ in $8D$. Gauging the conformal group yields $8D$ Conformal gravity and a Kaluza–Klein compactification from $8D \to 4D$ along an internal CP^2 space, a la Batakis [11], yields a $4D$ (Conformal) Gravity and a Yang–Mills theory based on the Standard Model group $SU(3)_c \times SU(2)_L \times U(1)_Y$. In this respect, the results (not our arguments) of this section are very similar to those of Smith's model of gravity and particle physics which includes the determination of coupling constants, particle masses, mixing angles, etc. [7]. A thorough analysis of the connections between Smith's model and Clifford Phase spaces associated with $8D$ was presented in [38, 54].

The *quasi* conformal groups (with *four* times), like the $SO(12,4)$ in $12D$ do arise in the $E_{8(-24)}$ symmetry breaking chain. It is one of the subgroups with 48 noncompact generators and 72 compact ones. $12D$ is the dimensions of Vafa's F-theory. The *quasi* conformal groups like $SO(12,4)$ have been studied by Gunaydin *et al.* [29]. To our knowledge we are not aware of anyone gauging the *quasi*-conformal group in order to obtain a *quasi*-conformal theory of Gravity in $12D$. By imposing suitable constraints; partially fixing some of the gauge symmetries and gauging the remaining symmetries we should be able to recover the ordinary Gravitational theory in $12D$ based on gauging the Poincare Group $ISO(D-1,1)$, the semi-direct product of the Lorentz group $SO(D-1,1)$ with the Translation group T_D in D-dim. Bars for the past years has studied in depth the *two* times 2T physics based on the conformal groups $SO(D,2)$ [36].

Metric affine theories of Gravity developed by Ne'eman and collaborators are based in gauging the semi-direct product of $GL(D,R)$ with the translation group in D-dim. $GL(D,R)$ admits *infinite*-dimensional spinorial representations but *not* finite-dimensional ones. $GL(D,R)$ spinors have an infinite number of components. For this reason one cannot realize the $GL(D,R)$ infinite-component spinors in terms of left/right ideal elements of the $Cl(D,R)$ algebra. One would require to have the $D \to \infty$ limit $Cl(D,R) = Cl(8) \times CL(8) \times \cdots$ by exploiting the modulo

8 periodicity of real Clifford algebras. Complex Clifford algebras have a modulo 2 periodicity. This infinite product of $Cl(8)$ has been discussed by Smith [7] in relation to von Neumann's type II_1 algebras. Ogievetsky has shown long ago that the algebra of $gl(D, R)$ and the conformal algebra $so(D, 2)$ does *not* close. However, upon taking an infinite succession of nested commutators, one generates the infinite-dim *diffeomorphisms* algebra in D-dim. This is a fundamental result because it implies that Einstein gravity may be an emergent theory after a symmetry breaking process. For example, the metric tensor $g_{\mu\nu}$ in D-dim has $D(D+1)/2$ components which match the dimensions of the noncompact coset space $GL(D, R)/SO(D, R)$.

To conclude Sec. **2**. In Subsec. **2.1** we have explained why the Chern–Simons E_8 Gauge theory of Gravity [31] *is a unified* field theory at the *Planck* scale of a Lanczos–Lovelock Gravity (LL) and a E_8 Generalized Yang–Mills (GYM) field theory and defined in the $15D$ boundary of a 16D bulk space. The dimensional reduction from $16D \to 4D$ yields ordinary Gravity *and* a E_8 Yang–Mills defined in the $3D$ boundary of a $4D$ bulk space after a freezing (truncation) of degrees freedom. Thus we have a Gravity-E_8 Yang–Mills unification at the *Planck* scale in $4D$. The rigorous details of the reduction of Chern–Simons gravitational theories from higher to lower dimensions was presented by [37].

In Subsec. **2.2** we explained the nature of the Exceptional E_8 Geometry associated to the 256-dim "*curved*" slice of the 256×256-dimensional flat $Cl(16)$ space. A more general E_8 gauge theory of gravity in such a 256-dim curved slice is obtained by introducing the spin connection Ω_M^{AB}, that gauges the generalized Lorentz transformations in the tangent space of the 256-dim curved slice; $M = 1, 2, 3, \ldots, 256$; $A, B = 1, 2, 3, \ldots, 256$. There are in addition 256×256 components of the vielbein one-form $E_M^A \mathcal{P}_A dX^M$, which encode in one-scoop, all the 248 E_8 gauge fields and 8 additional translations associated with the *vectorial* parts of the generators Υ_i; $i = 1, 2, 3, \ldots, 8$ of the diagonal subalgebra $[Cl(8) \otimes Cl(8)]_{\text{diag}} \subset Cl(16)$. Therefore, the E_8 gauge symmetry is just *part* of the $248 + 8 = 256$ generalized nonabelian translations along the tangent space of the 256-dim slice of the $Cl(16)$-space. We constructed the curvature, Ricci tensor and Ricci scalar; the torsion tensor and torsion vector, and finally we displayed the Einstein–Hilbert–Cartan action that represents a generalized Exceptional E_8 theory of gravity corresponding to the 256-dim curved slice in terms of its underlying "diagonal" embedding into a flat $Cl(16)$-space background. Finally we described how to construct a Clifford Superspace based on orthogonal, symplectic Clifford algebras (subalgebras of the super-Clifford algebras [47]) and extending the Clifford analysis approach to superspace of [49] to the one involving polyvectors.

At the beginning of Sec. **2.3** we reviewed briefly the E_8 Yang–Mills theory in $4D$, following the Bars–Gunaydin–Barr detailed analysis [29, 30] of the symmetry breaking process $E_{8(8)} \to SO(16) \to SO(10) \times SU(4)$ and leading to a $SO(10)$ GUT group and $SU(4)$ family unification group (4 fermion families plus 4 mirror families). Bars–Gunaydin and Barr, respectively, have analyzed in full detail how a symmetry breaking of $SO(10)$ and E_6 leaves only 3 *light* families and a super-heavy fourth family, which is what is observed. Two branchings of the $SO(10)$ GUT are possible furnishing the Patti–Salam and/or the Standard Model group at low energies.

At the end of Sec. **2.3**, we have explained how an E_8 ordinary Yang–Mills in $8D$, after a sequence of symmetry breaking processes $E_8 \to E_7 \to E_6 \to SO(8,2)$, and performing a Kaluza–Klein–Batakis compactification on CP^2, involving a nontrivial *torsion*, one is able to recover a (Conformal) Gravity *and* Yang–Mills theory based on the Standard Model in $4D$. These end result agrees with Smith's $Cl(8)$ model of gravity and particle physics [7].

3. Conclusion: Conformal Gravity–Yang–Mills Unification from Clifford Gauge Field Theory

It is known that within the framework of Quantum Field Theory (QFT), the Coleman-Mandula theorem requires that if symmetries are to be described in terms of Lie algebras, the symmetries of the proposed Grand Unified Field theory must be based on the direct product of the Poincare group with the internal symmetry group G. Haag-Lopuszanski-Sohnius extended the Coleman-Mandula result to Supersymmetric Quantum Field Theories by introducing spinorial generators. Nevertheless, higher order algebraic extensions of the Poincare and Clifford algebras in QFT, like ternary and higher order algebras, have been proposed more recently as another possibility, see [55] and references therein. Since the Poincare group is a natural subgroup of the Conformal group, we begin this conclusion by showing how Conformal Gravity can be obtained by gauging the Conformal group and, which in turn, is a very small sector of a more general Clifford algebra-valued gauge field theory.

Let us construct the Clifford $C(16)$ gauge field theory by writing the $Cl(16)$-valued gauge field

$$\begin{aligned}\mathbf{A}_\mu = \mathcal{A}_\mu^A \Gamma_A = \mathcal{A}_\mu \mathbf{1} + \mathcal{A}_\mu^a \Gamma_a + \mathcal{A}_\mu^{a_1 a_2} \Gamma_{a_1 a_2} + \mathcal{A}_\mu^{a_1 a_2 a_3} \Gamma_{a_1 a_2 a_3} + \cdots \\ + \mathcal{A}_\mu^{a_1 a_2 \cdots a_{16}} \Gamma_{a_1 a_2 \cdots a_{16}} \end{aligned} \quad (3.1)$$

and the $Cl(16)$-algebra-valued field strength (*omitting* numerical coefficients attached to the Γ's) is

$$\begin{aligned}\mathcal{F}_{\mu\nu}^A \Gamma_A = {} & \partial_{[\mu} A_{\nu]} \mathbf{1} + [\partial_{[\mu} A_{\nu]}^a + A_{[\mu}^{b_2} A_{\nu]}^{b_1 a} \eta_{b_1 b_2} + \cdots] \Gamma_a \\ & + [\partial_{[\mu} A_{\nu]}^{ab} + A_{[\mu}^a A_{\nu]}^b - A_{[\mu}^{a_1 a} A_{\nu]}^{b_1 b} \eta_{a_1 b_1} - A_{[\mu}^{a_1 a_2 a} A_{\nu]}^{b_1 b_2 b} \eta_{a_1 b_1 a_2 b_2} + \cdots] \Gamma_{ab} \\ & + [\partial_{[\mu} A_{\nu]}^{abc} + A_{[\mu}^{a_1 a} A_{\nu]}^{b_1 bc} \eta_{a_1 b_1} + \cdots] \Gamma_{abc} \\ & + [\partial_{[\mu} A_{\nu]}^{abcd} - A_{[\mu}^{a_1 a} A_{\nu]}^{b_1 bcd} \eta_{a_1 b_1} + \cdots] \Gamma_{abcd} \\ & + [\partial_{[\mu} A_{\nu]}^{a_1 a_2 \cdots a_5 b_1 b_2 \cdots b_5} + A_{[\mu}^{a_1 a_2 \cdots a_5} A_{\nu]}^{b_1 b_2 \cdots b_5} + \cdots] \Gamma_{a_1 a_2 \cdots a_5 b_1 b_2 \cdots b_5} + \cdots \end{aligned} \quad (3.2)$$

and is obtained from the evaluation of the commutators of the Clifford-algebra generators appearing in (3.1). The most general formulae for all commutators and anti-commutators of $\Gamma^\mu, \Gamma^{\mu_1\mu_2}, \ldots$, with the appropriate numerical coefficients, can be found in [50].

Therefore, E_8 Yang–Mills actions (are part of) can be embedded onto more general $Cl(16)$ actions with a much larger number of terms given by

$$S_{\text{Topological}}[Cl(16)] = \int d^4x \langle F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} \Gamma_{\mathcal{A}} \Gamma_{\mathcal{B}} \rangle_{\text{scalar}} \epsilon^{\mu\nu\rho\tau}$$

$$= \int d^4x F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} G_{\mathcal{AB}} \epsilon^{\mu\nu\rho\tau} \tag{3.3}$$

and

$$S_{YM}[Cl(16)] = \int d^4x \sqrt{g} \langle F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} \Gamma_{\mathcal{A}} \Gamma_{\mathcal{B}} \rangle_{\text{scalar}} g^{\mu\rho} g^{\nu\tau}$$

$$= \int d^4x \sqrt{g} F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} G_{\mathcal{AB}} g^{\mu\rho} g^{\nu\tau} \tag{3.4}$$

where $\langle \Gamma_{\mathcal{A}} \Gamma_{\mathcal{B}} \rangle = G_{\mathcal{AB}} \mathbf{1}$ denotes the *scalar* part of the Clifford geometric product of the gammas Γ. Notice that there are a total of 65536 terms in

$$F^{\mathcal{A}}_{\mu\nu} F^{\mathcal{B}}_{\rho\tau} G_{\mathcal{AB}} = F_{\mu\nu} F_{\rho\tau} + F^a_{\mu\nu} F^a_{\rho\tau} + F^{a_1a_2}_{\mu\nu} F^{a_1a_2}_{\rho\tau}$$

$$+ \cdots + F^{a_1a_2\cdots a_{16}}_{\mu\nu} F^{a_1a_2\cdots a_{16}}_{\rho\tau} \tag{3.5}$$

where the indices run as $a = 1, 2, \ldots, 16$. The Clifford algebra $Cl(16) = Cl(8) \otimes Cl(8)$ has the graded structure (scalars, bivectors, trivectors, ..., pseudoscalar) given by

$$\begin{array}{c} 1\ 16\ 120\ 560\ 1820\ 4368\ 8008\ 11440\ 12870 \\ 11440\ 8008\ 4368\ 1820\ 560\ 120\ 16\ 1 \end{array} \tag{3.6}$$

consistent with the dimension of the $Cl(16)$ algebra $2^{16} = 256 \times 256 = 65536$.

We proceed by showing how the conformal algebra $so(D, 2)$ in D-dimensions and the $so(D + 2)$ algebras are particular subalgebras of the $Cl(D - 1, 1)$ and $Cl(D)$ algebras, respectively. We may focus on the $Cl(7, 1), Cl(8)$ algebras as examples. A thorough discussion of Conformal transformations in ordinary spacetime from the Clifford space perspective can be found in [3].

The $8D$ Lorentz $SO(7, 1)$ is a subalgebra of $Cl(7, 1)$ which may be realized in terms of the 28 Clifford *bivector* generators $\mathcal{M}_{\mu\nu} = -\frac{i}{4}[\Gamma_\mu, \Gamma_\nu]$ when $\mu, \nu = 1, 2, 3, \ldots, 8$. The Clifford $Cl(7, 1)$ algebra is defined in terms of the anti-commutators

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}; \quad \Gamma^9 = \Gamma^1\Gamma^2\Gamma^3 \cdots \Gamma^8, \quad \{\Gamma^\mu, \Gamma^9\} = 0; \quad (\Gamma^9)^2 = -1. \tag{3.7}$$

A Euclidean signature Clifford algebra $Cl(8)$ would lead instead to $(\Gamma^9)^2 = 1$.

It is now when we borrow the results in [3] showing why the $8D$ Conformal algebra $SO(8, 2)$ (that contains the Lorentz algebra $SO(7, 1)$) is just a *subalgebra* of the Clifford algebra $Cl(7,1)$ after introducing the following realization of the Conformal Group in $D = 8$, $\mu, \nu = 1, 2, 3, \ldots, 7, 8$. The translation generator in $D = 8$ is

$$P_\mu = -\frac{1}{2}\Gamma_\mu(\mathbf{1} - \Gamma_9) \tag{3.8}$$

and can be interpreted as a linear combination of the rotation generators $\mathcal{M}_{\mu 9} + \mathcal{M}_{\mu 10}$, which from the C-space perspective involve the 8 Γ^μ generators, the unit $\mathbf{1}$ element of the $Cl(7,1)$ algebra which represents a particular direction in C-space, and the antisymmetrized product of all gammas with unit weight $\Gamma^{[123\cdots 8]} \sim \Gamma^9\epsilon^{123\cdots 8}$, that represents another direction in C-space [3], such that the effective number of directions involved in C-space is $8+2 = 10$, consistent with the fact that the Conformal group in $8D$ is $SO(8,2)$.

The Conformal boost generator is

$$K_\mu = -\frac{1}{2}\Gamma_\mu(\mathbf{1} + \Gamma_9) \tag{3.9}$$

and can be interpreted as the other independent linear combination of the rotation generators $\mathcal{M}_{\mu 9} - \mathcal{M}_{\mu 10}$. The Dilation generator is

$$D = -\frac{1}{2}\Gamma_9 \tag{3.10}$$

and can be interpreted as rotation $\mathcal{M}_{9,10}$. The Lorentz generators are

$$\mathcal{M}_{\mu\nu} = -\frac{1}{4}[\Gamma_\mu, \Gamma_\nu] \tag{3.11}$$

and can be interpreted as the usual rotations (boosts) around the axes perpendicular to the $x_\mu - x_\nu$ planes.

Equipped with a Clifford algebraic realization of the (anti-Hermitian) generators and after recurring to the $Cl(7,1)$ algebraic relations, it is a straightforward exercise to find

$$[\mathcal{M}_{\mu\nu}, \mathcal{M}_{\rho\sigma}] = -g_{\nu\sigma}\mathcal{M}_{\mu\rho} + g_{\nu\rho}\mathcal{M}_{\mu\sigma} + g_{\mu\sigma}\mathcal{M}_{\nu\rho} - g_{\mu\rho}\mathcal{M}_{\nu\sigma}. \tag{3.12}$$

$$[\mathcal{M}_{\mu\nu}, P_\rho] = g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu; \quad [\mathcal{M}_{\mu\nu}, K_\rho] = g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu. \tag{3.13}$$

$$[P_\mu, P_\nu] = [K_\mu, K_\nu] = 0; \quad [K_\mu, P_\nu] = -2(g_{\mu\nu}D + M_{\mu\nu}). \tag{3.14}$$

$$[K_\mu, D] = -K_\mu; \quad [P_\mu, D] = P_\mu. \tag{3.15}$$

Notice that despite that the generators of the $8D$ Conformal Algebra $SO(8,2)$ can be recast trivially in terms of the 45 *bivectors* of the $Cl(8,2)$ algebra, the *key* result of the above Eqs. (3.8)–(3.11) is that it allows us to recast the conformal $SO(8,2)$ algebra Eqs. (3.12)–(3.15) in $D = 8$ *directly as* a *subalgebra* of the algebra associated to the $Cl(7,1)$ group which is $2^D = 2^8 = 256$-dimensional. The $Cl(7,1)$ group is spanned by the antisymmetrized products of the $2^8 = 256$ generators

$$\mathbf{1}, \Gamma^\mu, \Gamma^{\mu_1} \wedge \Gamma^{\mu_2}, \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3}, \ldots, \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \cdots \Gamma^{\mu_8}. \tag{3.16}$$

Similar results follow for the Euclidean Clifford algebra $Cl(8)$ defined in terms of the generators $\Upsilon_i, i = 1, 2, 3, \ldots, 8$ as

$$\{\Upsilon^i, \Upsilon^j\} = 2\delta^{ij}\mathbf{1}; \quad \Upsilon^9 = \Upsilon^1\Upsilon^2\Upsilon^3\cdots\Upsilon^8, \quad \{\Upsilon^i, \Upsilon^9\} = 0; \quad (\Upsilon^9)^2 = 1. \quad (3.17)$$

One may construct the $SO(10)$ algebra realized in terms of the Υ^1 generators in the same way.

Gamma matrices can be used also to find a realization of Superconformal algebras. In particular the realization of the superconformal algebra $SU(2,2|1)$ in $D = 4$ can be found in [32]. The superconformal algebra $SU(2,2|1)$ leaves the superspace metric invariant

$$ds^2 = dz_\mu d\bar{z}^\mu + d\theta^*_\alpha (C^{-1})_{\alpha\beta} d\theta_\beta \quad (3.18)$$

where C is the charge conjugation matrix.

To sum up, the $so(10)$ algebra is a natural subalgebra of $Cl(8)$, and the $8D$ Conformal algebra $so(8,2)$ is a subalgebra of the $Cl(7,1)$ algebra. It is the $SO(10)$ group which provides one of the GUT groups. While the Conformal group $SO(8,2)$ associated to the 8-dim spacetime is the symmetry group associated with the $8D$ Conformal Gravity. Similar arguments apply to all Conformal groups $SO(D,2)$. In particular, the conformal gravity action in $4D$ reads

$$S = \frac{1}{4g^2}\int_{M^4} d^4x \kappa_{mnpq} F^{mn}_{\mu\nu} F^{pq}_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \quad (3.19)$$

where κ_{mnpq} is the Killing $SO(4,2)$ invariant metric defined in terms of the structure constants of the $so(4,2)$ algebra as

$$\kappa_{mnpq} = f^{[ab]}_{[mn][rs]} f^{[rs]}_{[ab][pq]}. \quad (3.20)$$

It is antisymmetric under the exchange of $m \leftrightarrow n$, $p \leftrightarrow q$ indices and it is symmetric under the exchange of pairs of indices $\kappa_{mnpq} = \kappa_{pqmn}$. One should not use ϵ_{mnpq} to contract internal indices because it is *not* an $SO(4,2)$ invariant tensor. In $8D$ one may construct the following actions

$$S_{YM} = \int_{M^8} [(\mathbf{F} \wedge \mathbf{F}) \wedge {}^*(\mathbf{F} \wedge \mathbf{F})], \quad (3.21)$$

$$S_{\text{Topological}} = \int_{M^8} [\mathbf{F} \wedge \mathbf{F} \wedge \mathbf{F} \wedge \mathbf{F}] \quad (3.22)$$

which are the $8D$ counterparts of the $4D$ actions. In order to contract internal indices once again one needs to construct the proper group-invariant tensors.

The gist of all this discussion is that one can embed the gauge theories based on the conformal group $SO(D,2)$ in D-dim, and the group $SO(D'+2)$, into a single gauge theory based on the Clifford group $Cl(D+D')$ by selecting judiciously the proper values of D, D' and recurring to the modulo 8 periodicity of Clifford algebras defined over the reals. We display some examples:

$$D = 4; \quad \text{since } SO(3,1) \times SO(8) \subset SO(4,2) \times SO(10) \subset Cl(3,1) \otimes Cl(8) \quad (3.23)$$

the $Cl(11,1)$-algebra gauge field theory defined over the $4D$ spacetime (base manifold) contains the desired $4D$ Conformal Gravity (based on $SO(4,2)$) and $SO(10)$ Yang–Mills field theory.

$$D = 4; \quad \text{since } SO(3,1) \times E_8 \subset SO(4,2) \times E_8 \subset Cl(3,1) \otimes Cl(8) \otimes Cl(8) \tag{3.24}$$

the $Cl(19,1)$-algebra gauge field theory defined over the $4D$ spacetime (base manifold) contains the desired $4D$ Conformal Gravity theory (based on $SO(4,2)$) and E_8 Yang–Mills theory.

$$D = 8; \quad SO(7,1) \times SO(8) \subset SO(8,2) \times SO(10) \subset Cl(7,1) \otimes Cl(8) \tag{3.25}$$

the $Cl(15,1)$-algebra gauge field theory defined over the $8D$ spacetime (base manifold) contains $8D$ Conformal Gravity (based on $SO(8,2)$) and the $SO(10)$ Yang–Mills theory.

The main idea of this concluding section is that upon extending this construction to the super-Clifford algebras case, *all* Grand-Unified field theories of Conformal supergravity and Super–Yang–Mills in D-dimensions, with structure groups $G = E_8, E_7, E_6, SO(10), SU(5), \ldots$, may be embedded into a super-Clifford-gauge field theory defined over a D-spacetime after exploiting the modulo 8 periodicity of Clifford algebras. The bosons and fermions are encoded into the various components of the super-Clifford polyvector-valued super-connection. This will be the subject of future investigation.

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Appendix A. The $GL(8,R)$ Decomposition of E_8

We shall reproduce the full appendix in [51] for the convenience of the reader. To recover the $SL(8,R)$ basis of [52], we will further decompose the above representations into representations of the subgroup $SO(8) \equiv (SO(8) \times SO(8))_{\text{diag}} \subset SO(16)$. The indices corresponding to the $\mathbf{8}_v, \mathbf{8}_s$ and $\mathbf{8}_c$ representations of $SO(8)$, respectively, will be denoted by a, α and $\dot{\alpha}$. After a triality rotation the $SO(8)$ vector and spinor representations decompose as

$$\begin{aligned} \mathbf{16} &\to \mathbf{8_s} \oplus \mathbf{8_c} \\ \mathbf{128_s} &\to (\mathbf{8_s} \otimes \mathbf{8_c}) \oplus (\mathbf{8_v} \otimes \mathbf{8_v}) = \mathbf{8_v} \oplus \mathbf{56_v} \oplus \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35_v} \\ \mathbf{128_c} &\to (\mathbf{8_v} \otimes \mathbf{8_s}) \oplus (\mathbf{8_c} \otimes \mathbf{8_v}) = \mathbf{8_s} \oplus \mathbf{56_s} \oplus \mathbf{8_c} \oplus \mathbf{56_c}. \end{aligned} \tag{A.1}$$

respectively. We thus have $I = (\alpha, \dot{\alpha})$ and $A = (\alpha\dot{\beta}, ab)$, and the E_8 generators decompose as

$$X^{[IJ]} \to (X^{[\alpha\beta]}, X^{[\dot{\alpha}\dot{\beta}]}, X^{\alpha\dot{\beta}}); \quad Y^A \to (Y^{\alpha\dot{\alpha}}, Y^{ab}). \tag{A.2}$$

Next we regroup these generators as follows. The 63 generators

$$E_a^b := \frac{1}{8}(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]} + \Gamma^{ab}_{\dot\alpha\dot\beta}X^{[\dot\alpha\dot\beta]}) + Y^{(ab)} - \frac{1}{8}\delta^{ab}Y^{cc}. \tag{A.3}$$

for $1 \leq a, b \leq 8$ span an $SL(8,R)$ subalgebra of E_8. The generator $N := Y^{cc}$ extends this subalgebra to $GL(8,R)$. $\Gamma^{ab}, \Gamma^{abc}, \ldots$ are antisymmetrized products of gammas. The remainder of the E_8 Lie algebra then decomposes into the following representations of $SL(8,R)$:

$$Z^a := \frac{1}{4}\Gamma^a_{\alpha\dot\alpha}(X^{\alpha\dot\alpha} + Y^{\alpha\dot\alpha}). \tag{A.4}$$

$$Z_{[ab]} = Z_{ab} := \frac{1}{8}(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]} - \Gamma^{ab}_{\dot\alpha\dot\beta}X^{[\dot\alpha\dot\beta]}) + Y^{[ab]}. \tag{A.5}$$

$$E^{[abc]} = E^{abc} := -\frac{1}{4}\Gamma^{abc}_{\alpha\dot\alpha}(X^{\alpha\dot\alpha} - Y^{\alpha\dot\alpha}). \tag{A.6}$$

and

$$Z_a := -\frac{1}{4}\Gamma^a_{\alpha\dot\alpha}(X^{\alpha\dot\alpha} - Y^{\alpha\dot\alpha}). \tag{A.7}$$

$$Z^{[ab]} = Z^{ab} := -\frac{1}{8}(\Gamma^{ab}_{\alpha\beta}X^{[\alpha\beta]} - \Gamma^{ab}_{\dot\alpha\dot\beta}X^{[\dot\alpha\dot\beta]}) + Y^{[ab]}. \tag{A.8}$$

$$E_{[abc]} = E_{abc} := -\frac{1}{4}\Gamma^{abc}_{\alpha\dot\alpha}(X^{\alpha\dot\alpha} + Y^{\alpha\dot\alpha}). \tag{A.9}$$

It is important to emphasize that $Z_a \neq \eta_{ab}Z^b$, $Z_{ab} \neq \eta_{ac}\eta_{db}Z^{cd}, \ldots$ for these reasons one should use the following notation for the generators

$$\mathcal{Z}^a_\pm \equiv (Z^a, Z_a); \quad \mathcal{Z}^{ab}_\pm \equiv (Z^{ab}, Z_{ab}); \quad \mathcal{Z}^{abc}_\pm \equiv (E^{abc}, E_{abc}). \tag{A.10a}$$

and

$$\mathcal{E}^{ab} = \mathcal{E}^{(ab)} + \mathcal{E}^{[ab]} \equiv E_a^b. \tag{A.10b}$$

in Eq. (A.3).

The Cartan subalgebra is spanned by the diagonal elements $E_1^1, \ldots, E_7^7$ and N, or, equivalently, by $Y^{11}, \ldots, Y^{88}$. The elements E_a^b for $a < b$ (or $a > b$) together with the elements for $a < b < c$ generate the Borel subalgebra of E_8 associated with the positive (negative) roots of E_8. Furthermore, these generators are graded w.r.t. the number of times the root α_8 (corresponding to the element N in the Cartan subalgebra) appears, such that for any basis generator X we have $[N, X] = deg(X) \cdot X$.

The degree can be read off from

$$[N, Z^a] = 3Z^a, \ \ [N, Z_a] = -3Z_a, \ \ [N, Z_{ab}] = 2Z_{ab}; \quad [N, Z^{ab}] = -2Z^{ab}$$
$$[N, E^{abc}] = E^{abc}, \ \ [N, E_{abc}] = -E_{abc}; \quad [N, E_a^b] = 0. \tag{A.11}$$

The remaining commutation relations are given by

$$[Z^a, Z^b] = 0; \quad [Z_a, Z_b] = 0; \quad [Z_a, Z^b] = E_a^b - \frac{3}{8}\delta_a^b N. \tag{A.12}$$

$$[Z_{ab}, Z^c] = 0; \quad [Z_{ab}, Z_c] = -E_{abc}; \quad [Z_{ab}, Z_{cd}] = 0;$$
$$[Z_{ab}, Z^{cd}] = 4\delta^{[c}_{[a}E^{d]}_{b]} + \frac{1}{2}\delta^{cd}_{ab}N; \quad [Z^{ab}, Z^c] = -E^{abc}; \quad [Z^{ab}, Z_c] = 0. \tag{A.13}$$

$$[E^{abc}, Z^d] = 0; \quad [E_{abc}, Z^d] = 3\delta^d_{[a} Z_{bc]}; \quad [E^{abc}, Z_{de}] = -6\delta^{[ab}_{de} Z^{c]}; \quad [E_{abc}, Z_{de}] = 0. \tag{A.14}$$

$$[E^{abc}, E^{def}] = -\frac{1}{32}\epsilon^{abcdefgh} Z_{gh}; \quad [E_{abc}, E_{def}] = \frac{1}{32}\epsilon_{abcdefgh} Z^{gh}. \tag{A.15}$$

$$[E^{abc}, Z_d] = 3\delta^{[a}_d Z^{bc]}; \quad [E_{abc}, Z_d] = 0; \quad [E^{abc}, Z^{de}] = 0; \quad [E_{abc}, Z^{de}] = 6\delta^{de}_{[ab} Z_{c]}. \tag{A.16}$$

$$[E^{abc}, E_{def}] = -\frac{1}{8}\delta^{[ab}_{[de} E_{f]}{}^{c]} - \frac{3}{4}\delta^{abc}_{def} N. \tag{A.17}$$

$$[E^b_a, Z^c] = -\delta^c_a Z^b + \frac{1}{8}\delta^b_a Z^c; \quad [E^b_a, Z_c] = \delta^b_c Z_a - \frac{1}{8}\delta^b_a Z_c. \tag{A.18}$$

$$[E^b_a, Z_{cd}] = -2\delta^b_{[c} Z_{d]a} - \frac{1}{4}\delta^b_a Z_{cd}; \quad [E^b_a, Z^{cd}] = 2\delta^{[c}_a Z^{d]b} + \frac{1}{4}\delta^b_a Z^{cd}. \tag{A.19}$$

$$[E^b_a, E^{cde}] = -3\delta^{[c}_a E^{de]b} + \frac{3}{8}\delta^b_a E^{cde} : [E_a{}^b, E_{cde}] = 3\delta^b_{[c} E_{de]a} - \frac{3}{8}\delta^b_a E_{cde}. \tag{A.20}$$

$$[E^b_a, E^d_c] = \delta^b_c E^d_a - \delta^d_a E^b_c. \tag{A.21}$$

The elements $\{Z^a, Z_{ab}\}$ (or equivalently $\{Z_a, Z^{ab}\}$) span the maximal 36-dimensional abelian nilpotent subalgebra of E_8 [53, 52]. Finally, the generators are normalized according to the values of the traces given by

$$Tr(NN) = 60 \cdot 8; \quad Tr(Z^a Z_b) = 60\delta^a_b, \quad Tr(Z^{ab} Z_{cd}) = 60 \cdot 2!\delta^{ab}_{cd}$$

$$Tr(E_{abc} E^{def}) = 60 \cdot 3!\delta^{def}_{abc}, \quad Tr(E^b_a E^d_c) = 60\delta^d_a \delta^b_c - \frac{15}{2}\delta^b_a \delta^d_c. \tag{A.22}$$

with all other traces vanishing.

Appendix B

The commutators involving the $\mathcal{J}_{AB}$ generators of the 256-dim Clifford space associated with the $8D$ Clifford algebra $Cl(8)$ that is defined by the anti-commutators $\{\gamma_b, \gamma^a\} = 2\delta^a_b \mathbf{1}$, for $a, b = 1, 2, \ldots, 8$, are obtained as follows

$$\mathcal{J}^a_b = [\gamma_b, \gamma^a] = 2\gamma^a_b; \quad \mathcal{J}^{a_1 a_2}_{b_1 b_2}, = [\gamma_{b_1 b_2}, \gamma^{a_1 a_2}] = -8\delta^{[a_1}_{[b_1} \gamma^{a_2]}_{b_2]}. \tag{B.1}$$

$$\mathcal{J}^{a_1 a_2 a_3}_{b_1 b_2 b_3} = [\gamma_{b_1 b_2 b_3}, \gamma^{a_1 a_2 a_3}] = 2\gamma^{a_1 a_2 a_3}_{b_1 b_2 b_3} - 36\delta^{[a_1 a_2}_{[b_1 b_2} \gamma^{a_3]}_{b_3]}. \tag{B.2}$$

$$\mathcal{J}^{a_1 a_2 a_3 a_4}_{b_1 b_2 b_3 b_4} = [\gamma_{b_1 b_2 b_3 b_4}, \gamma^{a_1 a_2 a_3 a_4}] = -32\delta^{[a_1}_{[b_1} \gamma^{a_2 a_3 a_4]}_{b_2 b_3 b_4]} + 192\delta^{[a_1 a_2 a_3}_{[b_1 b_2 b_3} \gamma^{a_4]}_{b_4]}. \tag{B.3}$$

etc.

In general for $pq = odd$ one has [50]

$$\mathcal{J}^{a_1 a_2 \cdots a_q}_{b_1 b_2 \cdots b_p} = [\gamma_{b_1 b_2 \cdots b_p}, \gamma^{a_1 a_2 \cdots a_q}] = 2\gamma^{a_1 a_2 \cdots a_q}_{b_1 b_2 \cdots b_p} - \frac{2p!q!}{2!(p-2)!(q-2)!}\delta^{[a_1 a_2}_{[b_1 b_2} \gamma^{a_3 \cdots a_q]}_{b_3 \cdots b_p]}$$
$$+ \frac{2p!q!}{4!(p-4)!(q-4)!}\delta^{[a_1 \cdots a_4}_{[b_1 \cdots b_4} \gamma^{a_5 \cdots a_q]}_{b_5 \cdots b_p]} - \cdots \tag{B.4}$$

for $pq = even$ one has

$$\mathcal{J}^{a_1 a_2 \cdots a_q}_{b_1 b_2 \cdots b_p} = [\gamma_{b_1 b_2 \cdots b_p}, \gamma^{a_1 a_2 \cdots a_q}] = -\frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \delta^{[a_1}_{[b_1} \gamma^{a_2 a_3 \cdots a_q]}_{b_2 b_3 \cdots b_p]} - \frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \delta^{[a_1 \cdots a_3}_{[b_1 \cdots b_3} \gamma^{a_4 \cdots a_q]}_{b_4 \cdots b_p]} + \cdots \tag{B.5}$$

The commutators involving the $\mathcal{J}^A_B$ generators are

$$[\mathcal{J}^{a_1 b_1}, \mathcal{J}^{a_2 b_2}] = \eta^{b_1 a_2} \mathcal{J}^{a_1 b_2} - \eta^{a_1 a_2} \mathcal{J}^{b_1 b_2} - \eta^{b_1 b_2} \mathcal{J}^{a_1 a_2} + \eta^{a_1 b_2} \mathcal{J}^{b_1 a_2}. \tag{B.6}$$

$$[\mathcal{J}^{a_1 b_1 a_2 b_2}, \mathcal{J}^{c_1 d_1 c_2 d_2}] = [\eta^{b_1 a_2} \mathcal{J}^{a_1 b_2} \pm \cdots, \eta^{d_1 c_2} \mathcal{J}^{c_1 d_2} \pm \cdots] = \eta^{b_1 a_2} \eta^{d_1 c_2} \eta^{b_2 c_1} \mathcal{J}^{a_1 d_2} \pm \cdots \tag{B.7}$$

etc.

The $[\mathcal{J}^{AB}, E_8]$ commutators are obtained by using the proper notation for the E_8 generators given by Eq. (A.10)

$$[\mathcal{J}^{ab}, \mathcal{Z}^c_\pm] = -\eta^{ac} \mathcal{Z}^b_\pm + \eta^{bc} \mathcal{Z}^a_\pm; \tag{B.8}$$

$$[\mathcal{J}^{ab}, \mathcal{Z}^{cd}_\pm] = \eta^{bc} \mathcal{Z}^{ad}_\pm - \eta^{ac} \mathcal{Z}^{bd}_\pm - \eta^{bd} \mathcal{Z}^{ac}_\pm + \eta^{ad} \mathcal{Z}^{bc}_\pm. \tag{B.9}$$

$$[\mathcal{J}^{ab}, \mathcal{Z}^{cde}_\pm] = \eta^{b[c} \mathcal{Z}^{de]a}_\pm - \eta^{a[c} \mathcal{Z}^{de]b}_\pm. \tag{B.10}$$

$$[\mathcal{J}^{ab}, \mathcal{E}^{[cd]}] = \eta^{bc} \mathcal{E}^{[ad]} - \eta^{ac} \mathcal{E}^{[bd]} - \eta^{bd} \mathcal{E}^{[ac]} + \eta^{ad} \mathcal{E}^{[bc]}. \tag{B.11}$$

$$[\mathcal{J}^{ab}, \mathcal{E}^{(cd)}] = \eta^{bc} \mathcal{E}^{(ad)} - \eta^{ac} \mathcal{E}^{(bd)} + \eta^{bd} \mathcal{E}^{(ac)} - \eta^{ad} \mathcal{E}^{(bc)}. \tag{B.12}$$

$$[\mathcal{J}^{ab}, \mathcal{E}^{cd}] = [\mathcal{J}^{ab}, \mathcal{E}^{(cd)} + \mathcal{E}^{[cd]}] = \eta^{bc} \mathcal{E}^{ad} - \eta^{ac} \mathcal{E}^{bd} + \eta^{bd} \mathcal{E}^{ca} - \eta^{ad} \mathcal{E}^{cb}. \tag{B.13}$$

and $[\mathcal{J}^b_a, \mathcal{N}] = 0$.

The commutators of the abelian translations are $[P_a, P_b] = 0$ and all the commutators of the E_8 generators with the P_a generators are *zero* $[E_8, P_a] = 0$, such that the Jacobi identities involving the E_8 and P_a generators will be trivially satisfied, while the $[\mathcal{J}^{AB}, P^a] \neq 0$:

$$[\mathcal{J}^{ab}, P^c] = -\eta^{ac} P^b + \eta^{bc} P^a. \tag{B.14}$$

There are other nonzero commutators like

$$[\mathcal{J}^{a_1 b_1 a_2 b_2}, \mathcal{Z}^{c_1 c_2}_\pm] = [\eta^{a_1 a_2} \mathcal{J}^{b_1 b_2} \pm \cdots, \mathcal{Z}^{c_1 c_2}_\pm] = \eta^{a_1 a_2} \eta^{b_2 c_1} \mathcal{J}^{b_1 c_2} \pm \cdots \tag{B.15}$$

etc. From Appendix **A, B** one has all the commutators needed to evaluate the field strengths in Eqs. (2.20).

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