Quantum signatures of Solar System dynamics

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Abstract Let $\omega(i)$ be period of rotation of the i-th planet around the Sun (or $\omega_j(i)$ be period of rotation of j-th satellite around the i-th planet). From empirical observations it is known that within margins of experimental errors $\sum_i n_i \omega(i) = 0$ (or $\sum_j n_j \omega_j(i) = 0$) for some integers n_i (or n_j), different for different satellite systems. These conditions, known as resonance conditions, make uses of theories such as KAM difficult to implement. The resonances in Solar System are similar to those encountered in old quantum mechanics where applications of methods of celestial mechanics to atomic and molecular physics were highly successful. With such a successes, the birth of new quantum mechanics is difficult to understand. In short, the rationale for its birth lies in simplicity with which the same type of calculations can be done using methods of quantum mechanics capable of taking care of resonances. The solution of quantization puzzle was found by Heisenberg. In this paper new uses of Heisenberg's ideas are found. When superimposed with the equivalence principle of general relativity, they lead to quantum mechanical treatment of observed resonances in the Solar System. To test correctness of theoretical predictions the number of allowed stable orbits for planets and for equatorial stable orbits of satellites of heavy planets is calculated resulting in good agreement with observational data. In addition, the paper briefly discusses quantum mechanical nature of rings of heavy planets and potential usefulness of the obtained results for cosmology.

Key words Heisenberg honeycombs • Quantum and celestial mechanics • Group theory • Exactly solvable classical and quantum dynamical problems • Equivalence principle • Cosmological constant •(anti) de Sitter spaces

1 Introduction

1.1 General comments

The role of celestial mechanics in development of modern quantum mechanics is well described in the lecture notes by Born (1924). The usefulness of the atomic mechanics to problems of celestial mechanics has been recognized only very recently (Porter and Cvitanovič 2005; Marsden and Ross 2006). Closely related to these papers is the paper by Convay at al (2007) where methods of

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optimal control and genetic algorithms were used for mission planning problems. In this work we extend the emerging reverse trend. For this purpose we critically reanalyze the logical steps leading from old to new quantum mechanics in the light of available astronomical observational data. For the sake of uninterrupted reading, some not widely known facts from the history of quantum mechanics are presented in nontraditional way based on the latest results from mathematics and atomic physics. Thus, prior to formulation of the problems to be solved in this paper, we provide some needed historical background information.

1.2   Resonances in old atomic mechanics

In 1923-24 academic year in Göttingen Max Born replaced planned two- semester lecture course in celestial mechanics by the course in atomic mechanics. Contrary to the standard superficial descriptions of "old" quantum mechanics which can be found at the beginning of any textbook on quantum mechanics, the achievements of "old" quantum mechanics go far beyond calculation of spectra of the Hydrogen atom. In fact, the optical and X-ray spectra of almost all known at that time elements were found accounting even for the fine structure relativistic effects. The theory of quantum angular momenta was developed and used in the theory of polyatomic molecules. The effects of Zeemann and Stark were considered as well, etc. If one would make an itemized list of problems considered in "old" quantum theory and would compare it with that of "new" quantum theory, surprisingly, one would not be able to find an item which was not treated within the "old" formalism. With such an impressive list of achievements it is hard to understand why this formalism was suddenly abandoned in favour of "new" quantum mechanics in 1925.

To explain this, we would like to bring some excerpts from the paper by Pauli and Born (1922). Being thoroughly familiar with works by Poincare′ on celestial mechanics, they were trying to apply these methods to multielectron atoms. For this purpose they were using methods of theory of perturbations to account for electron-electron interactions. By doing so they obtained the same types of divergencies as were known already from calculations of planetary dynamics. By realizing the asymptotic nature of the obtained expressions, they decided that to keep just few terms in these expansions is the best way to proceed. By doing so a reasonably good agreement with experimentally known location of spectral lines was expected to be obtained. Such a state of affairs had caused frustration for Bohr who conceded that *only those dynamical systems which admit a complete separation of variables are quantizable*[2]. If such a separation is absent, according to the Bohr's current opinion, the system should not possess a discrete spectrum so that visible lines in spectra of elements *other* than Hydrogen should be much wider. On the theoretical side such an assumption calls for development of methods enabling to determine the widths of spectral lines and of distribution of the intensity within these widths. Such an intensity is expected to be connected with the underlying mechanical motion inside the atomic system.

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[2] E.g. read (Pauli and Born 1922).

The spectroscopic data for almost entire periodic system were readily available at the turn of the 20th century (Baly 1905). Bohr was well aware of these data and used them for his search for correct atomic model (along with Rutherford's results of 1912 on scattering from the Hydrogen atom). In particular, he looked at the data for Helium in 1913 and published his findings in Nature (Bohr 1913). For the sake of arguments which will follow, we found it helpful to reproduce some of the data from his Table 1 below.

Table 1

Spectral series	$\lambda \cdot 10^8$	%error	$\lambda(\frac{1}{n_1^2} - \frac{1}{n_2^2}) \cdot 10^{10}$	$(n_1; n_2)$
P_1	4685.98	0.01	22779.1	(3;4)
P_2	3203.30	0.05	22779.0	(3;5)
P_2	2306.20	0.10	22777.3	(3;9)
P_1	2252.88	0.10	22779.1	(3;10)
S	5410.5	1.0	22774	(4;7)
S	4541.3	0.25	22777	(4;9)
S	4200.3	0.5	22781	(4;11)

These data were compared with those for the Hydrogen for which he used the analogous table (Table 2)[3]. For some reason, the data in his Table 2 *did not* contain the error column. Since the wavelength λ in both cases was measurable, it was possible to evaluate the ratio K_H/K_{He}, where $K = \lambda(\frac{1}{n_1^2} - \frac{1}{n_2^2}) \cdot 10^{10}$, which was found to be 4.0016. At the same time, Bohr's own calculations gave for K the following value: $K = \frac{c(M+m)h^3}{2\pi^2 Z^2 e^2 Mm}$, with h being the Planck's constant, Z and M being the charge and the mass of the nucleus, c being the speed of light, e and m being the charge and the mass of the electron. By assuming $M_{He} = 4M_H$ and $Z_{He} = 2Z_H$, one readily obtains for K_H/K_{He} the result: 4.00163. It is in good agreement with that obtained experimentally. In doing such calculations Bohr assumed that each electron in Helium can be treated as if it is a Hydrogen-like. This surely implies that the width of spectral lines for Helium should be practically the same as those for the Hydrogen atom. Nowadays we know (Ingle and Crouch 1988) that all atomic spectra have some finite linewidth. This linewidth is determined by factors such as: a) the collisional broadening, b) the Doppler broadening and c) the natural broadening. Each of these is having some further ramifications. Hence, from the standpoint of modern knowledge one can interpret Bohr's conclusions made in 1922 as acknowledgement of the fact that spectra of elements *other* than Hydrogen are broader because of natural reasons (so that one should take into consideration the data from the error column in Table 1) without invalidation of Bohr's major quantization assumptions. It should be noted though that such a conclusion leads to the question: why the very same factors are affecting the Hydrogen atoms much less?

Unhappy with his conclusions, Bohr asked Born and Heisenberg to make more rigorous calculations for Helium using perturbational methods analogous

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[3] Which we do not reproduce.

to those developed in the paper by Pauli and Born in 1922. Their findings were published in 1923 and resulted in practically total failure in accurate determination of energies of the ground and excited states for Helium atom. This fact is documented in Born's lecture notes (Born 1924).

Since the Helium atom calculations were made by Heisenberg (under Born's supervision) it may be not too surprising that, after all, it was Heisenberg who found the way out of the existing difficulties. The logic of his reasonings is discussed from the modern mathematical point of view in Section 2. In this introductory section we would like only to put his work in some historical perspective. For this, we need to make few comments on his joint work with Born. For the sake of space, we refer our readers to the cited literature for details.

1.2.1 3-body problem and the He atom (old quantum mechanics results)

The unperturbed Hamiltonian for He was chosen as $H = -A(\mathrm{J}_1^{-2} + \mathrm{J}_2^{-2})$ with the constant $A = 2\pi^2 e^4 m Z^2$ while the perturbation was chosen as $H_1 = e^2/R$ with J_1 and J_2 being the Bohr-Sommerfeld (B-S) adiabatic action integrals for electron 1 and 2 and R being the Euclidean distance between them. After H, He is the first nontrivial 3-body mechanical system whose behavior is amended in accordance with the rules of old quantum mechanics by requiring both of these integrals (i.e. J_1 and J_2) to have their lowest value, i.e. h, so that the *unperturbed* energy for He is twice that for H. Since the energy W for H is known to be $W = -\frac{A}{\mathrm{J}^2}$, the frequency ω of rotation of the electron at its stationary orbit in the action-angle formulation of classical mechanics is obtained as $\omega = \frac{\partial W}{\partial \mathrm{J}} \sim n^{-3}$ [4]. In obtaining this result it was assumed that J is *continuous* variable and, only after the differentiation is made, J is assumed to be discrete: $\mathrm{J} = nh$. It is important to realize at this point that exactly *the same logic* was used in Heisenberg's paper on quantum mechanics to be discussed in Section 2[5]. Hence, for the Helium atom within the approximations made the rotation frequencies of both electrons are the same. This fact is known in mechanics literature as *accidental degeneration*. In view of its crucial importance for this paper, we would like to pause now in order to provide more accurate definitions.

In terms of the action -angle $(\mathbf{I}, \boldsymbol{\varphi})$ variables the Hamilton's equations of motion for a completely integrable system can be written as:

$$\frac{d\mathbf{I}}{dt} = 0, \quad \frac{d\boldsymbol{\varphi}}{dt} = \frac{\partial H}{\partial \mathbf{I}} \equiv \boldsymbol{\omega}(\mathbf{I}), \tag{1}$$

where the boldface indicates that the dynamical system with Hamiltonian H is multicomponent (in general case). Solutions of the system (1) are: $\mathrm{I}_i = c_i, \varphi_i = \omega_i(\mathbf{I})t + C_i$, $i = 1 - N$. It is assumed that c_i and C_i are some known constants. In view of this result, any mechanical observable $\mathcal{F}(\mathbf{p}, \mathbf{q})$ made of generalized

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[4] E.g. see page 140 of (Born 1924).
[5] E.g.see Eq.(19) below.

momenta $\mathbf{p}$ and generalized coordinate $\mathbf{q}$ can be Fourier decomposed as

$$\mathcal{F} = \sum_{\mathbf{n}=-\infty}^{\infty} A_{\mathbf{n}} \exp(i\mathbf{n} \cdot \boldsymbol{\varphi}), \qquad (2)$$

where $\mathbf{n} = \{n_1, ..., n_N\}$. Accordingly, $\mathbf{n} \cdot \boldsymbol{\varphi} = \sum_{i=1}^{N} n_i \varphi_i$. Such a Fourier decomposition is expected to exist even for those *perturbed* systems for which the *empirically observed* orbits are closed (as seen in the case of planets in our Solar System). Dynamical system is considered to be *accidentally degenerate* if the relation

$$\sum_{i=1}^{N} n_i \omega_i(\mathbf{I}) = 0 \qquad (3)$$

holds for some *fixed* set of integers $\mathbf{n}$ and is *degenerate* if Eq.(3) holds for *any* set of $\mathbf{n}$'s. In this work this condition will be alternatively called as *resonance* condition in accord with modern terminology.

1.2.2   3-body problem and the He atom (modern perspective)

Failure of methods of celestial mechanics (modified by the B-S quantization rule) to accurately compute the ground and excited states for He has led Heisenberg to discovery of his matrix quantum mechanics in 1925. The problem appeared to be completely solved until 1960 when some difficulties emerged when the standard Hartree-Fock type variational calculations become inadequate for description of doubly excited electron states (Tanner et al 2000). The same authors notice that by 1990 the improvements which were made in 60ies failed again, especially for the extreme excitation regime which cannot be described using single electron quantum numbers. The way out of the existing difficulties was associated with accurately designed semiclassical methods. The backbone of these semiclassical descriptions "are the periodic orbits of the full classical two-electron system without any approximations". Tanner at all notice (e.g. read page 523) that "The classical two-electron atom is neither integrable nor fully chaotic. The apparently regular spectrum as well as the breakdown of approximate quantum numbers for highly doubly excited states and the enormous variation in the decay widths for resonances can be understood by studyng classical mechanics in detail. *Qualitative* results can be obtained by exploiting semiclassical periodic orbit theory." In other words, use of classical mechanics is quite sufficient for determination of both the ground and excited states of He and only for extreme case of highly doubly excited He the description becomes qualitative. Classical dynamics of He used for calculations of spectra is essentially the dynamics of the restricted 3-body problem[6] and, as such, exhibits chaotic and regular regimes. Since the experimental information which can be deduced from the low lying spectral excitations of He does not allow to disentangle regular and chaotic parts of the dynamics, it has become possible to simplify things further by recalculating spectra of He, Li, Be, and diatomic molecules made out of these and other atoms, and also of H_2, by revisiting Bohr's 1913 calculations (Svidzinsky et al

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[6]Modified by the fact that in the atomic case electrons repel each other and have the same mass.

2005a,b; Muravski and Svidzinsky 2006). These calculations involve a simple minded minimization of classical functionals of the type considered by Bohr in 1913. Mathematical justification of such a procedure was found by Chen et al (2006). The accuracy of the results obtained with help of such classical calculations (employing however the B-S quantization rule !) compares well with incomparably more elaborate traditional quantum mechanical calculations. It should be noted though that in his Nobel prize winning address Bohr (1923) was talking about his great success in calculating spectra of almost all elements of the periodic table using the B-S quantization rule and simple minimization procedure. It took another 80 years or so to bring these calculations to the level comparable with the best known quantum mechanical calculations!

1.3 Resonances in celestial mechanics

On page 265 of his lecture notes Born writes: "Accidental degeneration is a rare and remarkable exception in astronomy; the odds against (Eq.(3)) being exactly fulfilled are *infinite*. A close approach to it is found in the case of perturbations of some minor planets (Achilles, Patroclus, Hector, Nestor) which have very nearly the same period of revolution as Jupiter. *In atomic theory*, on the other hand, where J_k's can have only discrete values, *accidental degeneration is very common*." As result of such an accidental degeneracy Heisenberg's attempt at perturbative calculations for He failed miserably. Such a failure caused him to reconsider the whole computational scheme resulting in an ultimate breakthrough in 1925 leading to new quantum mechanics.

Before discussing his contributions from the modern perspective, we would like to make few remarks regarding the accuracy of astronomical data in Born's lectures. In 1968 Molchanov (1968), while analyzing the astronomical data, came to conclusion that the accidental degeneracy for Solar (and, very recently, Solar-like (Ferraz-Mello et al 2005; Ferraz-Mello et al 2006)) System(s) is as common as in atomic systems. In Table 2 (below) taken from his work (Molchanov's Table 1) we reproduce some data taken from this reference.

Table 2

	Planet	ω_i^O	ω_i^T	$\Delta\omega/\omega$	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9
1	Mercury	49.22	49.20	0.0004	1	-1	-2	-1	0	0	0	0	0
2	Venus	19.29	19.26	0.0015	0	1	0	-3	0	-1	0	0	0
3	Earth	11.862	11.828	0.0031	0	0	1	-2	1	-1	1	0	0
4	Mars	6.306	6.287	0.0031	0	0	0	1	-6	0	-2	0	0
5	Jupiter	1.000	1.000	0.000	0	0	0	0	0	2	-5	0	0
6	Saturn	0.4027	0.400	0.0068	0	0	0	0	1	0	-7	0	0
7	Uranus	0.14119	0.14286	-0.0118	0	0	0	0	0	0	1	-2	0
8	Neptune	0.07197	0.07143	0.0075	0	0	0	0	0	0	1	0	-3
9	Pluto	0.04750	0.04762	-0.0025	0	0	0	0	0	1	0	-5	1

For satellite systems of Jupiter, Saturn and Uranus Molchanov's paper also contains tables similar to our Table 2. According to the book by Beletsky

(2001), in view of the resonance nature of our Solar System, uses of KAM theory (Arnol'd et al 2006) for explanation of planetary stability typically fail.

To understand the data in Table 2 several comments are in order. First, the displayed frequencies are measured in the system of units in which the Jupiter's frequency was chosen as the unit of measurement. Second, in view of Eq.(3), the first row of data from Table 2 should be actually read as $\omega_1 - \omega_2 - 2\omega_3 - \omega_4 = 0$. All other rows should be treated accordingly. The theoretical frequencies ω_i^T are those which satisfy the resonance conditions exactly while ω_i^O denote the observed frequencies. The data for Pluto should *not* to be considered in terms of resonances for the following reason.

Consider a scalar product $\mathbf{n} \cdot \boldsymbol{\varphi} \equiv \boldsymbol{\Lambda}$ in Eq.(2). This can be looked upon as representation of the vector $\boldsymbol{\Lambda}$ in the coordinate basis $\{\boldsymbol{\varphi}\}$. The coordinate basis can be changed with help of some matrix $\mathbf{A}$ so that $\boldsymbol{\Lambda} = \mathbf{n} \cdot \mathbf{A} \cdot \tilde{\boldsymbol{\varphi}}$. It can be argued (Born 1924; Molchanov 1968) that $\det \mathbf{A} = 1$, so that the matrix $\mathbf{A}$ must be a unimodular square matrix. Only for the sake of this requirement the data for Pluto in Table 2 were assigned in a way given in the Table 2. Next, $\Delta\omega/\omega$ should be understood as $(\omega_i^O - \omega_i^T)/\omega_i^O$. After this, the obtained error margins can be compared against those for He in Table 1. Such a comparison indicates that the accuracy in both cases is essentially the same. It is such that Bohr was able to obtain using his old quantum mechanical theory a reasonably accurate ratio $\mathrm{K}_H/\mathrm{K}_{He}$ in agreement with experiment. It makes physical sense to blame the intrinsic inaccuracy of the collected data (e.g. that in Table 1) for observed frequency discrepancies. Hence, along with Bohr, it is reasonable to claim that, with exception of H, other atomic systems are not quantizable because of these discrepancies [7]. The same reasoning should then be applied to the planetary systems, especially in view of critique of Molchanov's work by Henon (1969) and Backus (1969). These authors argued that the error margins in Molchanov's tables are too large for the resonances to be considered seriously. The comparison between the Tables 1 and 2 indicates that even though the arguments by Henon and Backus may be mathematically correct, they do not have sound physical support due to intrinsic inaccuracies in measurements which cannot be substantially improved[8]. Thus, Molchanov's data and their interpretation remain correct even without additional explanations made by Molchanov (1969a,b) in defence of his results. Furthermore, subsequently obtained results by Brin (1973) and Patterson (1987) strongly suggest that, at least pairwise, planets are in resonance with each other. This is true in particular for the heavy planets in our Solar System as demonstrated by Ferraz-Mello et al (2005). Evidently, the linear combination of such pairwise resonances leads back to the Molchanov-type results.

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[7] Subsequent developments of quantum mechanics have demonstrated that Bohr was apparently wrong. We say "apparently" in view of the results of Section 2 where correct quantization prescription is discussed based on improvement of Heisenberg's ideas. In view of results of Section 2, it is reasonable to say that Bohr's intuition was nevertheless correct but the situation can be improved rigorously using methods which were not available in Bohr's time.

[8] E.g.read Sections 3 and 4 where these measurements are discussed for objects such as Solar System, etc.

It should be noted though that quantization prescriptions discovered by Heisenberg remain correct even in the case when there are no resonances: *They rely only on the existence of the closed stable orbits.* The existence of resonances, in fact, simplifies matters considerably since it makes the task of establishing the quantum-classical correspondence much easier as explained in the rest of this paper.

Prior to Molchanov's 1968 work an effort to explain the ubiquity of resonances in Solar System using methods of classical mechanics was made by Goldreich (1965) who demonstrated that "special cases of commensurate mean motions are not disrupted by tidal forces". Moreover, he proposed that it is the tidal forces which drive otherwise incommensurate system to commensurability.

Thus, the problem of stability of our Solar System is very much the same as that for the multielectron atoms. In both cases the accidental degeneracies (resonances) preclude the systematic use of standard perturbational methods. Unlike more traditional classical mechanics treatments (Murray and Dermott 1999), we apply Heisenberg-style arguments ultimately aimed at explanation of Solar System stability. For the sake of space, we are not discussing in this work the spin-orbit- type resonances also ubiquitous in the Solar System (Murray and Dermot 1999).

Finally, we would like to mention that development of our quantum mechanical formalism proceeds in historical accord with that for atomic systems for which the static (spectral) problems were considered first. The dynamical problems of atoms/molecules formation as well as their stability towards disintegration were considered only afterwards. Hence, only the spectral-type problems will be discussed in this work.

1.4 Organization of the rest of this paper

Existence of stable closed orbits, of resonances, as well as the lack of dissipation (in spite of presence of tidal effects) in Solar (and Solar-like) System(s) are indicative of the quantum nature of the orbital motions in the Solar System. Nevertheless, the formalism of quantum mechanics in the traditional form present in textbooks cannot be used. To apply methods of quantum mechanics to celestial mechanics is possible with use of Heisenberg's original ideas updated with help of the latest mathematical results. Sections 2 and 3 as well as Appendices A and B provide a self-contained overview of quantum mechanics based on Heisenberg's ideas. They provide needed background for the actual quantum calculations in celestial mechanics which are performed in Section 4 (supplemented with Appendix C). The main results of this section (and the whole paper) are summarized in Table 4. In this table the number of stable orbits for planets of Solar System as well as the number of stable orbits for satellites of heavy planets (Jupiter, Saturn, Uranus and Neptune) is calculated and compared against the observed numbers. Unusually good agreement between the calculated and observed numbers for Solar System, and the satellite systems of Jupiter and Saturn is obtained resulting in further suggestions for observational astronomy of the Solar System. In Table 3 a comparative sum-

mary of main theoretical assumptions of both quantum atomic and quantum celestial mechanics is given. Using it, the motion of rings around heavy planets is briefly studied resulting in the same conclusions about the quantum nature of such type of motion. Some auxiliary mathematical results needed for these calculations are presented in Appendix D. In study of all cases, including dynamics of planetary rings, it follows that the equivalence principle of general relativity plays the decisive role in development of quantum celestial mechanics. This fact caused us to write Section 5 in which the effects of general relativity on quantum mechanics of Solar System are studied further. In it we discuss how the obtained results (for which the importance of the Lorentz group SO(2,1) is emphasized) should be amended if we are interested in knowing to what extent the (larger scale) symmetries of space-time typically considered in cosmological models of general relativity may affect the quantum dynamics of Solar System. Such an information can be used in reverse for probing symmetries of space-time at scales comparable or larger than that for our Solar System. Finally, Section 6 is devoted to some concluding remarks.

2 Heisenberg's honeycombs and resonances

2.1 General comments

In this section we discuss Heisenberg's ground breaking paper (Heisenberg 1925) on quantum mechanics from perspective of modern mathematics. We begin with observation that the Schrödinger equation cannot be reduced to something else which is related to our macroscopic experience. *It has to be postulated.*[9] On the contrary, Heisenberg's basic equation from which all quantum mechanics can be recovered is directly connected with experimental data and looks almost trivial. Indeed, following Bohr, Heisenberg looked at the famous equations for energy levels difference

$$\omega(n, n - \alpha) = \frac{1}{\hbar}(E(n) - E(n - \alpha)), \tag{4}$$

where both n and $n - \alpha$ are some integers. He noticed that this definition leads to the following fundamental composition law:

$$\omega(n - \beta, n - \alpha - \beta) + \omega(n, n - \beta) = \omega(n, n - \alpha - \beta). \tag{5a}$$

Since by design $\omega(k, n) = -\omega(n, k)$, the above equation can be rewritten in a symmetric form as

$$\omega(n, m) + \omega(m, k) + \omega(k, n) = 0. \tag{5b}$$

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[9] Usually used appeal to the DeBroigle wave-particle duality is of no help since the wave function in the Schrödinger's equation plays an auxiliary role while the De Broigle waves are assumed to exist in real space-time.

In such a form it is known as the honeycomb equation (condition) in current mathematics literature (Knutson and Tao 2001) where it was rediscovered totally independently of Heisenberg's key quantum mechanical paper and, apparently, with different purposes. Connections between mathematical results of Knutson and Tao and those of Heisenberg were discovered in the recent paper by Kholodenko (2006a). In this work some results of this paper will be used.

In particular, we begin by noticing that Eq.(5b) due to its purely combinatorial origin does not contain the Plank's constant $\hbar$. Such fact is of central importance for this work. In particular, the simplest resonance condition encountered in celestial mechanics

$$n_1\omega_1 + n_2\omega_2 + n_3\omega_3 = 0 \tag{6}$$

can be equivalently rewritten in the form of (5b), where $\omega(n,m) = \omega_n - \omega_m$. It would be quite unnatural to think of the Planck's constant for this case. Even though, the resonance condition is equivalent to the Heisenberg's quantization condition, Eq.(5b), the reverse may not be true since frequencies in Eq.(5b) may be irrational. It should be noted though that such irrationality would be very difficult to detect experimentally in view of natural causes leading to the line broadening mentioned in the Introduction. Thus, from the experimental standpoint Eq.s (5b) and (6) are equivalent[10]. Furthermore, by assuming irrationality we would run into difficulty with obtaining the semiclassical limit in which (it is believed) the old quantum mechanics based on the Bohr-Sommerfeld method of quantization should be applicable.*These arguments imply that, at least semiclassically, dynamics of all quantum mechanical systems is resonant.*

Equation (5b) is the basic building block of the honeycomb structure encoding all information about the spectra of quantum system. Details leading to construction of this combinatorial structure are summarized in the paper by Knutson and Tao (2001). They were used in Kholodenko's (2006a) paper in which some physical applications absent in Knutson's-Tao paper are discussed.

To describe such honeycomb structure in the nutshell, let us choose the basic Y- shaped tripod whose edges are labeled by frequencies $\omega(n,m)'s$ is such a way that the total sum of these labels is equal to zero, as in Eq.(5b). The honeycomb is made of collection of such tripods placed on a 2- dimensional plane and joined with each other in such a way that the frequencies at the edges match. Several additional rules were set up by Knutson and Tao and are given in their original papers (Knutson and Tao 1999; Knutson, Tao and Woodward 2004). Our readers encouraged at this point to consult the interactive web site designed by Tao (2001) in order to get a feeling of honeycombs as combinatorial objects. For physical applications, other than those discussed in this paper, our readers are referred to the paper by Kholodenko (2006a). Provided references allow us

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[10] Although to make the frequencies independent (and, hence, irrational) is easy mathematically, it is unrealistic to detect such fact experimentally. In other words, even though the critique of Molchanov's (1968) paper by Henon (1969) and Backus (1969) could be mathematically justified in spite of Molchanov's counter arguments (Molchanov 1969a,b), it is of academic value only.

to squeeze to the absolute minimum the amount of mathematical information in this paper.

With account of these remarks, we proceed with development of Heisenberg's arguments. In his paper of October 7th of 1925, Dirac (1926), aware of Heisenberg's key paper, streamlined Heisenberg's results and introduced notations which are in use up to this day. He noticed that the combinatorial law, Eq.(5a), for frequencies, when used in the Fourier expansions for composition of observables, leads to the multiplication rule $a(nm)b(mk) = ab(nk)$ for the Fourier amplitudes for these observables. In general, in accord with Heisenberg, one expects that $ab(nk) \neq ba(nk)$. Such multiplication rule is typical for matrices. In the modern quantum mechanical language such matrix elements are written as $< n \mid \hat{O} \mid m > \exp(i\omega(n, m)t)$ so that Eq.(5b) is equivalent to the matrix statement

$$
\begin{aligned}
\sum_m \quad & < \quad n \mid \hat{O}_1 \mid m >< m \mid \hat{O}_2 \mid k > \exp(i\omega(n, m)t) \exp(i\omega(m, k)t) \\
& = \quad < n \mid \hat{O}_1\hat{O}_2 \mid k > \exp(i\omega(n, k)t)
\end{aligned}
\tag{7}
$$

for some operator (observables) $\hat{O}_1$ and $\hat{O}_2$ evolving according to the rule: $\hat{O}_k(t) = U\hat{O}_k U^{-1}, k = 1, 2$, provided that $U^{-1} = \exp(-i\frac{\hat{H}}{\hbar}t)$. From here it follows that $U^{-1} \mid m >= \exp(-\frac{E_m}{\hbar}t) \mid m >$ if one identifies $\hat{H}$ with the Hamiltonian operator. Clearly, upon such an identification the Schrödinger equation can be obtained at once as is well known (Dirac 1958). We shall avoid such a pathway (at least at this stage), however. Moreover, we also shall avoid use of Heisenberg's equations of motion

$$
i\hbar\frac{\partial}{\partial t}\hat{O} = [\hat{O}, \hat{H}].
\tag{8}
$$

Our readers may ask at this point: why it is necessary to do so? And, if this is the case, what else is left from the traditional formulations of quantum mechanics which still can be used? The answers can be found in Kholodenko (2006 a,b). For the sake of uninterrupted reading they are summarized below as well. Following Heisenberg's philosophy, we shall assume that there is a set of classical observables $\{O_i(t)\}$ which is assumed to be complete in the sense that the composition of any two of these observables is given by the classical fusion rule:

$$
\{O_i, O_j\} = \sum_k C_{ij}^k O_k,
\tag{9}
$$

where C_{ij}^k are some known constants and $\{,\}$ represents the Poisson brackets of classical mechanics. Accordingly, quantum mechanically, instead of Eq.(8), we need to consider the decomposition

$$
[\hat{O}, \hat{H}] = \sum_k \tilde{C}_{ij}^k \hat{O}_k
\tag{10}
$$

valid for any t ! Under such circumstances (quantum) dynamics formally disappears! This is, of course, an exaggeration since not all systems possess needed

symmetry so that the fusion rule, Eq.(9), may not exist[11]. When it does exist, such an observation can be strengthened due to the following chain of arguments. In mathematics, expressions like $\hat{O}_i(t) = \hat{U}\hat{O}_i\hat{U}^{-1} \equiv Ad_{\hat{U}}\hat{O}_i$ define an *orbit* for the operator $\hat{O}_i$ in the Lie algebra (made of operators $\{\hat{O}_i\}$) so that the motion is caused by the action of elements $\hat{U}$ from the Lie group associated with such an algebra. Following the existing rules and notations in mathematics of Lie groups and Lie algebras (Kirillov 1976), we write $ad_{\hat{H}}\hat{O}$ for $[\hat{O}, \hat{H}]$. This requires us to use the r.h.s. of Eq.(10) instead of the formal symbol $i\hbar\frac{\partial}{\partial t}\hat{O}$ used in Heisenberg's mechanics. Evidently, we can obtain the same (or even greater) information by working with *Ad* operators instead of *ad*. In particular, it is useful to consider the trace, i.e. $tr\{Ad_{\hat{U}}\hat{O}_i\} = \chi(\hat{O}_i)$, which is just the character of $\hat{O}_i$. It is time-independent by design. If there is no time evolution then, superficially, nothing happens. This is not true, however, as was recognized long time ago by Dirac (Dirac 1958). In Chapter 9 of his book he writes : " The Hamiltonian is symmetrical function of the dynamic variables and thus commutes with every permutation. It follows that each permutation is a constant of motion. This happens even if the Hamiltonian is not constant[12]." At this point it is important to recall the famous theorem by Caley (Kargapolov and Merzlyakov 1979) which states that "every finite group is isomorphic to some permutation group". It should be noted that in mathematics literature the "permutation" group has the same meaning as "symmetric" group S_n[13]. In physics and, especially, in quantum mechanics, the symmetric group can be infinite dimensional. The theory of such groups was unknown to Dirac since it was developed only quite recently (Vershik 2003). This fact explains why it have not been in use in the traditionally written textbooks on quantum mechanics. Fortunately, for the purposes of this work, it is sufficient to use only a tiny fraction from the theory of symmetric groups.

2.2 Some useful facts about S_n

In view of the fact that the character $\chi(\hat{O}_i) = tr\{Ad_{\hat{U}}\hat{O}_i\}$ is manifestly time-independent, the orbit $Ad_{\hat{U}}\hat{O}_i$ is caused by permutations[14]. These can be analyzed using methods of algebraic geometry (Fulton 1997) and theory of linear algebraic groups (Borel 1991). A brief and self-contained introduction to these topics can be found in Kholodenko (2006b). The key concept in this field is the notion of the torus action T. It is directly connected with the notion of the Weyl-Coxeter reflection group $W = N/T$ in which the numerator N refers to

---

[11] Since all observables are made of **p** and **q** variables, such a rule does exist if we decompose these observables into power series in **p** and **q**. In those cases when such a series is infinite, normally, one should expect loss of integrability and, hence, loss of quantization. For one dimensional many body systems the situation might be repairable for suitably chosen interaction potentials. We shall elaborate on this remark further below, in Section 2.5.

[12] I.e. time-dependent.

[13] Here n denotes the number of elements in the group.

[14] Since, according to Dirac, the permutation operator commutes with the Hamiltonian.

some permutation group and the denominator T refers to those group elements (fixed points) which remain unaffected by permutations. Representations of Lie algebras (including the affine Lie algebras) associated with these Weyl-Coxeter reflection groups produce all Lie algebras known in quantum mechanics and in conformal field theories (Di Francesco, Mathieu and Senechal 1997). To simplify matters, we choose another pathway in this work to arrive at the same results. It is better adapted for connecting the experimental data with theoretical constructions.

We begin with observation that the representation theory for S_n can be built using representation theory for general linear group $GL(\mathrm{N},\mathbf{C})$ acting in the complex space made of n copies of $\mathbf{C}^N$, i.e. $\mathbf{C}^N \otimes \mathbf{C}^N \otimes \cdots \otimes \mathbf{C}^N$. This fact is known as the Schur-Weyl duality (Vershik and Okounkov 2005). The Schur functions (to be defined below) are characters of $GL(\mathrm{N},\mathbf{C})$. They play the key role in developing the representation theory of S_n in which both N and n can become infinite.

Next, we recall that a *partition* λ (finite or infinite) is a sequence

$$\lambda = \{\lambda_1, \lambda_2, \lambda_3, ...\} \tag{11}$$

made of integers[15] such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0$. The *weight* of λ is denoted by $|\lambda| = \sum_i \lambda_i$. If $\lambda's$ are integers, and if $|\lambda| = n$ we say that λ is a *partition* of n. Let λ be a partition. It is useful to associate with it a monomial $\mathbf{x}^\lambda \equiv x_1^{\lambda_1} x_2^{\lambda_2} \cdots$. Next, we introduce a symmetric function m_λ as a sum of all distinct monomials that can be obtained from $\mathbf{x}^\lambda$ by permuting of all arguments. Using these results it is possible to prove (Macdonald 1998) that the Schur function s_λ can be represented with help of m_λ as follows

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu. \tag{12}$$

To explain the meaning of the Kostka number $K_{\lambda\mu}$ in (12) we should mention the one-to -one correspondence between the partitions and the Young tableaux (Fulton 1997). In terms of such a correspondence the Kostka number $K_{\lambda\mu}$ is just the number of semistandard tableaux with shape λ and weight μ. Hence, for not too large tableaux such a number can be straightforwardly computed. The Schur functions possess a remarkable orthogonality property. For partitions λ and μ and properly defined scalar product $<,>$ one can write

$$< s_\lambda, s_\mu >= \delta_{\lambda,\mu} \tag{13}$$

in accord with general theory of characters and, in particular, of characters of S_n (Macdonald 1998). With such defined orthogonality property of $s'_\lambda s$ one can proceed with the composition (fusion) law for Schur functions. It is given by

$$s_\lambda \cdot s_\mu = \sum_\nu C^\nu_{\lambda\mu} s_\nu, \tag{14}$$

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[15] Since these numbers normally are identified with the eigenvalues of some matrix (finite or infinite) (Knutson and Tao 2001; Kholodenko 2006a), one can relax the condition that $\lambda'_i s$ are integers and make them rational or even irrational numbers but the nonnegativity and the ordering are essential.

where $|\lambda| + |\mu| = |\nu|$ and, in view of (13), the Littlewood-Richardson (L-R) coefficient $C_{\lambda\mu}^{\nu}$ can be formally defined as $C_{\lambda\mu}^{\nu} = < s_\lambda \cdot s_\mu, s_\nu >$ .These coefficients play an important role in representation theory of S_n analogous to the role the Clebsch-Gordan coefficients play in the representation theory for spin and angular momenta. The L-R coefficients can be obtained very easily with help of the honeycomb construction as discussed by Knutson and Tao (2001) and by Kholodenko (2006a).   For completeness, we provide a brief sketch of how this can be done.

We begin with the 1-honeycomb. It is just the Y-shaped tripod as discussed already. When constructing the 2-honeycomb in the plane we shall follow the rule that the labels for the edges of this new honeycomb should be geometrically and combinatorially arranged in the same way as those for the 1-honeycomb. This requires us to use yet another two tripods which can be joined together and with the third tripod only in one way in view of the imposed rules[16]. Thus, instead of just one boundary label, e.g. λ_1, in the North-West direction, now we shall have two, say, λ_1 and λ_2. The same applies for the South and the North-East directions. Thus, all larger honeycombs will have only the boundaries in the directions just mentioned which are labeled by the partitions λ, ν and μ. Unlike the 2- honeycomb for which the boundary labels determine such a honeycomb uniquely, for larger honeycombs this is no longer true. For the fixed set of boundary labels, normally, there will be more than one honeycomb with these labels. On page 1053 of Knutson and Tao (1999) the following theorem is proven: Let λ, μ and ν be three pre assigned (boundary) partitions for the k-honeycomb. Then the number of different honeycombs with such pre assigned boundary conditions is given by the L-R coefficient $C_{\lambda\mu}^{\nu}$.

Summarizing, we have defined a set (finite or not) of mutually orthogonal Schur polynomials which by design forms the Hilbert space. The partitions and the energy levels can be put into one-to-one correspondence by using the honeycomb condition, Eq.(5). Such Hilbert space is designed using experimental data. We can look at different portions (segments) of the spectra and study their overlaps thanks to the composition rule, Eq.(14)[17]. Unlike more traditional formulations of quantum mechanics requiring objects of classical mechanics as an input, no reference to the objects of classical mechanics was made thus far. In the next (sub)sections we shall discuss the extent to which such a way of developing quantum mechanics is advantageous as compared with more traditional formulations.

2.3   From combinatorics to physics

In this subsection we follow the logic of Heisenberg's paper once again. In this paper Heisenberg was concerned also with the proper interrelation between the

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[16]E.g. see Fig.2 in (Kholodenko 2006a) paper.

[17]Very much like it is done in the case of determination of the entire DNA structure from its fragments.

objects of classical and quantum mechanics. Naturally, he focused his attention at the Bohr-Sommerfeld (B-S) quantization rule

$$\oint p\,dq = nh, \quad n = 0, 1, 2, ..., \tag{15}$$

since this rule was the only one available link between the new and old mechanics. He argued that such a rule is not exact! It is determined with accuracy up to a constant (unknown at the time of writing of his paper). He argued that if such a constant would be known, the B-S rule *would become exact*, that is valid for any n's. From the point of view of our present understanding of quantum mechanics Heisenberg's intuition was correct: the old fashioned B-S rule is valid rigorously only in the limit of large n's while the calculation of the constant can be done, for instance, with help of either the WKB or considerably more sophisticated theory of Maslov indices (Dittrich and Reuter 1992). As much as these arguments are plausible, they are nevertheless superficial as can be found from reading the page 246 of the book by Arnol'd (1974). Using this reference it follows that already at the classical level the adiabatic invariant $\oint p\,dq$ is determined only up to some constant. This observation makes Heisenberg's arguments less convincing. Nevertheless, following Heisenberg, we assume that if the B-S quantization rule is corrected, it would make sense fully quantum mechanically. Presumably, under such circumstances one can get an additional information out of it. For this purpose, Heisenberg introduces the Fourier decomposition of the generalized coordinate q as

$$q(n,t) = \sum_{\alpha=-\infty}^{\infty} a_\alpha(n) \exp(i\omega(n,\alpha)t), \tag{16}$$

where in anticipation of its quantum mechanical use it is written with respect to some pre assigned energy level n. Using Eq.(16) the velocity can be readily obtained as follows

$$\dot{q}(n,t) = \sum_{\alpha=-\infty}^{\infty} i a_\alpha(n)\omega(n,\alpha) \exp(i\omega(n,\alpha)t). \tag{17a}$$

The calculation of the velocity square over the total period is given therefore by

$$\oint dt [\dot{q}(n,t)]^2 = 2\pi \sum_{\alpha=-\infty}^{\infty} |a_\alpha(n)|^2 \omega(n,\alpha)^2. \tag{17b}$$

In view of this result, the B-S adiabatic invariant can be rewritten as

$$\oint p\,dq = \oint m\dot{q}\,dq = \oint m\dot{q}^2 dt = 2\pi m \sum_{\alpha=-\infty}^{\infty} |a_\alpha(n)|^2 \omega(n,\alpha)^2 = nh + const. \tag{18}$$

Next, Heisenberg proceeds as follows. Since the *const* is unknown, it is of interest to obtain results which are constant-independent. At the same time, since the result, Eq.(18), is assumed to be exact, we have to use instead of scalars $|a_\alpha(n)|^2$ the matrices in accord with Eq.(7). This causes us to use matrices of the type

$|a(n, n + \alpha)|^2$ and $|a(n, n - \alpha)|^2$ depending on the actual sign of α. In addition, he had silently assumed that the n -dependence for amplitudes is much weaker than that for the frequencies $\omega(n, n + \alpha)$ and $\omega(n, n - \alpha)$ so that it can be neglected completely. Under such conditions he treats n as continuous variable and differentiates both sides of Eq.(18) with respect to n thus obtaining the following result:

$$h = 4\pi m \sum_{\alpha=0}^{\infty} \{|a(n, n + \alpha)|^2 \, \omega(n, n + \alpha) - |a(n, n - \alpha)|^2 \, \omega(n, n - \alpha)\}. \quad (19)$$

Obtained result takes into account that $\omega(mn) = -\omega(nm)$. The validity of this result depends upon additional assumption about the ground state energy. If n_0 represents such a state, then one must require that $a(n_0, n_0 - \alpha) = 0$ for all $\alpha > 0$. When (19) is used in combination with the results from Appendix A, the famous commutation rule

$$[\hat{x}, \hat{p}] = i\hbar \quad (20)$$

is obtained. From the above derivation and results of Appendix A several conclusions can be drawn.

First, the number of x-p commutators by construction is in one-to one correspondence with the number of the B-S adiabatic invariants. This means that the system which is completely integrable classically can be completely quantized. However, if classically system is nonintegrable, one cannot write the classical Hamiltonian and to replace x's and p's in it by the corresponding operators obeying commutation relations, Eq.(20), for each generalized degree of freedom. Although this prescription is used routinely in the existing textbooks on quantum mechanics, rigorously speaking, one cannot write down the Schrödinger's equation in such a case so that formally Bohr's intuition was correct.

Second, Eq.(19) assumes that the underlying mechanical system, when it is written in terms of the action-angle variables, is essentially the set of independent harmonic oscillators. Heisenberg's derivation explicitly assumes that quantum mechanically there is a ground state-typical for the harmonic oscillator-but otherwise the spectrum is boundless. If the system is nonintegrable, again, the commutation rule, Eq.(20), is not justified. Hence, once again, one cannot write the Schrödinger's equation. To by pass this difficulty Heisenberg developed perturbation theory (for one dimensional case only!) which uses the classical perturbation theory as an input modified by the imposed (quantum) requirements on amplitudes and frequencies. Although he did not discuss the resonances, he did checked that the obtained perturbative expressions for energies are in agreement with the basic equations (4) and (5) assuring correct quantization. No attempts to study the multidimensional case was made.

Third, as results of Appendix A demonstrate, the experimental justification of the commutator rule, Eq.(20), is based on the validity of results of the first order perturbational calculations. Mathematically, such a procedure is questionable or, better, may be totally unacceptable.

16

Furthermore, the B-S quantization cannot be used for spin quantization (since formally there is no classical analog of spin, i.e. the B-S rule does not account for half integers). The spin has no place in the Schrödinger formalism, and, *apparently*, there is no room for spin in the Heisenberg's formalism as well. Fortunately, this happens only apparently as we would like to discuss now. This is possible only because the facts just listed do not affect the main Heisenberg's quantization postulate, Eq.(5), which, as recognized already by Heisenberg, is more fundamental than the x-p commutator identity.

2.4   From physics back to combinatorics

To find a way out of the difficulties just described let us return back to the expression $< n \mid \hat{O} \mid k > \exp(i\omega(n,k)t)$. Suppose that the algebra of observables contains an identity element (operator). Then, by replacing $\hat{O}$ by this operator we obtain, $< n \mid k >=< n(t) \mid k(t) >$. This makes sense only if we require $< n \mid k >= const\delta_{nk}$. Clearly, we can always adsorb the constant into the definition of the scalar product. In this work, following (Kholodenko 2006a) we suggest to replace the basic commutators, Eq.(20), by the requirement of orthogonality. This requirement is compatible with the requirement that the operators describing observables are Hermitian whose eigenfunctions are mutually orthogonal. Instead of operators whose explicit form is difficult to obtain we shall focus our attention on the properties of orthogonal functions and, more generally, on the properties of orthogonal polynomials (e.g. s_λ, etc.). Development of theory of orthogonal polynomials of several variables in connection with quantum exactly solvable model systems is an active area of current research (Vershik 2003; Kuznetsov 2006; Baik et al 2007). Such an approach makes sense since it is known (Orlik and Terrao 2001) that *all* one- variable orthogonal functions of exactly solvable problems in quantum mechanics (Flugge 1971) are obtainable as various limiting cases of the Gauss-type hypergeometric functions[18]. Following ideas by Aomoto, Orlik and Terrao demonstrate that the hypergeometric functions of multiple arguments (of which the Gauss-type is just a special case) are expressible in the form of period integrals[19]. By the principle of complementarity all many-body exactly solvable quantum mechanical problems should have hypergeometric functions of multiple arguments as eigenfunctions. The most important fact for our developments lies in the observation that when these functions become eigenfunctions (as it is known in one component case), this produces orthogonal polynomials-different for different many-body quantum mechanical problems. This fact can be formulated as a problem : for a given set of orthogonal polynomials find the corresponding many-body operator for which such a set of orthogonal polynomials forms a complete set of eigenfunctions.

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[18] This fact will be discussed in detail in Section 3.

[19] Periods can be associated with the homology basis -different for different (algebraic) manifolds. Interested readers may consult either (Orlik and Terrao 2001) or (Vassiliev 2002) for more details.

After these general remarks, we are ready to provide more concrete evidence that this is indeed the case. The symmetric group S_n has the following presentation in terms of generators s_i and Coxeter relations:

$$
\begin{aligned}
s_i^2 &= 1, \\
s_i s_j &= s_j s_i \quad \text{for} \quad |i - j| \geq 2, \\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}.
\end{aligned}
\tag{21}
$$

If there is a set of n elements of whatever kind the generator s_i interchanges an element i with $i + 1$ so that $s_1, ..., s_{n-1}$ generate S_n. There are $n!$ permutations in the set of n elements. If we assign the initial ordered state, then any other state can be reached by successful application of permutational generators to this state. The word $w = s_{a_1} s_{a_2} \cdots s_{a_l}$ (where the indices $a_1, ..., a_l$ represent a subset of the set of $n - 1$ elements) can be identified with such a state. Since one can reach this state in many ways, it makes sense to introduce the *reduced word* w whose length $l(w)$ is minimal. We would like the generators of S_n to act on monomials $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. For this purpose, following Lascoux and Schützenberger (1983) (L-S) we introduce an operator ∂_i via rule:

$$
\partial_i := \frac{1 - s_i}{x_i - x_{i+1}}.
\tag{22}
$$

It acts on monomials such as $\mathbf{x}^{\mathbf{a}}$ in such a way that the generator s_i acting on the combination $x_i^{a_i} x_{i+1}^{a_{i+1}}$ converts it into $x_i^{a_{i+1}} x_{i+1}^{a_i}$. By construction, the action of this operator on monomial is zero if $a_i = a_{i+1}$, otherwise it diminishes the degree of the monomial by 1. In addition, the same authors introduce operators

$$
\bar{\pi}_i = \frac{(1 - s_i)}{x_i - x_{i+1}} x_{i+1}
\tag{23}
$$

and $\pi_i = 1 + \bar{\pi}_i$. Finally, being armed with such definitions, we can introduce an operator $D_i(p, q, r) = p \partial_i + q \bar{\pi}_i + r s_i$[20], where p, q and r are some numbers. L-S demonstrate that such defined operator, while acting on monomials, obeys the braid-type relations (the 2nd and the 3-rd lines in Eq.(21)) while the relation $s_i^2 = 1$ is replaced by

$$
D_i^2 = q D_i + r(q + r).
\tag{24a}
$$

With constants p, q and r properly chosen, such a relationship defines the Hecke algebra H_n of the symmetric group S_n. Usually, it is written as

$$
D_i^2 = (1 - Q) D_i + Q
\tag{24b}
$$

with Q being some number (effectively playing the same role as p, q, r). H_n should be considered as a deformation of S_n. The rationale for its introduction lies in its direct connections with the knot and link theory so that quantum

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[20] By doing so, the operators $\partial_i, \bar{\pi}_i$ and s_i become equivalent in the sense which we shall explain shortly.

mechanics can be considered as some branch of this theory (Kholodenko 2006a). This fact will have its impact on quantization. To demonstrate this, following Kirillov Jr.(1997), by relabeling earlier defined operator ∂_i as b_{ij} we reserve the notation $\partial_i = \frac{\partial}{\partial x_i}$ for the usual operator of differentiation. With its help we introduce the so called Dunkl operator $\mathcal{D}_i$ via

$$\mathcal{D}_i = \partial_i + k \sum_{j \neq i} b_{ij} \qquad (25)$$

with k being some (known) constant. Such defined operator acts on monomials (polynomials). It possesses the property $w\mathcal{D}_i w^{-1} = \mathcal{D}_{w(i)} \; \forall i \in S_n$. Consider now the commutator $[\mathcal{D}_i, \mathcal{D}_j]$. Kirillov demonstrated that such a commutator is zero if b_{ij} satisfy the classical Yang-Baxter equation (CYBE)

$$[b_{12}, b_{13}] + [b_{12}, b_{23}] + [b_{13}, b_{23}] = 0. \qquad (26)$$

Alternatively, Eq.(26) can be taken as the definition for b_{ij}. This is facilitated by designing of the *degenerate affine Hecke algebra* (Cherednik 2005). The purpose of this algebra from the physical point of view is to introduce the Heisenberg commutation rule Eq.(20) without reference to the B-S quanization prescription or to the (optical) sum rule described in Appendix A. Such an algebra is made up as a semidirect product of S_n with the commutator algebra

$$x_{i+1}s_i - s_i x_i = h; \; x_i s_j = s_j x_i \; \; \forall i \neq j, j+1; \; x_i x_j = x_j x_i, \qquad (27a)$$

where the constant h is playing essentially the same role as the Plank's constant $\hbar$. From the above definitions it follows that Eq.(27a) is the discrete analog of the Heisenberg's commutation rule, Eq.(20). Furthermore, in view of the remark made after introduction of $D_i(p,q,r)$, it is possible to rewrite the commutator in Eq.(27a) in the equivalent form. This indeed was accomplished in the paper by Adin et al (2000). Hence, we can rewrite Eq.(27a) equivalently as

$$x_i \partial_i - \partial_i x_{i+1} = h; \quad \partial_i x_i - x_{i+1} \partial_i = h ; \; x_i \partial_j = \partial_j x_i \; \; \forall i \neq j, j+1; \; x_i x_j = x_j x_i,$$
$$(27b)$$

where ∂_i should be understood in the sense of Eq.(22). At this point it is useful to introduce yet another operator $\hat{s}_i = s_i + hb_{i,i+1}$. It is designed in such a way that it obeys the braid relations:

$$\hat{s}_1 \hat{s}_2 \hat{s}_1 = \hat{s}_2 \hat{s}_1 \hat{s}_2. \qquad (28)$$

Furthermore, if now we define the operators $R_{12} = s_1 \hat{s}_1, R_{23} = s_2 \hat{s}_2, R_{13} = s_1 R_{23} s_1 = s_2 R_{12} s_2$, then the Eq.(28) becomes equivalent to the standard Yang-Baxter (Y-B) equation for $R_{ij} = 1 + hb_{ij}$ (or $R_{ij} \simeq \exp(hb_{ij})$ for $h \to 0$). Explicitly, we obtain: $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$.

All this discussion looks a bit formal at this point. Indeed, why to introduce the operator $\mathcal{D}_i$? Why to be concerned about the commutator $[D_i, D_j]$?

19

What the Yang-Baxter equations have to do with the results of this paper? We would like to provide the answers to these questions now and in the following subsection.

First, consider an equation $\mathcal{D}_i f = 0$. It can be written alternatively as

$$\kappa \frac{\partial}{\partial z_i} f(\mathbf{z}) = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} f(\mathbf{z}) \tag{29}$$

which is just the celebrated Knizhnik-Zamolodchikov (K-Z) equation[21]. This means that: a) the operator $\mathcal{D}_i$ is effectively a covariant derivative (the Gauss-Manin connection in the formalism of fiber bundles) and, b) that the vanishing of commutator $[\mathcal{D}_i, \mathcal{D}_j]$ is just the zero curvature condition (Kassel 1995) essential for all known exactly integrable systems. The question still remains: how Ω_{ij} in Eq.(29) is related to b_{ij} in Eq.(25)? The answer was found by Belavin and Drinfel'd (1982). In the simplest (rational) case we have $b_{ij}(z) = \frac{\Omega_{ij}}{z}$, as expected. More complicated trigonometric and elliptic cases found by Belavin and Drinfel'd are summarized in the book by Etingof with collaborators (Etingof et al 1998). From the references just provided, it should be clear that since solutions of the K-Z equations are expressible in terms of hypergeometric functions of single and multiple arguments, all examples of exactly solvable quantum mechanical problems (including those involving the Dirac equation, and, hence, the spin) found in the textbooks on quantum mechanics are covered by the formalism we have just described. In the next subsection we would like to illustrate these results by concrete physical examples taken from current physical literature.

2.5 Latest developments in atomic physics illustrating general principles

In the review paper by Tanner et al (2000) as well as in the book by Cvitanovic'(1998) it is explained in detail that in order to calculate the He spectrum it is sufficient either: a) to consider the classical dynamics of two electrons and the nucleus *on the line* and to use this information in the semiclassical trace formula producing very accurate results for the spectrum or, b) to restrict quantum mechanical calculations to the spherical approximation (the so called s- wave approximation) in order to arrive at the exactly solvable *radial* Schrodinger-type equation for two electrons and massive nucleus (Draeger et al 1994, Howard and March 2005) producing very reasonable results for the He spectrum. To these achievements we would like to add those by Svidzinsky et al (2005 a,b) and Muravski and Swidzinsky (2006) where the same type or even better accuracy for He and other atoms and diatomic molecules is obtained using the so called d-scaling. In this method the multielectron Schrödinger equation is analyzed in various dimensions. Upon proper rescaling, the limiting case: d→ ∞, is reduced again to the

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[21] In fact, in general case (Orlik and Terrao 2001) the scalar function $f(\mathbf{z})$ is replaced by the vector function $\mathbf{f}(\mathbf{z})$. This fact should be kept in mind in actual calculations.

exactly solvable radial-type multielectron equation which in the present case becomes classical equation considered already by Bohr in 1913. Thus again, the zeroth order exactly solvable problem is one dimensional. Corrections in powers of 1/d are easily calculable producing results which compare extremely well with much more cumbersome (and time consuming) Hartree-Fock type calculations. To this list of examples it is appropriate to add the work by Ostrovsky and Prudov (1995) which uses essentially the same averaging and perturbation methods as developed in celestial mechanics (Ferraz-Mello 2007) superimposed with the Bohr-Sommerfeld quantization prescription. All examples discussed in this subsection were done without theoretical guidance (other than the proof of the existence of minimizers for Bohr-type functionals (Chen et al 2006)). The theoretical framework developed in this section naturally explains why these results are actually working so well. This framework is essential for developing applications in celestial mechanics to be discussed in the rest of this paper.

3 Space, time and space-time in classical and quantum mechanics

3.1 General comments

If one contemplates quantization of dynamics of celestial objects using traditional textbook prescriptions, one will run into myriad of small and large problems immediately. Unlike the atomic systems in which all electrons repel each other, have the same masses and are indistinguishable, in the case of, say, Solar System all planets (and satellites) attract each other, have different masses and visibly distinguishable. Besides, in the case of atomic systems the Planck constant $\hbar$ plays the prominent role while no such a role can be given to the Planck constant in the sky. The only thing which remains in common between both atomic and celestial dynamic systems is the existence of stable closed orbits. In the previous section we demonstrated that this fact is absolutely essential for quantization. Nevertheless, the formalism developed thus far resembles more the existence theorem in mathematics rather then the actual manual describing the computational protocol. The task now lies in developing necessary constructive steps leading to actual implementation of general principles. This task is accomplished below and in the following section

3.2 Space and time in classical and quantum mechanics

Although celestial mechanics based on the Newton's law of gravity is considered to be classical (i.e.nonquantum), with such an assumption one easily runs into serious problem. Indeed, such an assumption implies that the speed with which the interaction propagates is infinite and that the time is the same everywhere. Wether this is true or false can be decided only experimentally. Since at scales of our Solar System one has to use radio signals to check the correctness of

Newton's celestial mechanics one is faced immediately with all kind of wave mechanics effects such as retardation, the Doppler effect, etc. Because of this, the measurements are necessarily having some error margins. The error margins naturally will be larger for more distant objects. Accordingly, even at the level of classical mechanics applied to the motion of celestial bodies we have to deal with certain inaccuracies similar in nature to those in atomic mechanics. To make formalisms of both atomic and celestial mechanics look the same one has to think about the space, time and space-time transformations already at the level of classical mechanics.

We begin with observation that in the traditional precursor of quantum mechanics-the Hamiltonian mechanics-the Hamiltonian equations *by design* remain invariant with respect to the canonical transformations (Pars 1966). That is if sets $\{q_i\}$ and $\{p_i\}$ represent the "old" canonical coordinates and momenta while $Q_i = Q_i(\{q_i\}, \{p_i\})$ and $P_i = P_i(\{q_i\}, \{p_i\})$, $i = 1 - N$, represent the "new" set of canonical coordinates and momenta, the Hamiltonian equations in the old variables given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \text{ and } \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{30}$$

and those rewritten in "new" variables will have the same form. Here we used the commonly accepted notations, e.g. $\dot{q}_i = \frac{d}{dt}q_i$, etc. Quantum mechanics uses this form-invariance essentially since the Poisson brackets introduced in Eq.(9) by design will also have the same form in terms of both "old" and "new" canonical variables.

We would like to complicate matters by investigating the possibility of the "canonical " time changes in classical mechanics. Fortunately, such a possibility was explored to a some extent already. This is described in the monograph by Pars (1966) thus making our task considerably simpler. For the sake of space, we refer our readers to pages 535-540 of this monograph. Furthermore, following Dirac (1950) , we notice that a good quantization procedure should always begin with the Lagrangian formulation of mechanics since it is not always possible to make a transition from the Lagrangian to Hamiltonian form of mechanics (and, thus, to quantum mechanics) due to the presence of some essential constraints (typical for mechanics of gauge fields, etc.). Hence, we also begin with the Lagrangian functional $\mathcal{L} = \mathcal{L}(\{q_i\}, \{\dot{q}_i\})$. The Lagrangian equations of motion can be written in the form of Newton's equations given by $\dot{p}_i = F_i$, where the generalized momenta p_i are given by $p_i = \delta\mathcal{L}/\delta\dot{q}_i$ and the generalized forces F_i are given by $F_i = -\delta\mathcal{L}/\delta q_i$. In the case if the total energy E is conserved, it is possible instead of "real" time t to introduce the fictitious time θ via relation $dt = u(\{q_i\})d\theta$ where the function $u(\{q_i\})$ is assumed to be nonnegative and is sufficiently differentiable with respect to its arguments. At this point we can enquire if Newton's equations can be written in terms of new time variable so that they remain form- invariant. To do so, following Pars, we must: a) to replace $\mathcal{L}$ by $u\mathcal{L}$, b) to replace $\dot{q}_i$ by q'_i /u, where $q'_i = \frac{d}{d\theta}q_i$, c) to rewrite the new Lagrangian in terms of such defined new time variables and, finally, d) to obtain Newton's equations according to the described rules, provided that now we have

to use p'_i instead of $\dot{p}_i$. In the case if the total energy of the system is conserved, we shall obtain back the same form of Newton's equations rewritten in terms of new variables. This means that by going from the Lagrangian to Hamiltonian formalism of classical mechanics we can write the Hamilton's Eq.(30) in which the dotted variables are replaced by primed. These arguments demonstrate connections between space and time already at the level of classical mechanics. Situation here is similar to that encountered in thermodynamics where instead of absolute temperature one can use any nonegative function of absolute temperature as new temperature (Fermi 1937). Using these arguments we notice that since the temperature is conjugate to energy in thermodynamics, the time is conjugate to energy in mechanics and, accordingly, in quantum mechanics. This means that for the nondissipative (i.e. energy conserving) Hamiltonian system[22] the Hamiltonian equations of motion, Eq.(30), will remain form- invariant if we replace the Hamiltonian H by some nonnegative function $f(H)$ while changing time t to time θ according to the rule $d\theta/dt = df(H)/dH \mid_{H=E}$. Such a change will affect the quantum mechanics where now the Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi \tag{31a}$$

is to be replaced by

$$i\hbar\frac{\partial}{\partial \theta}\Psi = f(\hat{H})\Psi. \tag{31b}$$

With such an information at our hands, we would like to discuss the extent to which symmetries of our (empty) space-time affect dynamics of particles "living" in it.

3.3 Space-time in quantum mechanics

3.3.1 General comments

Use of group-theoretic methods in quantum mechanics had began almost immediately after its birth. It was initiated by Pauli in1926. He obtained a complete quantum mechanical solution for the Hydrogen atom employing symmetry arguments. His efforts were not left without appreciation. Our readers can find many historically important references in two comprehensive review papers by Bander and Itzykson (1966). In this subsection we pose and solve the following problem: Provided that the symmetry of the (classical or quantum) system is known, will this information be sufficient for determination of this system uniquely? Below, we shall provide simple and concrete examples illustrating

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[22]It should be kept in mind that the concept of nondissipativity is actually of quantum origin (e.g. recall superconductors or superfluids). In classical mechanics such a concept is just a convenient idealization similar to the notion of a material point in Newton's mechanics or the notion of thermodynamics when it is applied to the real heat engines, etc. The truly nondissipative mechanical systems thus should behave quantum mechanically. This observation provides the hint that some stable motions in our Solar System are of quantum nature. In view of Eq.(6) this option makes sense.

meaning of the word "determination". In the case of quantum mechanics this problem is known as the problem about hearing of the "shape of the drum". It was formulated by Mark Kac (Kac 1966). The problem can be formulated as follows. Suppose that the spectrum of the drum is known, will such an information determine the shape of the drum uniquely? The answer is "No" (Dhar et al 2003). Our readers may argue at this point that nonuniqueness could come as result of our incomplete knowledge of symmetry or, may be, as result of the actual lack of true symmetry (e.g. the Jahn-Teller effect in molecules, etc. in the case of quantum mechanics). These factors do play some role but they cannot be considered as decisive as the basic example below demonstrates.

3.3.2   Difficulties with the correspondence principle for Hydrogen atom

In this subsection we even do not use arguments by Kac. Since our arguments are straightforward they are more intuitively appealing: we choose the most studied case of the Hydrogen atom as an example.

As it is well known, the classical mechanical problem about motion of the particle in centrally symmetric field is planar and is exactly solvable for both the scattering and bound states (Pars 1966, Landau and Lifshitz 1960). The result of such a solution depends on two parameters: the energy and the angular momentum. The correspondence principle formulated by Bohr (1923) is expected to provide the bridge between the classical and quantum realities by requiring that in the limit of large quantum numbers the results of quantum and classical calculations for observables should coincide. Appendix A provides a good example of such kind of thinking. However, this requirement may or may not be possible to implement. It is violated already for the Hydrogen atom. Indeed, according to the naive canonical quantization prescriptions, one should begin with the *classical* Hamiltonian in which one has to replace the momenta and coordinates by their operator analogs. Next, one uses such constructed "quantum" Hamiltonian in the Schrödinger's equation, etc. Such a procedure breaks down at once for the Hamiltonian of the Hydrogen atom since the intrinsic planarity of the classical Kepler's problem is entirely ignored thus leaving the projection of the angular momentum without its classical analog. Accordingly, the scattering states of Hydrogen atom and the classical mechanically obtained Rutherford's formula obtained for planar configurations are reproduced quantum mechanically (within the 1st Born approximation) using the 3-d Schrödinger's equation ! Thus, even for the Hydrogen atom the classical and the quantum (or, better, pre quantum) Hamiltonians *do not* match thus formally violating the correspondence principle. Evidently, semiclassically we can only think of energy and the angular momentum thus leaving the angular momentum projection unobserved. Such a "sacrifice" is justified by the agreement between the observed and predicted Hydrogen atom spectra and by use of Hydrogen-like atomic orbitals for multielectron atoms. Although, to our knowledge, such a mismatch is not mentioned in any of the existing textbooks on

quantum mechanics, its existence is essential if we are interested in applications of quantum mechanical ideas to Solar System dynamics. In view of such an interest, we would like to reconsider traditional treatments of Hydrogen atom, this time being guided only by symmetry considerations. This is accomplished in the next subsection.

3.3.3 Emergence of the SO(2,1) symmetry group

In April of 1940 Jauch and Hill (1940) published a paper in which they studied the planar Kepler problem quantum mechanically. Their work was stimulated by earlier works by Fock of 1935 and by Bargmann of 1936 in which it was shown that the spectrum of bound states for the Hydrogen atom can be obtained by using representation theory of SO(4) group of rigid rotations of 4-dimensional Euclidean space while the spectrum of scattering states can be obtained by using the Lorentzian group SO(3,1). By adopting results of Fock and Bargmann to the planar configuration Jauch and Hill obtained the anticipated result. In the planar case one should use SO(3) group for the bound states and SO(2,1) group for the scattering states. Although this result will be reconsidered almost entirely, we mention about it now having several purposes in mind.

First, we would like to reverse arguments leading to the final results of Jauch and Hill in order to return to the problem posed at the beginning of this section. That is, the fact that the Kepler problem is planar (due to central symmetry of the force field) and the fact that the motion is restricted to the plane and takes place in (locally) Lorentzian space-time are the most general symmetry constraints imaginable. Thus, the fact that the Lorentz SO(2,1) group is related to the spectrum of Kepler problem should be anticipated. Nevertheless, the question remains: is Kepler's problem the only one exactly solvable classical and quantum mechanical problem associated with the SO(2,1) group? Below we demonstrate that, unfortunately, this is not the case. In anticipation of such negative result, we would like to develop our intuition by using some known results from quantum mechanics.

3.3.4 Classical-quantum correspondence allowed by SO(2,1) symme-
try: a gentle introduction

For the sake of space, we consider here only the most generic (for this work) example in some detail: the radial Schrödinger equation for the planar Kepler problem with the Coulombic potential. It is given by[23]

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{m^2}{\rho^2}\right)\Psi(\rho) - \frac{Ze^2}{\rho} = E\Psi(\rho). \qquad (32)$$

Here $|m| = 0, 1, 2, ..$ is the angular momentum quantum number as required. For $E < 0$ it is convenient to introduce the dimensionless variable x via $\rho = ax$ and

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[23]The rationale for discussing the Coulombic potential instead of gravitational will be fully explained in the next section.

to introduce the new wave function: $\psi(\rho) = \sqrt{\rho}\Psi(\rho)$. Next, by the appropriate choice of constant a and by redefining $\psi(\rho)$ as $\psi(\rho) = \gamma x^{\frac{1}{2}+|m|} \exp(-y)\varphi(y)$, where $y = \gamma x$, $-\gamma^2 = \frac{2\mu E}{\hbar^2}a^2$, $a = \frac{\hbar^2}{\mu Z E}$, the following hypergeometric equation can be eventually obtained:

$$\left\{ y\frac{d^2}{dy^2} + 2[|m| + \frac{1}{2} - y]\frac{d}{dy} + 2[\frac{1}{\gamma} - |m| - \frac{1}{2}] \right\} \varphi(y) = 0. \qquad (33)$$

Formal solution of such an equation can be written as $\varphi(y) = \mathcal{F}(-A(m), B(m), y)$, where $\mathcal{F}$ is the confluent hypergeometric function. Physical requirements imposed on this function reduce it to a polynomial leading to the spectrum of the planar Kepler problem. Furthermore, by looking into standard textbooks on quantum mechanics, one can easily find that *exactly the same type of hypergeometric equation* is obtained for problems such as one-dimensional Schrödinger's equation with the Morse-type potential,[24] three dimensional radial Schrödinger equation for the harmonic oscillator[25] and even three dimensional radial equation for the Hydrogen atom[26]. Since the two-dimensional Kepler problem is solvable with help of the representations of SO(2,1) Lorentz group, the same should be true for all quantum problems just listed. That this is the case is demonstrated, for example, in the book by Wybourne (1974). A sketch of the proof is provided in Appendix B. This proof indicates that, actually, the *discrete spectrum* of all problems just listed is obtainable with help of SO(2,1) group. The question remains: if the method outlined in Appendix B provides the spectra of several quantum mechanical problems listed above, can we be sure that these are the only exactly solvable quantum mechanical problems associated with the SO(2,1) Lorentz group? Unfortunately, the answer is "No". More details are given below.

3.3.5 Common properties of quantum mechanical problems related to SO(2,1) Lorentz group

In Appendix B we provide a sketch of the so called spectrum-generating algebras (SGA) method producing the exactly solvable one-variable quantum mechanical problems. In this subsection we would like to put these results in a broader perspective. In particular, in Section 2 we demonstrated that all exactly solvable quantum mechanical problem should involve hypergeometric functions of single or multiple arguments. We argued that the difference between different problems can be understood topologically in view of the discussed relationship with braid groups. On another hand, obtained results, even though rigorous, are not well adapted for immediate practical use. In this regard more useful

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[24] That is, $V(x) = A(exp(-2\alpha x) - 2exp(-\alpha x))$.

[25] That is, $V(r) = \dfrac{A}{r^2} + Br^2$.

[26] That is, $V(r) = \dfrac{A}{r^2} - \dfrac{B}{r}$.

would be to solve the following problem: For a given set of orthogonal polynomials find the corresponding many-body operator for which such a set of orthogonal polynomials forms a complete set of eigenfunctions. At the level of orthogonal polynomials of one variable relevant to all exactly solvable two-body problems of quantum mechanics, one can think about related problem of finding all potentials in the one-dimensional radial Schrödinger equation, e.g. Eq.(B.1), leading to the hypergeometric-type solutions. Such a task was accomplished by Natanzon (1979). Subsequently, his results were reinvestigated by many authors with help of different methods, including SGA. To our knowledge, the most complete recent summary of the results, including potentials and spectra can be found in the paper by Levai (1994). Even this (very comprehensive) paper does not cover all aspects of the problem. For instance, it does not mention the fact that these results had been extended to relativistic equations such as Dirac and Klein-Gordon for which similar analysis was made by Cordero with collaborators (1971). In all cited cases (relativistic and non relativistic) the underlying symmetry group was SO(2,1). The results of Appendix B as well as all other listed references can be traced back to the classically written papers by Bargmann (1947) and Barut and Fronsdal (1965) on representations of SO(2,1) Lorentz group. Furthermore, the discovered connection of this problematic with supersymmetric quantum mechanics (Cooper et al 1987, Junker and Roy 1998) can be traced back to the 19th century works by Gaston Darboux (Levai 1994).

Summarizing, the established in Section 2 rigorous connections between exactly solvable two-body quantum mechanical problems and hypergeometric functions and, by complementarity principle, between the exactly solvable many body problems and hypergeometric functions of many arguments are consequences of the locally Lorentzian group structure of our space-time. Such a structure allows many *but not infinitely many* exactly solvable problems to exist. The fact that the *planar* SO(2,1) is sufficient to cover all known exactly solvable two-body cases (instead of the full SO(3,1) Lorentz group) is quite remarkable. It is sufficient for the purposes of this work but leaves open the question : Will use of the full Lorentz group lead to the exactly solvable quantum mechanical problems not accounted by SO(2,1) group symmetry? This topic will be discussed in Section 5. In the meantime, we would like to address the problem of quantization of Solar System dynamics using results of Sections 2 and 3. This is done in the next section

4   Quantum celestial mechanics of Solar System

4.1   General remarks

We begin this section by returning back to Eq.(6) once again. Based on previous discussions, this equation provides us with the opportunity to think seriously about the quantum nature of our Solar System dynamics. Nevertheless, such an equation reveals only one aspect of quantization problem and, as such, provides

only a sufficient condition for quantization. The necessary condition in atomic and celestial mechanics lies in the *nondissipativity* of the dynamical systems in both cases[27]. Recall that Bohr introduced his quantization prescription to avoid dissipation caused by the emission of radiation by electrons in orbits in general position. As we demonstrated previously, new quantum mechanics have *not* explained absence of dissipation for stationary Bohr's orbits[28]. In fact, as our analysis of Heisenberg's work(s) indicates, new quantum mechanics have *not* added a single new element to the old atomic mechanics in terms of the new issues to be considered. In the nutshell, *new quantum mechanics provided a convenient computational tool for dealing with otherwise purely mechanical problems involving accidental degeneracy (that is resonances). By doing so, it made no attempt at explaining (using known results from mechanics and electrodynamics) the nondissipativity.* Nevertheless, the phenomenon of nondissipativity was explained quite convincingly in the case of superconductivity and superfluidity later on. Thanks to these intrinsically quantum phenomena, we can be sure that quantum mechanics did capture some truth. Regrettably, only some since, as we discussed in Section 3.2.2, even for the most studied case of Hydrogen atom the task of establishing the correspondence between the classical and quantum models of Hydrogen atom is nontrivial. The symmetry (and supersymmetry) arguments of Section 3 based on the locally Lorentzian space-time structure as well as the combinatorial arguments of Section 2 simplified the task of establishing the quantum-classical correspondence considerably. This happened because of firmly established finite number of exactly solvable quantum mechanical problems allowed by the Lorentzian-type symmetry whose spectra are known and documented. These facts allow us to think seriously about quantization of Solar System dynamics.

4.2   From Laplace to Poincare$'$ and Einstein

Before discussing this issue in some detail, we still need to make several remarks. First, although superficially the classical Hamiltonians for Coulombic and Newtonian potentials look almost the same, the naive textbook-style quantization will immediately run into major problems. For one thing, all electron masses are the same while all planetary/satellite masses are different. For other thing, filling of atoms by electrons is controlled by the electric charge of the nucleus so that stable atoms/molecules are electrically neutral. Apparently, no such restriction exists for the system of gravitating bodies. Next, apparent violation of planarity of Hydrogen atom treated at the level of classical mechanics is justified by the fact that the angular momentum projection does play an important role in chemistry. As far as we can see, nothing of that sort exists in the sky.

To deal with the mass differences for planetary systems we have to recall some facts from general relativity. We shall restrict ourself only by some illustrative examples meant to provide some feeling of problems we would like to

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[27] E.g. see the paper by Goldreich (1965) mentioned in Section 1.2.

[28] At the level of old Bohr theory absence of dissipation at the stationary Bohr orbit was explained by Boyer (1975). Subsequently his result was refined by Puthoff (1987).

discuss. To this purpose we would like to make some comments on the classical mechanical treatment of Kepler problem in representative physics textbooks, e.g. read (Landau and Lifshitz 1960, Goldstein et al 2002). Such treatments tend to ignore the equivalence principle essential for the gravitational Kepler problem and nonexistent for the Coulomb-type problems. This causes some significant inaccuracies to emerge. Specifically, according to Vol.2 of famous Landau-Lifshitz course in theoretical physics (Landau and Lifshitz 1975) if we take $\mathcal{L} = \dfrac{m\mathbf{v}^2}{2} - m\varphi$ as the Lagrangian for a particle in gravitational field (represented by a local potential φ),the Lagrangian (Newtonian) equations of motion can be written as

$$\dot{\mathbf{v}} = -\boldsymbol{\nabla}\varphi \tag{34}$$

so that the mass drops out of this equation making it possible to think about such an equation as equation for a geodesic in pseudo-Riemannian space. This observation had lead Einstein to full development of general relativity theory. By noticing that Newton's equation makes sense only for material points (that is for idealized *formally nonexisting* objects) the same must be true for Eq.(34). Hence, as such it is valid only for the well localized point-like objects. Using such idealized model, we need to discuss briefly the 2-body Kepler problem for particles with masses m_1 and m_2 interacting gravitationally. The Lagrangian for this problem is given by

$$\mathcal{L} = \frac{m_1}{2}\dot{\mathbf{r}}_1^2 + \frac{m_2}{2}\dot{\mathbf{r}}_2^2 + \gamma\frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{35a}$$

Introducing the center of mass and relative coordinates via $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the above Lagrangian can be rewritten as

$$\mathcal{L} = \frac{\mu}{2}\dot{\mathbf{r}}^2 + \gamma\frac{m_1 m_2}{|\mathbf{r}|} \equiv \frac{m_1 m_2}{m_1 + m_2}\left(\frac{\dot{\mathbf{r}}^2}{2} + \gamma\frac{(m_1 + m_2)}{|\mathbf{r}|}\right), \tag{35b}$$

where, as usual, we set $\mu = \frac{m_1 m_2}{m_1 + m_2}$.The constant $\frac{m_1 m_2}{m_1 + m_2}$ can be dropped and, after that, instead of the geodesic (34) we obtain the equation for a fictitious point-like object of unit mass moving in the field of gravity produced by the point-like body of mass $m_1 + m_2$. Clearly, in general, one cannot talk about geodesics in this case. Nevertheless, as it is usually done, if, say, $m_1 \gg m_2$ (as for the electron in Hydrogen atom or for the Mercury rotating around Sun) one can with a very good accuracy discard mass m_2 thus obtaining an equation for a geodesic. Such an approximation was indeed made by Einstein in his major work on general relativity (Einstein 1916) in which he ignored the mass of Mercury entirely when making his calculations of the perihelium shift for this planet. More recent results (Misner, Thorne and Wheeler 1973) show that such an approximation is expected to be quite satisfactory for other planets of our Solar System[29]. With the exception of Pluto-Charon system, where $\mu_2 = m_2/(m_1 + m_2)$ is of order 10^{-1}, and the Earth-Moon system, where μ_2 is of

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[29]E.g. read Box 40.3 of this reference as well as pages 1126-1129.

order 10^{-2} , all other planet-satellite and Sun- planet pairs have μ_2 of order 10^{-3} and less (Murray and Dermott 1999) so that use of geodesics is justifiable physically. Mathematically, however, this is not quite the case since, even in the case of Mercury considered by Einstein it is necessary to prove that the influence of the rest of planets of Solar System on its motion can be ignored (as well as finite size of the Sun, etc.).

The task of proving that motion of planets can be well approximated by geodesics can be traced back to works by Laplace on celestial mechanics. Lecture notes by Moser and Zehnder (2005) contain accessible discussion of Laplace's works[30] to which we refer our readers for details. In short, Laplace was interested in dynamics of the planar 4-body problem using Jupiter and its 3 satellites: Io, Europe and Ganymede, as an example. He noticed that the motion of these satellites obeys the resonance condition and he was able to reproduce this motion analytically by ignoring satellite masses (just like in Eq.(35b), but beginning with the full 4-body problem initially). Under these conditions, gravitational interactions between satellites can be neglected so that the motions become completely decupled but subject to the resonance condition. Furthermore, to study stability of such resonance motions Laplace (and Lagrange) assumed that the actual (Lagrangian) motions[31] of satellites oscillate about the respective stable orbits of these satellites. Thus, *effectively, Laplace and Lagrange were considering the effects of general relativity and quantum mechanics long before these disciplines have been officially inaugurated.* In their lectures, Moser and Zehnder also provide references to works by Poincare$'$ and de Sitter on further refinements of Laplace's results. Although according to Arnol'd et al (2006) extension of the work by Laplace to the full n+1 body planar problem was given in the monograph by Charlier (1927), the rigorous mathematical proofs have been obtained only quite recently by Fejoz (2004) and Biasco et al (2006). To realize the difficulties in providing such a proof it is sufficient, following Poincare$'$(1892 − 1898),to demonstrate that the results of the massless limit considered by Laplace will remain practically unchanged if the satellites would have some small but finite masses (so that they interact with each other). Such a philosophy lies at the heart of KAM theory used and improved in the works by Fejoz (2004) and Biasco et al (2006).

Even with these proofs being available one should take into account that, in view of experimental limitations, Newton's law of gravity should be amended by taking into account finite speed of propagation of gravitational interaction as well as the fact that all observations are made with some kind of light/radio sources causing retardation, Doppler and other effects. Thus, taking into account the experimental conditions, the *classical mechanics* description of celestial motions becomes very similar to that *encountered in quantum mechanics* where one has to use probabilities to account for incompleteness of information. Furthermore, these proofs do not account for dissipation effects playing major stabilizing role in both atomic and quantum celestial mechanics.

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[30] E.g. read pages 102-120.
[31] E.g. see Arnol'd et al (2006) page 261.

If we assume that the motion of bodies indeed takes place on geodesics then, formally, there are no interactions and the local time becomes proper time. In the case of, say, binary stars of comparable masses one cannot use the geodesics for description of their relative motion[32] so that one is confronted with the problem of matching the Einsteinian gravity with its Newtonian limit as discussed in the paper by Einstein, Infeld and Hoffmann (1938)[33]. In all other theories of gravity, including the Brans-Dicke-Jordan's theory, there are substantial departures from the geodesic motion. Details can be found on pages 1127-1129 of the book by Misner et al (1973).

Clearly, the difficulties of explaining motions using classical mechanics of the n+1 body problem are such that the assumption about truly geodesic motion looks suspicious. But these Newtonian mechanics theories do not account for dissipation and retardation effects. These facts also complicate the choices between different (alternative) theories of gravity. Hence, it is clear that at the present state of our knowledge the ultimate choice between competing theories can be made only based on additional information. Such an information is supplied, in part, in this work where uses of historical analogies between the quantum (atomic) and celestial mechanics provide some helpful guidance. For this purpose we compiled our Table 3 prior to actual computations.

Table 3

\Type of mechanics Properties	Quantum atomic mechanics	Quantum celestial mechanics
Dissipation (type of)\ (yes\no)\on stable orbits	electromagnetic friction\no\ Bohr orbits	tidal friction \no\Einstein's geodesics
Accidental degeneracy\ (yes\no)\origin	yes\Bohr-Sommerfeld condition	yes\closure of the Lagrangian orbits
Charge neutrality	yes	no(but see below)
Masses	electrons having the same masses	(up to validity of the equivalence principle) masses are the same
Minimal symmetry group	SO(2,1)	SO(2,1)
Correspondence principle	occasionally violated	occasionally violated
Discrete spectrum: finite or infinite\reason\ Pauli principle(yes\no)	finite and infinite\ charge neutrality\ yes	finite\ no charge neutrality\ yes

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[32] This case was discussed in papers by Einstein, Infeld and Hoffmann (1938) and Robertson (1938) with the outcome that it is possible to describe gravitational field *outside* such a binary system in terms of geodesics. This leaves open the question of dynamical stability of such binaries since their motion is controlled by the Newton's equations of motion. In view of the effects of tidal friction, which should be quite appreciable in this case, the dynamics of such binaries should be most likely unstable. For such systems one can safely neglect friction caused by the emission of gravitational waves since these are effects of fifth order in c^{-1} (Landau and Lifshitz 1975, paragraph 106).

[33] See also Section 6

Details related to this table are discussed further below.

4.3   Celestial spectroscopy, the Titius-Bode law of planetary distances and quantum celestial mechanics

The atomic spectroscopy was inaugurated by Newton, in the second half of 17th century. As discussed in Introduction, the results of atomic and molecular spectroscopy were used by Bohr in essential way resulting in the birth of new quantum mechanics. The celestial spectroscopy was inaugurated by Titius in the second half of 18th century and become famous after it was advertised by Johann Bode, the Editor of the "Berlin Astronomical Year-book" in the late 18th century. The book by Nieto (1972) provides extensive bibliography related to uses and interpretations of the Titius-Bode (T-B) law up to a second half of 20th century. Unlike the atomic spectroscopy, where the observed atomic and molecular spectra were expressed using simple empirical formulas which were (to our knowledge) never elevated to the status of "law", in celestial mechanics the empirical T-B formula

$$r_n = 0.4 + 03. \cdot 2^n, \quad n = -\infty, 0, 1, 2, 3, ... \tag{36}$$

for the orbital radii (semimajor axes) of planets acquired the status of a law in the following sense. In the case of atomic spectroscopy the empirical formulas used for description of the atomic/molecular spectra have not been used (to our knowledge) for making predictions. Their purpose was just to describe in mathematical terms what had been observed. Since the T-B empirical formula for planetary distances was used as the law, it was used in search for planets not yet discovered. In such a way Ceres, Uranus, Neptune and Pluto were found (Celletti 2007). However, the discrepancies for Neptune and Pluto were much larger than the error margins allowed by the T-B law[34].This fact divided the astronomical community into "believers" and "atheists" regarding to the meaning and uses of this law. Without going into historical details, we would like to jump to the very end of the Titius-Bode story in order to use its latest version which we found in the paper by Neslušan (2004) who, in turn, was motivated by work of Lynch (2003). Instead of Eq.(36) these authors use another empirical power law dependence

$$r_n = r_0 B^n, \; n = 1, 2, 3, .., 9. \tag{37}$$

For planets (except Pluto and including the asteroid belt) Neslušan obtained[35] $r_0(au) = 0.203$ and $B = 1.773$ with the rms deviation accuracy of 0.0534[36].

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[34]Chapter 10 of the book by Celletti (2007) provides very lively account of the present knowledge about various objects "living" in the Solar System.

[35]In astronomical units (to be defined below).

[36]This result gives for the Earth in astronomical (au) units the result $r_3 \simeq 1.13$. Much better result is obtained in case if we choose $B = 1.7$. In this case we obtain: $r_3 \simeq .997339$. Lynch (2003) provides $B = 1.706$ and $r_0 = 0.2139$.

Analogous power law dependencies were obtained previously in the work by Dermott (1968) for both planets and satellites of heavy planets such as Jupiter, Saturn and Uranus.

It should be noted that because of noticed discrepancies the attempts were made to prove or disprove the Titius-Bode law by using statistical analysis, e.g. see papers by Lynch (2003) and Hayes and Tremaine (1998), with purpose of finding out to which extent the observed dependencies can be considered as non accidental. In view of the Heisenberg quantization (honeycomb) condition, Eq.(5.a), it should be obvious by now that whatever distribution of frequencies can be measured, it can, in principle, lead to quantization. In principle, because to implement this in practice requires to identify possible models and the Hamiltonians for these models as discussed extensively in the previous sections. Hence, in the present case we are confronted with exactly the same task. To move forward some historical analogies are helpful at this time.

When Bohr was analyzing the data for He atom (Table 1) he had in mind a model of He made of two independent electrons rotating around the same nucleus. As results of Section 1 indicate, such an approximation produced quite reasonable results. Clearly, when dealing with dynamics of Solar System, one would like to follow the same philosophy. That is to assume first that the planets are noninteracting and move along the geodesics independently. In the case of atomic mechanics it was clear from the beginning that such an approximation should sooner or later fail even though it works well in some cases. For exactly the same reasons it is rather naive to expect that the T-B law makes always sense. Rather, it makes sense to assume that it works for as long as the assumption of noninteracting planets moving on geodesics can be checked quantum mechanically. Furthermore, the nonexisting electroneutrality in the sky provides strong hint that the T-B law must be of very limited use since the number of discrete levels for gravitating systems should be always finite. Otherwise we would observe the countable infinity of satellites around Sun or heavy planets which is both observationally and physically wrong.

To facilitate matters, we still have to make several additional observations. First, we have to find the analog of Planck constant. Second, we have to have some mechanical model in mind to make our search for correct answer meaningful. To accomplish the first task, we have to take into account the 3-rd Kepler's law. In accord with Eq.(35b), it can be written as $r_n^3/T_n^2 = \dfrac{4\pi^2}{\gamma(M+m)}$. In view of arguments presented in previous subsection, we can safely approximate this result by $4\pi^2/\gamma M$, where M is the mass of Sun. For the purposes of this work, it is convenient to restate this law as $3 ln\, r_n - 2 \ln T_n = \ln 4\pi^2/\gamma M = const$ Below, we choose the astronomical system of units in which $4\pi^2/\gamma M = 1$[37].

Consider now the Bohr result, Eq.(4), and take into account that $E = \hbar\omega \equiv \dfrac{h}{2\pi}\dfrac{2\pi}{T}$. Therefore, the Bohr's result can be conveniently restated as $\omega(n,m) = \omega(n) - \omega(m)$. Taking into account Eq.s(4),(31b),(37) and the third Kepler's law

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[37] Since, by definition, in this system we have for Earth $r_3 = T_3 = 1$

we obtain:

$$\omega(n,m) = \frac{1}{c\ln\tilde{A}}(nc\ln\tilde{A} - mc\ln\tilde{A}), \qquad (38)$$

where the role of Planck's constant is being played now by $c\ln\tilde{A}$ where $\tilde{A} = B^{\frac{3}{2}}$ and c is some constant which will be determined selfconsistently below[38].

At first, one may think that what we obtained is just a simple harmonic oscillator spectrum. After all, this should come as not too big a surprise since in terms of the action -angle variables all exactly integrable systems are reducible to the sets of harmonic oscillators. This result is also compatible with the results of Appendix B. The harmonic oscillator option is physically undesirable in the present case since for gravitating systems the charge neutrality constraint cannot be imposed, e.g. see Table 3. Evidently, allowing such a spectrum is equivalent to the correctness of the T-B law. But we know that this law is not working for larger numbers. In fact it would be very strange should it be working in this regime since the total mass of all harmonically bound planets could potentially become infinite.

To make a progress, we have to use the 3rd Kepler's law once again, i.e. we have to take into account that in chosen astronomical system of units $3lnr_n = 2\ln T_n$. In view of the arguments just discussed, a quick look at Eq.s B(13),(14) suggests that the underlying mechanical system is likely to be associated with that for the Morse potential. The low lying states of such a system cannot be distinguished from those for the harmonic oscillator. However, this system does have only a finite number of energy levels which makes sense physically. The task remains to connect this system with the planar Kepler's problem. Although in view of the results of Appendix B such a connection does indeed exist, we would like to demonstrate it explicitly at the level of classical mechanics.

Following Pars (1996), the motion of point of unit mass in the field of gravity is described by the following equation

$$\dot{r}^2 = (2Er^2 + 2\gamma Mr - \alpha^2)/r^2, \qquad (39)$$

where α is the angular momentum integral (e.g. see Eq.(5.2.55) of Pars book). We would like now to replace $r(t)$ by $r(\theta)$ in such a way that $dt = u(r(\theta))d\theta$. Let therefore $r(\theta) = r_0\exp(x(\theta))$, $-\infty < x < \infty$. Unless otherwise specified, we shall write $r_0 = 1$. In such (astronomical) system of units) we obtain, $\dot{r} = x'\dfrac{d\theta}{dt}\exp(x(\theta))$. This result can be further simplified by choosing $\dfrac{d\theta}{dt} = \exp(-x(\theta))$. With this choice Eq.(39) acquires the following form:

$$(x')^2 = 2E + 2\gamma M\exp(-x) - \alpha^2\exp(-2x). \qquad (40)$$

Consider points of equilibria for the potential $U(r) = -2\gamma Mr^{-1} + \alpha^2 r^{-2}$. From here we obtain: $r^* = \dfrac{\alpha^2}{\gamma M}$. According to Goldstein et al (2002) such defined r^* coincides with the major elliptic semiaxis. It can be also shown,

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[38] Not to be confused with the speed of light !

e.g. Pars, Eq.(5.4.14), that for the Kepler problem the following relation holds: $E = -\dfrac{\gamma M}{2r^*}$. Accordingly, $r^* = -\dfrac{\gamma M}{2E}$, and, furthermore, using condition $\frac{dU}{dr} = 0$ we obtain, $\dfrac{\alpha^2}{\gamma M} = -\dfrac{\gamma M}{2E}$ or, $\alpha^2 = -\dfrac{(\gamma M)^2}{2E}$. Since in the chosen system of units $r(\theta) = \exp(x(\theta))$, we obtain, $\dfrac{\alpha^2}{\gamma M} = \exp(x^*(\theta))$. It is convenient to choose $x^*(\theta) = 0$. This requirement makes the point $x^*(\theta) = 0$ as the origin and implies that with respect to such chosen origin $\alpha^2 = \gamma M$[39]. Using this fact Eq.(40) can then be conveniently rewritten as

$$\frac{1}{2}(x')^2 - \gamma M(\exp(-x) - \frac{1}{2}\exp(-2x)) = E \qquad (41a)$$

or, equivalently, as

$$\frac{p^2}{2} + A(\exp(-2x) - 2\exp(-x)) = E, \qquad (41b)$$

where $A = \dfrac{\gamma M}{2}$. Since this result is exact classical analog of the quantum Morse potential problem, transition to quantum mechanics can be done straightforwardly at this stage. By doing so we have to replace the Planck's constant $\hbar$ by $c\ln \tilde{A}$. After that, we can write the answer for the spectrum at once (Landau and Lifshitz 1962):

$$-\tilde{E}_n = \frac{\gamma M}{2}[1 - \frac{c\ln \tilde{A}}{\sqrt{\gamma M}}(n + \frac{1}{2})]^2. \qquad (42)$$

This result contains an unknown parameter c which we would like to determine now. To do so it is sufficient to expand the potential in Eq.(41b) and to keep terms up to quadratic. Such a procedure produces the anticipated harmonic oscillator result

$$\frac{p^2}{2} + Ax^2 = \tilde{E} \qquad (43)$$

with the spectrum given by $\tilde{E}_n = (n+\frac{1}{2})c\sqrt{2A}\ln \tilde{A}$. In the astronomical system of units the spectrum reads: $\tilde{E}_n = (n+\frac{1}{2})c2\pi \ln \tilde{A}$. This result is in agreement with Eq.(38). To proceed, we notice that in Eq.(38) the actual sign of the Planck-type constant is undetermined. Specifically, in our case (up to a constant) the energy $\tilde{E}_n$ is determined by $\ln\left(\frac{1}{T_n}\right) = -\ln \tilde{A}$ so that it makes sense to write $-\tilde{E}_n \sim n\ln \tilde{A}$. To relate the classical energy defined by the Kepler-type

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[39] In doing so some caution should be exercised since upon quantization equation $r^* = \dfrac{\alpha^2}{\gamma M}$ becomes $r_n^* = \dfrac{\alpha_n^2}{\gamma M}$. Selecting the astronomical scale $r_3^* = 1$ as the unit of length implies then that we can write the angular momentum α_n^2 as $\varkappa \dfrac{r_n^*}{r_3^*}$ and to define $\varkappa$ as $\alpha_3^2 \equiv \alpha^2$.

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equation $E = -\dfrac{\gamma M}{2r^*}$ to the energy we just have have obtained, we have to replace this Kepler-type equation by $-\tilde{E}_n \equiv -\ln|E| = -2\ln\sqrt{2}\pi + \ln r_n$ This is done in view of the 3rd Kepler's law and the fact that the new coordinate x is related to the old coordinate r via $r = e^x$. Using Eq.(37) (for $n = 1$) in previous equation and comparing it with the already obtained spectrum of harmonic oscillator we obtain:

$$-2\ln\sqrt{2}\pi + \ln r_0 B = -c2\pi\ln\tilde{A}, \qquad (44)$$

where in arriving at this result we had subtracted the nonphysical ground state energy. Thus, we obtain,

$$c = \frac{1}{2\pi\ln\tilde{A}}\ln\frac{2\pi^2}{r_0 B}. \qquad (45)$$

Substitution of this result back into Eq.(42) produces

$$
\begin{aligned}
-\tilde{E}_n &= 2\pi^2[1 - \frac{(n+\frac{1}{2})}{4\pi^2}\ln\left(\frac{2\pi^2}{r_0 B}\right)]^2 \simeq 2\pi^2[1 - \frac{1}{9.87}(n+\frac{1}{2})]^2 \\
&\simeq 2\pi^2 - 4(n+\frac{1}{2}) + 0.2(n+\frac{1}{2})^2.
\end{aligned}
\qquad (46)
$$

To determine the number of bound states, we follow the same procedure as developed in chemistry for the Morse potential. For this purpose[40] we introduce the energy difference $\Delta\tilde{E}_n = \tilde{E}_{n+1} - \tilde{E}_n = 4 - 0.4(n+1)$ first. Next, the maximum number of bound states is determined by requiring $\Delta\tilde{E}_n = 0$. In our case, we obtain: $n_{\max} = 9$. This number is in perfect accord with observable data for planets of Solar System (with Pluto being excluded and the asteroid belt included). In spite of such a good accord, some caution must be exercised while analyzing the obtained result. Should we not insist on physical grounds that the discrete spectrum must contain only finite number of levels, the obtained spectrum for harmonic oscillator would be sufficient (that is to say, that the validity of the T-B law would be confirmed). Formally, it solves the quantization problem completely in accord with the numerical data (Neslušan 2004). The problem lies however in the fact that these data were fitted to the power law, Eq.(37), in accord with the original T-B empirical guess. Heisenberg's honeycomb rule, Eq.(5), does *not* require the specific $n-$dependence. In fact, we have to consider the observed (the Titius-Bode-type) $n-$dependence only as a hint. With theoretical guidance emerging from this work, it is hoped, that the attempts will be made to fit the observational data to the Morse-like spectra in a way it is done routinely in chemical physics for the Morse-type potentials. In this work we intentionally avoid use of any adjustable parameters since the developed procedure when supplied with correctly interpreted numerical data should be sufficient for obtaining results without any adjustable parameters[41].

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[40] Recall that in chemistry the Morse potential is being routinely used for description of the vibrational spectra of diatomic molecules.

[41] Provided, of course, that the approximation of planetary motions by geodesics is valid.

In support of our conjectures we performed similar calculations for satellite systems of Jupiter, Saturn, Uranus and Neptune. To do such calculations the astronomical system of units is not immediately useful since in the case of heavy planets one cannot use the relation $4\pi^2/\gamma M_\odot = 1$. This is so because we have to replace the mass of the Sun $M_\odot$ by the mass of the respective heavy planet. To do so, we write $4\pi^2 = \gamma M_\odot$, multiply both sides by M_j (where j stands for the j-th heavy planet) and divide both sides by $M_\odot$. Thus, we obtain: $4\pi^2 q_j = \gamma M_j$, where $q_j = \dfrac{M_j}{M_\odot}$. Since the number q_j is of order $10^{-3} - 10^{-5}$, it is inconvenient in actual calculations. To by pass this difficulty, we need to readjust Eq.(40) by rescaling x coordinate as $x = \delta \bar{x}$ and, by choosing $\delta^2 = q_j$. After transition to quantum mechanics such a rescaling results in replacing Eq.(42) for the spectrum by the following result:

$$-\tilde{E}_n = \frac{\gamma M}{2}[1 - \frac{c\delta \ln \tilde{A}}{\sqrt{\gamma M}}(n + \frac{1}{2})]^2. \tag{47}$$

Since the constant c is undetermined initially, we can replace it by $\tilde{c} = c\delta$ so that we reobtain back equation almost identical to Eq.(46). That is

$$-\tilde{E}_n = 2\pi^2[1 - \frac{(n + \frac{1}{2})}{4\pi^2} \ln \left(\frac{\gamma M_j}{(r_j)_1}\right)]^2 \tag{48}$$

In this equation $\gamma M_j = 4\pi^2 q_j$ and $(r_j)_1$ is the semimajor axis of the satellite lying in the equatorial plane and closest to the j-th planet. Our calculations are summarized in the Table 4 below. Appendix C contains the input data used in our calculations of n^*_{theory}.

Table 4

Satellite system\n_{max}	n^*_{theory}	n^*_{obs}
Solar system	9	9
Jupiter system	11-12	8
Saturn system	20	20
Uranus system	40	18
Neptune system	33	6

Since the discrepancies for Uranus and Neptune systems may be genuine or not we come up with the following general pattern described below.

4.4 Further analogies with atomic mechanics

From atomic mechanics we know that the approximation of independent electrons used by Bohr fails rather quickly with increased number of electrons. For this reason to expect that the T-B law is going to hold for satellites of all heavy planets is rather naive as we have explained already. At the same time, for planets rotating around the Sun such an approximation is seemingly good. The SO(2,1) symmetry explains why the motion of all planets should be planar but

it does not explain why the motion of all planets is taking place in the plane almost coinciding with the equatorial plane of the Sun. The same is true for the regular satellites of all heavy planets as discussed by Dermott (1968). If we adopt the quantum mechanical point of view advocated in this work, then we should accept that such an arrangement of planets is the result of some kind of spin-orbital interaction whose exact quantum mechanical nature remains to be elucidated. Other rotational resonances ubiquitous in the Solar system could then be explained quantum mechanically as well. The equatorial plane in which planets (satellites) move can be considered as some kind of an orbital (in the atomic physics terminology). It is being filled in accordance with the equivalent of the Pauli principle: *each orbit can be occupied by no more than one planet*[42]. Once the orbital is filled, other orbitals associated with other planes will begin to be filled out. Some of orbitals can be empty. This is indeed being observed (Dermott 1968). It should be said though that it appears (according to the available data, e.g. see Celletti (2007)) that not all of observed satellites are moving on stable orbits. It appears also as if the "inner shell", when completely filled, acts as some kind of an s-type spherical orbital since orbits of other satellites lie strictly outside the sphere whose diameter is greater or equal to that corresponding to the last allowed energy level in the first shell. The location of the secondary planes appears to be quite arbitrary as well as the filling of their stable orbits. Furthermore, without account of spin-orbital interactions, quantum mechanics says nothing about the direction of the orbital rotation. Although for all planets it does coincide with the direction of rotation of the Sun's axis, in the case of Phoebe- the irregular satellite of Saturn-rotation takes place in the opposite direction to that of the axis of Saturn. If the spin-orbital interaction does exists, most likely, Phoebe's orbit is not a stable one.

It is tempting to extend the picture just sketched beyond the scope of our Solar System. If for a moment we would ignore relativistic effects (they will be discussed in the next section), we can then find out that our Sun is moving along almost circular orbit around our galaxy center with the period $T = 185 \cdot 10^6$ years (Chebotarev 1967). Our galaxy is also flat as our Solar System and the major mass is concentrated in the galaxy center. Hence, again, if we believe that the stable stellar motion is taking place along the geodesics in accordance with laws of Einstein's general relativity, then we have to accept that our galaxy is a quantum object. It would be very interesting to estimate the number of allowed energy levels for our galaxy and to check if the Pauli-like principle works for the galaxy as well.

4.5 Latest developments supporting our point of view

We begin with the following observation. The motion of a planet of mass m_0 in the field of two static centers of attraction with masses m_1 and m_2 was discussed by Legendre and Jacobi in 19th century (Pars 1966) in connection with

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[42]The meteorite belt can be looked upon as some kind of a ring. We shall briefly discuss the rings below.

their study of elliptic functions. Such an idealized problem is a precursor of the restricted 3-body problem to be discussed in the next subsection in connection with dynamics of planetary rings. As simple as it is, the full study of this problem is extremely complex. It involves classification of all points and lines of equilibria and motions in the domains restricted by these lines. In addition to the eight major types of bounded orbits there are many more coming from collision of equilibrium point/lines etc. Characterization of the unbound motion is also interesting but is less complex. In quantum mechanics the motion of an electron in the presence of two fixed positive ions is also a benchmark problem (in addition to study of He discussed in section 1). Normally, the charges of ions are assumed to be the same (e.g. for H_2^+) which makes such a problem somewhat different (since they repel each other) from the problem studied by Legendre and Jacobi. All classification of molecular spectra can be traced back to this problem (Landau and Lifshitz 1962). As in the case of H atom, the correspondence principle is not well established in this case since (to our knowledge) nobody studied the agreement between the quantum -mechanical calculations in the semiclassical limit and the results of Legendre-Jacobi theory modified due to the chemical requirements. Interestingly enough such a comparison was made to a larger extent between the classical restricted 3-body problem and its quantum analog. The quantum analog of the restricted 3-body problem exists in the form of the H atom placed in a strong crossed constant electric and magnetic fields (Milczewski et al 1997). Since semiclassical and classical analysis of such a system is sufficiently well understood, this fact allows such a system to be studied both theoretically and experimentally. These studies are well summarized in two recent reviews (Porter and Cvitanovich 2005; Marsden and Ross 2006) to which we refer our readers for details. For immediate purposes of this work the following quotation from Porter and Cvitanovič is helpful: " almost perfect parallel between the governing equations of atomic physics and celestial mechanics implies that the transport mechanism for these two situations is virtually identical: on the celestial scale, transport takes a spacecraft from one Lagrange point to another until it reaches its desired destination[43]. On atomic scale, the same type of trajectory transports an electron initially trapped near the atom across the escape threshold (in chemical parlance, across a "transition state"), never to return. The orbits used to design space missions thus also determine the ionization rates of atoms and chemical reaction rates of molecules". This statement is nicely illustrated in the paper by Jaffe et al (2002) in which it is obtained that the *transition state theory developed initially in chemistry* (to describe the rates of chemical reactions) *is working actually better in celestial mechanics* where the discrepancy between the chemical theory and numerical simulations (done for celestial mechanics transport problems) is less than 1%. It should be noted though that the calculations were done at the classical level only (that is for a very large quantum numbers). The current status of transition state theory at the quantum and classical levels in chemistry can be found in the recent book by Micha and Burghardt (2007).

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[43] E.g. see also paper by Convay at al (2007).

4.6 The restricted 3-body problem and planetary rings

Although the literature on the restricted 3-body problem is huge, we would like to discuss this problem from the point of view of its connection with general relativity and quantization of planetary orbits as developed in this paper. We begin with several remarks.

First, the existence of ring systems for all heavy planets is well known (Celletti 2007). Second, these ring systems are interspersed with satellites of heavy planets. Third, both rings and satellites lie in the respective equatorial planes so that the satellites move on stable orbits. From these observations it follows that:

a) While each of heavy planets is moving along the geodesics around the Sun, the respective satellites are moving along the geodesics around respective planets;

b) The motion of these satellites is almost circular.

The restricted 3-body problem can be formulated now as follows. Given that the rings are made of some kind of small objects whose masses can be neglected (Hill's problem (Arnol's et al (2006)) as compared to masses of both the satellite(s) and the particular heavy planet. Following previously discussed ideas by Laplace, we can ignore mutual gravitational interaction between these objects. Under such conditions we end up with the restricted three-body problem of motion of a given piece of a ring (of zero mass) in the presence of two bodies of masses m_1 and m_2 respectively. To simplify matters, one usually assumes that the motion of these two masses takes place on a circular orbit with respect to their center of mass. Complications associated with the eccentricity of such a motion are discussed in the book by Szebehely (1967) and can be taken into account if needed. They will be ignored nevertheless in our discussion since we shall assume that the satellites of heavy planets move on geodesics so that the center of mass coincides with the position of a heavy planet anyway thus making our computational scheme compatible with Einsteinian relativity. By assuming that ring pieces are massless we also are making their motion compatible with requirements of general relativity since whatever orbits they may have-these are geodesics.

Thus far only the motion of satellites in the equatorial planes (of respective planets) was considered as stable (and, hence, quantizable). The motion of ring pieces was not accounted thus far by these stable orbits. The task now lies in showing that the satellites lying inside the respective rings of heavy planets are essential for stability of these rings motion and hence, are making it quantizable.

For the sake of space, we would like only to provide a sketch of arguments leading to such a conclusion. Full details are left for subsequent publications. Our task is greatly simplified by the fact that very similar situation exist for the 3-body system such as Moon, Earth and Sun. Dynamics of such a system was studied very thoroughly by Hill whose work played profound role in Poincar$e^{\prime\prime}$ studies of celestial mechanics (Poincar$e^{\prime}$1892-1898). Recently, Avron and Simon (1981) have adopted Hill's ideas in order to develop a formal quantum

mechanical treatment of the Saturn rings. In this work we follow the original Hill's ideas concerning dynamics of the Earth-Moon-Sun system. We claim that, when these ideas are looked upon from the point of view of modern mathematics of exactly integrable systems, they enable us to describe not only the Earth-Moon-Sun system but also the dynamics of rings of heavy planets. These modern mathematical methods enable us to find a place for the Hill's theory within general quantization scheme discussed in previous sections.

4.6.1 Basics of the Hill's equation

To avoid repetitions, we refer our readers to the books of Pars (1966), Chebotarev (1967) and Brouwer and Clemence (1961) for detailed and clear account of the restricted 3-body problem and Hill's contributions to Lunar theory. Here we only summarize the ideas behind Hill's ground breaking work.

In a nutshell his method of studying Lunar problem can be considered as extremely sophisticated improvement of previously discussed Laplace and Lagrange method. Unlike Laplace, Hill realized that both Sun and Earth are surrounded by the rings of influence[44]. The same goes for all heavy planets: each of these planets and each satellite of such a planet will have its own domain of influence whose actual width is controlled by the Jacobi integral of motion. For the sake of argument, consider the Saturn as an example. It has Pan as its the innermost satellite. Both the Saturn and Pan have their respective domains of influence. Naturally, we have to look at the domain of influence for the Saturn. Within such a domain let us consider a hypothetical closed Kepler-like trajectory. Stability of such a Lagrangian trajectory is described by the Hill equation[45]. Since such an equation describes a wavy-type oscillations around the presumably stable trajectory, the parameters describing such a trajectory are used as an input (perhaps, with subsequent adjustment) in Hill's equation given by

$$\frac{d^2x}{dt^2} + (q_0 + 2q_1\cos 2t + 2q_2\cos 4t + \cdots)x = 0. \qquad (49)$$

If we would ignore all terms except q_0 first, we would naively obtain: $x_0(t) = A_0 cos(t\sqrt{q_0} + \varepsilon)$. This result describes oscillations around the equilibrium position along the trajectory with the constant q_0 carrying information about this trajectory and the amplitude A is expected to be larger or equal to the average distance between the pieces of the ring. This naive picture gets very complicated at once should we use the obtained result as an input into Eq.(49). In this case the following equation is obtained

$$\frac{d^2x}{dt^2} + q_0x + A_0q_1\{\cos[t(\sqrt{q_0} + 2) + \varepsilon] + \cos[t(\sqrt{q_0} - 2) - \varepsilon]\} = 0 \qquad (50)$$

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[44] Related to the so called Roche limit (Celletti 2007).

[45] In fact, there will be the system of Hill's equations in general (Chebotarev 1967). This is so since the disturbance of trajectory is normally decomposed into that which is perpendicular and that which is parallel to the Kepler's trajectory at given point. We shall avoid these complications in our work.

whose solution will enable us to determine q_1 and A_1 using the appropriate boundary conditions. Unfortunately, since such a procedure should be repeated infinitely many times it is obviously impractical. Hill was able to design much better method. Before discussing Hill's equation from the perspective of modern mathematics, it is useful to recall the very basic classical facts about this equation summarized in the book by Ince (1926). For this purpose, we shall assume that the solution of Eq.(49) can be presented in the form

$$x(t) = e^{\alpha t} \sum_{r=-\infty}^{\infty} b_r e^{irt}. \tag{51}$$

Substitution of this result into Eq.(49) leads to the following infinite system of linear equations

$$(\alpha + 2ri)^2 b_r + \sum_{k=-\infty}^{\infty} q_k b_{r-k} = 0, \ r \in \mathbf{Z}. \tag{52}$$

As in finite case, obtaining of nontrivial solution requires the infinite determinant $\Delta(\alpha)$ to be equal to zero. This problem can be looked upon from two directions: either all constants q_k are assigned and one is interested in bounded solution of Eq.(51) for $t \to \infty$, or one is interested in the relationship between constants such that $\alpha = 0$. In the last case it is important to know wether there is one or more than one of such solutions available. Although answers can be found in the book by Magnus and Winkler (1966), we follow McKean and Moerbeke (1975), Trubowitz (1977) and Moser (1980).

For this purpose we need to bring our notations in accord with those used in these references. Thus, the Hill operator is defined now as $Q(q) = -\frac{d^2}{dt^2} + q(t)$ with periodic potential $q(t) = q(t+1)$. Eq.(49) can now be rewritten as

$$Q(q)x = \lambda x. \tag{53}$$

This presentation makes sense since q_0 in Eq.(49) plays the role of λ in Eq.(53). Since this is the second order differential equation, it has formally 2 solutions. These solutions depend upon boundary conditions. For instance, for *periodic* solutions such that $x(t) = x(t+2)$ the "spectrum" of Eq.(53) is discrete and is given by
$$-\infty < \lambda_0 < \lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \cdots \uparrow +\infty.$$

We put the word spectrum in quotation marks because of the following. Eq.(53) does have a normalizable solution only if λ belongs to the (pre assigned) intervals $(\lambda_0, \lambda_1), (\lambda_2, \lambda_3), ..., (\lambda_{2i}, \lambda_{2i+1}), ...$ In such a case the eigenfunctions x_i are normalizable in the usual sense of quantum mechanics and form an orthogonal set. The periodic solutions make sense for the vertical displacement from the reference trajectory. For the horizontal displacement the boundary condition should be chosen as $x(0) = x(1) = 0$. For such chosen boundary condition the discrete spectrum also exists but it lies exactly in the gaps between the intervals just described, i.e. $\lambda_1 \le \mu_1 \le \lambda_2 < \lambda_3 \le \mu_2 \le \lambda_4 < \cdots$. For such a spectrum there is also set of normalized mutually orthogonal eigenfunctions.

Thus in both cases quantum mechanical description is assured. One can do much more however. In particular, Trubowitz (1977) designed an explicit procedure of recovering the potential $q(t)$ from the $\mu-$spectrum supplemented by information about normalization constants.

The Hill's equation can be interpreted in terms of the auxiliary dynamical (Neumann) problem. Such an interpretation is very helpful for us since it allows us to include the quantum mechanics of Hill's equation into general formalism developed in Sections 2 and 3.

4.6.2 Connection with the dynamical Neumann problem and the Korteweg -de Vries equation

Before doing so, we would like to add few details to results of previous subsection. First, the number of the pre assigned intervals is always finite. This means that, beginning with some pre assigned $\hat{\imath}$, we would be left with $\lambda_{2i} = \lambda_{2i+1} \forall i > \hat{\imath}$. These *double* eigenvalues do not have independent physical significance since they can be determined by the set of *single* eigenvalues (for which $\lambda_{2i} \neq \lambda_{2i+1}$) as demonstrated by Hochstadt (1963). Because of this, the potentials $q(t)$ in the Hill's equation are called *finite gap* potentials[46]. Hence, physically, it is sufficient to discuss only such potentials which possess finite single spectrum. The auxiliary $\mu-$spectrum is then determined by the gaps of the single spectrum as explained above. With this information in our hands, we are ready to discuss the exactly solvable Neumann dynamical problem. It is the problem about dynamics of a particle moving on the $n-$dimensional sphere $< \boldsymbol{\xi}, \boldsymbol{\xi} > \equiv \xi_1^2 + \cdots + \xi_n^2 = 1$ under the influence of a quadratic potential $\phi(\boldsymbol{\xi}) = < \boldsymbol{\xi}, \mathbf{A}\boldsymbol{\xi} >$. Equations of motion describing the motion on $n-$ sphere are given by

$$\ddot{\boldsymbol{\xi}} = -\mathbf{A}\boldsymbol{\xi} + u(\boldsymbol{\xi})\boldsymbol{\xi} \ \text{ with } u(\boldsymbol{\xi}) = \phi(\boldsymbol{\xi}) - < \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}} > . \tag{54}$$

Without loss of generality, we assume that the matrix $\mathbf{A}$ is already in the diagonal form: $\mathbf{A} := diag(\alpha_1, ..., \alpha_n)$. With such an assumption we can equivalently rewrite (54) in the following suggestive form

$$\left(-\frac{d^2}{dt^2} + u(\boldsymbol{\xi}(t)) \right) \xi_k = \alpha_k \xi_k \ ; \ \ k = 1, ..., n. \tag{55}$$

Thus, in the case if we can prove that $u(\boldsymbol{\xi}(t))$ in (55) is the same as $q(t)$ in (53), the connection between the Hill and Neumann's problems will be established. The proof is presented in Appendix D. It is different from that given in the lectures by Moser (1980) since it is more direct and much shorter.

This proof brought us the unexpected connection with hydrodynamics through the static version of the Korteweg-de Vries equation. Attempts to describe the Saturnian rings using equations of hydrodynamic are described in the recent monograph by Esposito (2006). This time, however, we can accomplish more using obtained information. This is the subject of the next subsection.

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[46]Since there is only finite number of gaps $[\lambda_1, \lambda_2], [\lambda_3, \lambda_4], ...$ where the spectrum is forbidden.

4.6.3 Connections with SO(2,1) group and the K-Z equations

Following Kirillov (1982), we introduce the commutator for the fields (operators) ξ and η as follows: $[\xi, \eta] = \xi \partial \eta - \eta \partial \xi$. Using the KdV, Eq.(D.10), let us consider 3 of its independent solutions: ξ_0, ξ_{-1} and ξ_1. All these can be obtained from general result: $\xi_k = t^{k+1} + O(t^2)$, valid near zero. Consider now a commutator $[\xi_0, \xi_1]$. Straightforwardly, we obtain, $[\xi_0, \xi_1] = \xi_1$. Analogously, we obtain, $[\xi_0, \xi_{-1}] = -\xi_{-1}$ and, finally, $[\xi_1, \xi_{-1}] = -2\xi_0$. According to Kirillov, such a Lie algebra is isomorphic to that for the group $SL(2, R)$. Vilenkin (1991) have demonstrated that the group $SL(2, R)$ is isomorphic to $SU(1, 1)$. Indeed, by means of transformation: $w = \dfrac{z - i}{z + i}$, it is possible to transform the upper half plane (on which $SL(2, R)$ acts) into the interior of unit circle on which $SU(1, 1)$ acts. Since, according to Appendix B, the group $SU(1, 1)$ is the connected component of $SO(2, 1)$, the anticipated connection with $SO(2, 1)$ group is established.

In Appendix D we noticed connections between the Picard-Fuchs, Hill and Neumann-type equations. In a recent paper by Veselov et al (2001) such a connection was developed much further resulting in the K-Z type equations[47] for Neumann-type dynamical systems. We refer our readers to the original literature, especially to the well written lecture notes by Moser(1980). These notes as well and his notes in collaboration with Zehnder (2005) provide an excellent background for the whole circle of ideas relating Hill's equation to integrable models.

5 Solar System at larger scales: de Sitter, anti -de Sitter and conformal symmetries compatible with orbital quantization

The obtained results demonstrate a remarkable interplay between the Newtonian and Einsteinian mechanics already at the scale of our Solar System. Since quantization of stable orbits described in this paper is possible only with use of the basic experimental facts assuring correctness of results of general relativity, it is only natural to reverse this statement and to say that *the correctness of general relativity is assured by the observed pattern of stable (quantum) orbits.*

Since quantum mechanics can be developed group-theoretically, the same should be true for relativity. Quoting Einstein, Infeld and Hoffmann (1938): "Actually, the only equations of gravitation which follow without ambiguity from the fundamental assumptions of the general theory of relativity are the equations for empty space[48], and it is important to know whether they alone are

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[47] E.g. see Eq.(29) of Section 2.
[48] See also Section 6.

capable of determining the motion of bodies". In this work we have argued that this is certainly correct locally when the Lorentzian-type symmetry holds true. Now we would like to discuss how such locally Lorentzian space-time embeds into space-times of general relativity possessing larger symmetry groups[49]. Since this topic is extremely large, we shall discuss only the most basic facts from the point of view of results obtained in this paper.

To our knowledge, Dirac (1935) was the first who recognized the role of space-time symmetry in quantum mechanics. In his paper he wrote: "The equations of atomic physics are usually formulated in terms of space-time of special relativity. They then have to form a scheme which remains invariant under all transformations which carry the space-time over into itself. These transformations consist of the Lorentz rotations about a point combined with arbitrary translations, and form a group.... Nearly all of more general spaces have only trivial groups[50] of operations which carry the spaces into themselves....There is one exception, however, namely the de Sitter space (with no local gravitational fields). This space is associated with a very interesting group, and so the study of the equations of atomic physics in this space is of special interest, *from mathematical point of view.*" Subsequent studies indicate that the symmetry of space-time could be important even at the atomic scale (Bros et al 2006, Aldrovandi et al 2004). Another reason to look at larger symmetry groups is associated with the cosmological constant problem (Peebles and Ratra 2002) and, associated with it the problem of existence of cold dark energy (CDE) (Copeland at al 2006), cold dark matter (CDM) (Kay et al 2002) and the modified Newtonian dynamics (MOND) (de Blok et al 2001). Clearly, we are unable to discuss these issues within the scope of this paper since they are more relevant to processes at the galactic scales. Nevertheless, we would like to notice that, for instance, the MOND presupposes use of Newtonian and the modified Newtonian mechanics at the galactic scales which, as discussed in Section 4, strictly speaking, is not permissible even at the scales of our Solar System. The rationale for the dark energy and dark matter is explained in our recent papers (Kholodenko and Ballard 2007) and (Kholodenko 2007a) based on mathematical arguments consistent with that used by Grigory Perelman in his proof of the Poincare$'$ conjecture.

Hence, we proceed with description of the de Sitter and anti -de Sitter spaces based on results of our recent works. We begin with the following Hilbert-Einstein functional

$$S^c(g) = \int_{\mathcal{M}} d^d x R \sqrt{g} + \Lambda \int_{\mathcal{M}} d^d x \sqrt{g} \qquad (56)$$

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[49] At the level of quantum field theory Utiyama (1956) demonstrated that the requirement of the local gauge invariance implemented for the non Abelian Lorentz group produces Einstein's equations for the gravitational field. This result implies that any "improvements" of Einsteinian relativity should involve changes in the local Lorentzian structure of space-time which is very unlikely. Independent arguments supporting this point of view are presented in Section 6.

[50] This statement of Dirac is not correct. However, it is correct at the time of writing of his paper.

defined for some (pseudo) Riemannian manifold $\mathcal{M}$ of total space-time dimension d. The (cosmological) constant Λ is determined as follows.

Using R_{ij}, the Ricci curvature tensor, the *Einstein space* is defined as solution of the following vacuum Einstein equation

$$R_{ij} = \lambda g_{ij} \tag{57}$$

with λ being a constant. From this definition it follows that

$$R = d\lambda. \tag{58}$$

At the same time, variation of the action $S^c(g)$ produces

$$G_{ij} + \frac{1}{2}\Lambda g_{ij} = 0, \tag{59}$$

where the Einstein tensor G_{ij} is defined as $G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R$ with R being the scalar curvature determined by the metric tensor g_{ij}[51]. Combined use of Eq.s(58) and (59) produces: $\Lambda = \lambda(d-2)$. Substitution of this result back into Eq.(59) produces:

$$G^i_j = (\frac{1}{d} - \frac{1}{2})\delta^i_j R. \tag{60}$$

Since by design $G^i_{j,h} = 0$, we obtain our major result:

$$(\frac{1}{d} - \frac{1}{2})R_{,j} = 0, \tag{61}$$

implying that scalar curvature R is constant.

For isotropic homogenous spaces the Riemann curvature tensor can be presented in the following known form (Landau and Lifshitz 1975)

$$R_{ijkl} = k(x)(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{62}$$

Accordingly, the Ricci tensor is obtained as: $R_{ij} = k(x)g_{ij}(d-1)$. The Schur's theorem (Willmore 1993) guarantees that for $d \geq 3$ we must have $k(x) = k = const$ for the entire space. Therefore, we obtain: $\lambda = (d-1)k$ and, furthermore, $R = d(d-1)k$. The spatial coordinates can always be rescaled so that $R = k$ or, alternatively, the constant k can be normalized to unity. For $k > 0$, $k = 0$ and $k < 0$ we obtain respectively de Sitter, flat and anti-de Sitter spaces. Thus, we just have demonstrated that homogeneity and isotropy of space-time is synonymous with spaces being de Sitter, flat and anti-de Sitter very much like in ordinary Riemannian geometry there are spaces of positive, negative and zero curvature. This fact can be used to give the alternative description of just obtained results.

We begin with simple observation that the surface of constant positive curvature is conformally equivalent to a sphere embedded in the Euclidean space

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[51] Eq.(59) illustrates the meaning of the term "dark matter". The constant Λ enters into the stress-energy tensor (in the present case given by $-\frac{1}{2}\Lambda g_{ij}$) typically associated with the matter, Einstein (1916).

(Kholodenko 2007a). In particular, let us consider a 3-sphere embedded into 4d Euclidean space. It is described by the equation

$$S^3 = \{x \in E_4,\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2\}. \tag{63}$$

S^3 is homogenous isotropic space with positive scalar curvature whose value is $6/R^2$. The group of motions associated with this homogenous space is the rotation group $SO(4)$. The space of constant negative curvature H^3 is obtained analogously. For this purpose it is sufficient, following Dirac (1935), to make x_1 purely imaginary and to replace R^2 by $-R^2$ in Eq.(63). Such replacements produce:

$$H^3 = \{x \in M_4,\ x_1^2 - x_2^2 - x_3^2 - x_4^2 = R^2\}. \tag{64}$$

In writing this result we have replaced the Euclidean space E_4 by the Minkowski space M_4 so that the rotation group $SO(4)$ is now replaced by the Lorentz group $SO(3,1)$. The de Sitter space can now be obtained according to Dirac (1935) as follows. In Eq.(63) we replace E_4 by E_5 and make x_1 purely imaginary thus converting E_5 into M_5. The obtained space is the de Sitter space whose group of symmetry is $SO(4,1)$

$$dS_4 = \{x \in M_5,\ x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 = R^2\}. \tag{65}$$

It has a constant positive scalar curvature whose value is $12/R^2$. Very nice description of such a space is contained in the book by Hawking and Ellis (1973). The connection between the parameter R and the cosmological constant Λ is given by $R = \sqrt{\dfrac{3}{\Lambda}}$. The anti-de Sitter space is determined analogously as also discussed by Hawking and Ellis and by Dirac. Specifically, it is given by

$$adS_4 = \{x \in E_{3,2},\ x_1^2 - x_2^2 - x_3^2 - x_4^2 + x_5^2 = R^2\}, \tag{66}$$

where the five dimensional space $E_{3,2}$ is constructed by adding the time-like direction to M_4. Hence, the symmetry group of adS_4 is SO(3,2). All these groups can be described simultaneously if, following Dirac (1935), we introduce the quadratic form

$$\sum_{\mu=1}^{5} x_\mu x_\mu = R^2 \tag{67}$$

in which some of the arguments are allowed to be purely imaginary. Transformations preserving such a quadratic form are appropriate respectively for groups SO(5), SO(4,1) and SO(3,2). We still can embed all these groups into a larger (conformal) group SO(4,2) by increasing summation from 5 to 6 in Eq.(67). In such a case all groups discussed in this work, starting from SO(2,1), can be embedded into this conformal group as subgroups as discussed in great detail by Wybourne (1974)[52]. Comprehensive group-theoretic description of the

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[52]Incidentally, the work by Graner and Dubrulle (1994), when translated into group-theoretic language, becomes just a corollary of conformal invariance implied by the conformal group .

Einstein spaces, e.g. see Eq.(57), including those which are invariant with respect to the conformal group, can be found in the monograph by Petrov (1969). The significance and use of conformal symmetry in both gravity and conformal field theories has been recently extended in Kholodenko (2007a). All existing cosmological models in the limit $R \to \infty$ approach one of the Einstein's spaces whose group of symmetry belongs to the types just described. The de Sitter and anti-de Sitter spaces are the simplest examples of such spaces (Weinberg 1972)[53].

The task still remains to find out if representations of these larger groups, e.g. see Vilenkin (1991) and Wybourne (1974) for mathematical details, can produce the exact solutions of radial Schrödinger equations not listed in the Natanzon-style classification, e.g. see Levai (1994), for SO(2,1). If such solutions do exist, one might be able to find those of them which are of relevance to celestial quantum mechanics and, hence, to cosmology.

6   Conclusions

It is a remarkable historical fact (discussed in Section 4.2.) that Laplace was the first who studied resonance dynamics of known satellites of Jupiter (effectively) using geodesics while Lagrange analyzing the motion of these satellites along these stable geodesics had (also effectively) arrived at the Bohr-Sommerfeld quantization conditions. The validity of the geodesic-type approximation is based on the equivalence principle of general relativity correct in the limit of vanishingly small masses as compared to the mass of the central body. Using this principle Einstein (1916) was able to calculate the perihelium shift for Mercury. Within approximations he made all interactions of Mercury with the rest of planets were ignored. Because of this, the same type of calculations in the spirit of Laplace can be made for all planets as discussed by Misner et al (1973). The fundamental problem then lies in proving that such stable motions will survive in the case if masses of planets are small but nonzero. In the case of satellites of Jupiter such a task was completed by Poincare$'$ and de Sitter as mentioned in Section 4. The latest advancements are also discussed in the same section.

In this (concluding) section we argue that it is possible to arrive at the same field equations of general relativity *by entirely by passing the equivalence principle.* This can be achieved by studying the limiting case of dynamics of 2+1 gravity as discussed in Kholodenko (2000a,b) and Kholodenko (2001). In such a limit one studies surface dynamics of the fictitious 2 dimensional gravitating bodies[54]. Simple topological arguments applied to this case indicate that the Einsteinian field equations survive such a reduction while Newtonian (actually, the Poissonian-type) equations do not survive this reduction. As result, the

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[53] The most recent mathematically rigorous description of both de Sitter and anti-de Sitter spaces can be found in the paper by Andersson and collaborators (2007).

[54] Which can be actually visualized by the crossections (lying on the surface) of infinitely long and thin massive 3 dimensional rods. According to the imposed rules we are allowed to watch only the motion of crossections.

dynamics of 2+1 gravity *is strictly Einsteinian.* In mathematics this type of dynamics (the dynamics of measured foliations) was discovered totally independently of gravity-related considerations by Thurston (1982) in his study of 3-manifolds. Physically, such type of dynamics is realized in dynamics of some 2 dimensional liquid crystals. All this is explained in Kholodenko (2000a,b) and Kholodenko (2001) whose works, in turn, were inspired by earlier work by Deser**,** Jackiw and t'Hooft (1984). The theory of foliations (for surfaces) is thoroughly discussed in monograph by Nikolaev (2001). The book by Hehl and Obukhov (2003) uses foliations for description of classical electrodynamics in 3+1 space. Some basics of foliation theory from the point of view of Lie groups and Lie algebras are discussed in the very readable book by Moerijk and Mrčun (2003). By reversing reduction arguments it is possible to arrive at dynamics of full 3+1 gravity in the presence of matter beginning from the dynamics of measured foliations for 2+1 gravity. The topological properties of dynamics of 2+1 gravity are described in terms of polynomials of knots and links as explained in detail in Kholodenko (2001). Accordingly, dynamics of full 3+1 gravity should be associated with time dynamics of 3-manifolds foliating 3+1 space (Kholodenko 2007a). These remarks provide needed physical justification to works by Witten (2007) and Kholodenko (2007b) connecting statics and dynamics of 3 (or 2+1) gravity with conformal and string theories.

Appendix A. Details of Heisenberg's derivation of the commutator identity $[\hat{x}, \hat{p}] = i\hbar$

In this appendix we would like to provide some details of Heisenberg's reasoning leading to the discovery of $[\hat{x}, \hat{p}] = i\hbar$. This would be unnecessary should his original paper (Heisenberg 1925) contain all details.

At the classical level consider a gas of noninteracting atoms, better just one atom containing N electrons which are assumed to scatter light independently. The interaction between the incoming light and individual electron is described with help of the combination $\mathbf{d} = \beta\mathbf{E}$ where $\mathbf{d}$ is the dipole moment of the electron in the atom, $\mathbf{E}$ is the strength of the external electric field which is assumed to be time-dependent, and β is the polarization tensor (in the simplest case it is assumed to be a scalar). In the medium the strength of the electric field changes as compared to the vacuum. By denoting it as $\mathbf{D}$ it is known that $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ where $\mathbf{P} = N\mathbf{d}$. Since, at the same time, by definition, $\mathbf{d} = e\mathbf{r}$ we

have to have an equation for **r.** It is given by

$$\ddot{\mathbf{r}} + \omega_0^2 \mathbf{r} + \boldsymbol{\gamma}\dot{\mathbf{r}} = \frac{e}{m}\mathbf{E}(t) \tag{A.1}$$

where e is electron's charge and m is its mass. In writing this equation it is assumed that our electron is bound harmonically (with the basic frequency ω_0^2) and that the friction is of known (electromagnetic) nature and is assumed to be small. Using Fourier decomposition of $\mathbf{r}(\mathrm{t})$ we obtain,

$$\mathbf{r}(\omega) = \frac{e}{m}\frac{\mathbf{E}}{\omega_0^2 - \omega^2 + i\omega\gamma}. \tag{A.2}$$

This equation allows us to obtain **P** and, hence, **D** as follows :

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = (1 + 4\pi N \frac{e^2}{m}\frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma})\mathbf{E} \equiv \varepsilon(\omega)\mathbf{E}. \tag{A.3}$$

This equation defines a complex frequency-dependent dielectric constant $\varepsilon(\omega)$. From electrodynamics it can be equivalently rewritten as $\varepsilon(\omega) = (n(\omega) - i\varkappa(\omega))^2$ where $n(\omega)$ is the refractive index while $\varkappa(\omega)$ is the coefficient of absorption. Using these facts we can write approximately

$$n(\omega) = 1 + 2\pi N \frac{e^2}{m}\frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}. \tag{A.4}$$

By ignoring friction in the high frequency limit we obtain,

$$n(\omega) = 1 - 2\pi N \frac{e^2}{m\omega^2}. \tag{A.5.}$$

To account for quantum mechanical effects, Thomas, Reich and Kuhn in 1925 (just *before* the quantum mechanics was born !) have suggested to replace Eq.(A.4) by

$$n(\omega) = 1 + 2\pi N \frac{e^2}{m}\sum_i \frac{f_i}{\omega_{i0}^2 - \omega^2} \tag{A.6}$$

where, following these authors, we ignored friction and introduced the *oscillator strength* f_i. To reconcile Eq.(A.6) with (A.5) we have to require $\sum_i f_i = 1$. This requirement is known as the *sum rule*. These facts were known to Kramers and Heisenberg[55] where our readers can find additional details. To make our point and to save space, we would like to reobtain the result, Eq.(A.6), quantum mechanically using modern formalism. We refer our reader to the book by Davydov (1965) for additional details. Basically, we need to calculate quantum mechanically the dipole moment **d,** that is

$$\mathbf{d}_m = \int \psi_m^* e\mathbf{r}\psi_m d^3\mathbf{r}. \tag{A.7.}$$

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[55] E.g. see the reference in Heisenberg's paper.

In this expression the wave function ψ_m is calculated with help of the stationary perturbation theory with accuracy up to the first order in perturbation (which is e$\mathbf{r}\cdot\mathbf{E}$). A short calculation produces the following result for the oscillator strength,

$$f_{km} = \frac{2m\omega_{km}}{\hbar} |\langle k \mid \hat{x} \mid m \rangle|^2 .$$ (A.8)

This result can be equivalently rewritten as

$$f_{km} = \frac{m\omega_{km}}{\hbar} \{\langle k \mid \hat{x} \mid m \rangle^* \langle k \mid \hat{x} \mid m \rangle + \langle k \mid \hat{x} \mid m \rangle^* \langle k \mid \hat{x} \mid m \rangle\}.$$ (A.9)

Since, however,

$$im\omega_{km} \langle k \mid \hat{x} \mid m \rangle = \langle k \mid \hat{p}_x \mid m \rangle$$ (A.10)

we can rewrite Eq.(A.9) as

$$f_{km} = \frac{1}{i\hbar} \{\langle m \mid \hat{x} \mid k \rangle \langle k \mid \hat{p}_x \mid m \rangle - \langle m \mid \hat{p}_x \mid k \rangle \langle k \mid \hat{x} \mid m \rangle\}$$ (A.11)

since $\omega_{km} = -\omega_{mk}$. Finally, we have to require $\sum_k f_{km} = 1$. This is possible only if

$$\frac{1}{i\hbar} \langle m \mid \hat{x}\hat{p}_x - p_x\hat{x} \mid m \rangle = 1,$$ (A.12)

QED.

Appendix B. Some quantum mechanical problems associated with the Lie algebra of SO(2,1) group

Following Wybourne (1974) consider the second order differential equation of the type

$$\frac{d^2Y}{dx^2} + V(x)Y(x) = 0$$ (B.1)

where $V(x) = a/x^2 + bx^2 + c$. Consider as well the Lie algebra of the noncompact group SO(2,1) or, better, its connected component SU(1,1). It is given by the following commutation relations

$$[X_1, X_2] = -iX_3; \ [X_2, X_3] = iX_1; \ [X_3, X_1] = iX_2$$ (B.2)

We shall seek the realization of this Lie algebra in terms of the following generators

$$X_1 := \frac{d^2}{dx^2} + a_1(x); \ \ X_2 := i[k(x)\frac{d}{dx} + a_2(x)]; \ \ X_3 := \frac{d^2}{dx^2} + a_3(x).$$ (B.3)

The unknown functions $a_1(x), a_2(x), a_3(x)$ and $k(x)$ are determined upon substitution of Eq.s(B.3) into Eq.s(B.2). After some calculations, the following result is obtained

$$X_1 := \frac{d^2}{dx^2} + \frac{a}{x^2} + \frac{x^2}{16}; \ X_2 := \frac{-i}{2}[x\frac{d}{dx} + \frac{1}{2}]; \ X_3 := \frac{d^2}{dx^2} + \frac{a}{x^2} - \frac{x^2}{16}.$$ (B.4)

51

In view of this, Eq.(B.1) can be rewritten as follows

$$[(\frac{1}{2} + 8b)X_1 + (\frac{1}{2} - 8b)X_3 + c]Y(x) = 0 \tag{B.5}$$

This expression can be further simplified by the unitary transformation $UX_1U^{-1} = X_1\cosh\theta + X_3\sinh\theta$; $UX_3U^{-1} = X_1\sinh\theta + X_3\cosh\theta$ with $U = exp(-i\theta X_2)$. By choosing $\tanh\theta = -(1/2 + 8b)/(1/2 - 8b)$ Eq.(B.5) is reduced to

$$X_3\tilde{Y}(x) = \frac{c}{4\sqrt{-b}}\tilde{Y}(x) \tag{B.6}$$

where the eigenfunction $\tilde{Y}(x) = UY(x)$ is an eigenfunction of both X_3 and the Casimir operator $\mathbf{X}^2 = X_3^2 - X_2^2 - X_1^2$ so that by analogy with the Lie algebra of the angular momentum we obtain,

$$\mathbf{X}^2\tilde{Y}_{jn}(x) = J(J+1)\tilde{Y}_{Jn}(x) \quad \text{and} \tag{B.7a}$$

$$X_3\tilde{Y}_{Jn}(x) = \frac{c}{4\sqrt{-b}}\tilde{Y}_{Jn}(x) \equiv (-J+n)\tilde{Y}_{Jn}(x); \quad n = 0,1,2,.... \tag{B7b}$$

It can be shown that $J(J+1) = -a/4 - 3/16$. From here we obtain : $J = -\frac{1}{2}(1 \pm \sqrt{\frac{1}{4} - a})$; $\frac{1}{4} - a \geq 0$. In the case of discrete spectrum one should choose the plus sign in the expression for J. Using this result in Eq.(B.7) we obtain the following result of major importance

$$4n + 2 + \sqrt{1 - 4a} = \frac{c}{\sqrt{-b}}. \tag{B.8}$$

Consider now the planar Kepler problem. In this case, in view of Eq.(32), the radial Schrödinger equation can be written in the following symbolic form

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{v}{r} + \frac{u}{r^2} + g\right]R(r) = 0 \tag{B.9}$$

By writing $r = x^2$ and $R(r) = x^{-\frac{1}{2}}\mathcal{R}(x)$ This equation is reduced to the canonical form given by Eq(B.1), e.g. to

$$(\frac{d^2}{dx^2} + \frac{4u + 1/4}{x^2} + 4gx^2 + 4v)\mathcal{R}(x) = 0 \tag{B.10}$$

so that the rest of arguments go through. Analogously, in the case of Morse-type potential we have the following Schrodinger-type equation initially:

$$\left[\frac{d^2}{dz^2} + pe^{2\alpha z} + qe^{\alpha z} + k\right]R(z) = 0 \tag{B.12}$$

By choosing $z = lnx^2$ and $R(z) = x^{-\frac{1}{2}}\mathcal{R}(x)$ Eq.(B12) is reduced to the canonical form

$$(\frac{d^2}{dx^2} + \frac{16k + \alpha^2}{4\alpha^2 x^2} + \frac{4p}{\alpha^2}x^2 + \frac{4q}{\alpha^2})\mathcal{R}(x) = 0 \tag{B.13}$$

By analogous manipulations one can reduce to the canonical form the radial equations for Hydrogen atom and for the 3-dimensional harmonic oscillator.

Appendix C. Numerical data used for claculations of n^*_{theory} (Table 4).

1 au=$149.598 \cdot 10^6 km$
Masses (in kg): Sun $1.988 \cdot 10^{30}$, Jupiter $1.8986 \cdot 10^{27}$, Saturn $5.6846 \cdot 10^{26}$, Uranus $8.6832 \cdot 10^{25}$, Neptune $10.243 \cdot 10^{25}$.
q_j : Jupiter $0.955 \cdot 10^{-3}$, Saturn $2.86 \cdot 10^{-4}$, Uranus $4.37 \cdot 10^{-5}$, Neptune $5.15 \cdot 10^{-5}$.
$(r_j)_1 (km)$: Jupiter $127.69 \cdot 10^3$, Saturn $133.58 \cdot 10^3$, Uranus $49.77 \cdot 10^3$, Neptune $48.23 \cdot 10^3$.
$\ln\left(\dfrac{\gamma M}{2r_1}\right)$: Earth 4.0062, Jupiter 3.095, Saturn 1.844, Uranus 0.9513, Neptune 1.15.

Appendix D. Connections between the Hill and Neumann's dynamical problems.

We follow our paper (Kholodenko 2002) where some mathematical of the results of the paper by Lazutkin and Pankratova (1975) were used for solution of concrete physical problems. In particular, following our paper, let us consider the Fuchsian-type equation given by

$$y'' + \frac{1}{2}\phi y = 0, \qquad (D.1)$$

where the potential ϕ is determined by the equation $\phi = [f]$ with $f = y_1/y_2$ and y_1, y_2 being two independent solutions of Eq.(D.1) normalized by the requirement $y_1' y_2 - y_2' y_1 = 1$. The symbol $[f]$ denotes the Schwarzian derivative of f. Such a derivative is defined as follows

$$[f] = \frac{f' f''' - \frac{3}{2}(f'')^2}{(f')^2}. \qquad (D.2)$$

Consider Eq.(D.1) on the circle S^1 and consider some map of the circle given by $F(t+1) = F(t)+1$. Let $t = F(\xi)$ so that $y(t) = Y(\xi)\sqrt{F'(\xi)}$ leaves Eq.(D.1) form -invariant, i.e. in the form $Y'' + \frac{1}{2}\Phi Y = 0$ with potential Φ being defined now as $\Phi(\xi) = \phi(F(\xi))[F'(\xi)]^2 + [F(\xi)]$. Consider next the infinitesimal transformation $F(\xi) = \xi + \delta\varphi(\xi)$ with δ being some small parameter and $\varphi(\xi)$ being some function to be determined. Then, $\Phi(\xi + \delta\varphi(\xi)) = \phi(\xi) + \delta(\hat{T}\varphi)(\xi) + O(\delta^2)$. Here $(\hat{T}\varphi)(\xi) = \phi(\xi)\varphi'(\xi) + \frac{1}{2}\varphi'''(\xi) + 2\phi'(\xi)\varphi(\xi)$. Next, we assume that the parameter δ plays the same role as time. Then, we obtain

$$\lim_{t \to 0}\frac{\Phi - \phi}{t} = \frac{\partial \phi}{\partial t} = \frac{1}{2}\varphi'''(\xi) + \phi(\xi)\varphi'(\xi) + 2\phi'(\xi)\varphi(\xi) \qquad (D.3)$$

Since thus far the perturbing function $\varphi(\xi)$ was left undetermined, we can choose it now as $\varphi(\xi) = \phi(\xi)$. Then, we obtain the Korteweg -de Vriez (KdV) equation

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\phi'''(\xi) + 3\phi(\xi)\phi'(\xi) \tag{D.4}$$

determining the potential $\phi(\xi)$. For reasons which will be explained in the text, it is sufficient to consider only the static case of KdV, i.e.

$$\phi'''(\xi) + 6\phi(\xi)\phi'(\xi) = 0. \tag{D.5}$$

We shall use this result as a reference for our main task of connecting the Hill and the Neumann's problems. Using Eq.(54) we write

$$u(\xi) = \phi(\xi) - <\dot{\xi}, \dot{\xi}> . \tag{D.6}$$

Consider an auxiliary functional $\varphi(\xi) = <\xi, A^{-1}\xi>$. Suppose that $\varphi(\xi) = u(\xi)$. Then,

$$\frac{du}{dt} = 2 <\dot{\xi}, A\xi> -2 <\ddot{\xi}, \dot{\xi}> . \tag{D.7}$$

But $<\ddot{\xi}, \dot{\xi}> = 0$ because of the normalization constraint $<\xi, \xi> = 1$. Hence, $\frac{du}{dt} = 2 <\dot{\xi}, A\xi>$. Consider as well $\frac{d\varphi}{dt}$. By using Eq.s (54) it is straightforward to show that $\frac{d\varphi}{dt} = 2 <\dot{\xi}, A^{-1}\xi>$. Because by assumption $\varphi(\xi) = u(\xi)$ we have to demand that $<\dot{\xi}, A^{-1}\xi> = <\dot{\xi}, A\xi>$ as well. If this is the case, consider furthermore

$$\frac{d^2 u}{dt^2} = 2 <\ddot{\xi}, A^{-1}\xi> +2 <\dot{\xi}, A^{-1}\dot{\xi}> \tag{D.8}$$

Using Eq.s(54) once again we obtain

$$\frac{d^2 u}{dt^2} = -2 + 2u\varphi + 2 <\dot{\xi}, A^{-1}\dot{\xi}> . \tag{D.9}$$

Finally, consider as well $\frac{d^3 u}{dt^3}$. Using Eq.(D.9) as well as Eq.(54) and (D.7) we obtain,

$$\frac{d^3 u}{dt^3} = 2\frac{du}{dt}\varphi + 4u\frac{du}{dt} = 6u\frac{du}{dt} \tag{D.10}$$

By noticing that in Eq.(D.5) we can always make a rescaling $\phi(\xi) \rightarrow \lambda\phi(\xi)$ we always can choose $\lambda = -1$. Therefore Eq.s (D.5) and (D.10) coincide. This establishes the correspondence between the Neumann and Hill-type problems.
QED

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