Primary proof that the torque exerted on a stationary body is zero

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Abstract

In a primary manner it is shown that if a body is stationary the torque exerted on it is zero, while at present avoiding the analytical proof of this theorem, this theorem is presented to the student unprovenly and almost as an axiom.

1 Introduction

We know that in the analytical mechanics, defining the torque as $\mathbf{r} \times \mathbf{F}$, it is easily proven that when the angular momentum of a rigid body does not change with time, inevitably the torque exerted on it is zero. When teaching the subject of torque to a novice student, since he or she is not at that level that the analytical proof of the above theorem can be presented to him or her, after giving a definition like the above one for the torque, it is by no means proven that the torque exerted on a stationary body is zero, but it is stated that it must be so. In simpler words supposing that the bar AB shown in Fig. 1, which is hinged at A, is stationary, the relation $F_1 \cdot AB = F_2 \cdot AC$ is not proven as a theorem, but is presented as an obligatory axiom to the student.

Acceptance of this unproven obligation is difficult for a curious and perspicacious student, and he or she asks himself or herself whether a new law in mechanics is being revealed, a law stating that the torque exerted on a stationary body must be zero.

Therefore, the necessity of proving the above theorem in a simple manner understandable for a novice student displays itself. In the next section we proceed to such a proof.

2 The proof

Suppose that while the string of Fig. 2 is tied to a fixed support in A and B, the force F is exerted on it in the point O due to which the string exerts the forces F_1 and F_2 on the support. The string is in equilibrium, then the support exerts forces equal to F_1 and F_2, in the directions shown in Fig. 2, on the string in A and B. Each of these two forces, exerted by the support, has two components having directions as shown in the figure with the magnitudes of $F_1' = F_1 \cdot OO'/OA$, $F_1'' = F_1 \cdot AO'/OA$, $F_2' = F_2 \cdot OO'/OB$ and $F_2'' = F_2 \cdot BO'/OB$. Since the string has no horizontal (leftward or rightward) translational motion, we have $F_1'' = F_2''$ or $F_1 \cdot AO'/OA = F_2 \cdot BO'/OB$ which we write it as $(F_1 \cdot OO'/OA) \cdot AO' = (F_2 \cdot OO'/OB) \cdot BO'$ or $F_1' \cdot AO' = F_2' \cdot BO'$. Notice that we have reached the definition of torque, because $F_2' = F_2 \cdot OO'/OB$ is the normal force exerted on B and BO' is the torque arm relative to O', and also $F_1' = F_1 \cdot OO'/OA$ is the normal force exerted on A and AO' is the torque arm relative to O', which as we saw the product of the first two was equal to the product of the second two (that was in fact because of absence of any rotational motion).

Now imagine that the angles θ_1 and θ_2 in Fig. 2 approach zero. In this state since F_1 and F_2 must have normal components for balancing the constant force F, inevitably their horizontal components (balancing each other) approach infinity. It is quite obvious that no substance can stand infinite force and then in practice θ_1 and θ_2 will not be zero although can be very small. But notice that even if we suppose that $\theta_1 = \theta_2 = 0$, the above-mentioned law for torque will be still true, ie we shall have $F_1' \cdot OA = F_2' \cdot OB$ for the case shown in Fig. 3, because obviously this is a limit state for a process in each stage of which the above law of torque has been true.

The above material is true for every rigid body (eg a rigid rod each end of which is borne by a person and a weight is hung from a point of it between these two ends). In this manner the law of torque has been proven generally. Considering Fig. 3 and this fact that we saw $F_1' \cdot OA = F_2' \cdot OB$, we can write: $F_2' \cdot OB = F_1' \cdot OA \Rightarrow F_2' \cdot OB + F_2' \cdot OA = F_1' \cdot OA + F_2' \cdot OA$ or $(F_2' \cdot (OB + OA) = F_2' \cdot AB) = ((F_1' + F_2') \cdot OA = F \cdot OA)$ and also $F_2' \cdot OB = F_1' \cdot OA \Rightarrow F_2' \cdot OB + F_1' \cdot OB = F_1' \cdot OA + F_1' \cdot OB$ or $((F_1' + F_2') \cdot OB = F \cdot OB) = (F_1' \cdot (OA + OB) = F_1' \cdot AB)$

In an almost similar manner it can be proven that the torque of F relative to each point in space is absolutely equal to the sum of the torques of F_1' and F_2' relative to that point. For example for the point C in Fig. 3 we have: $F \cdot OB = F_1' \cdot AB \Rightarrow F \cdot OB + F \cdot BC = F_1' \cdot AB + F \cdot BC$ or $F \cdot (OB + BC) = F_1' \cdot AB + (F_1' + F_2') \cdot BC$ or $F \cdot OC = F_1' \cdot (AB + BC) + F_2' \cdot BC$ or $F \cdot OC = F_1' \cdot AC + F_2' \cdot BC$

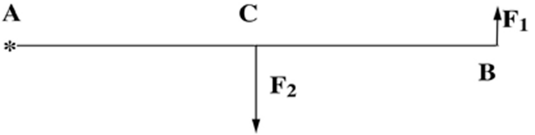

Fig. 1. *A stationary bar AB, hingend at A, on which the forces F_1 and F_2 are exerted.*

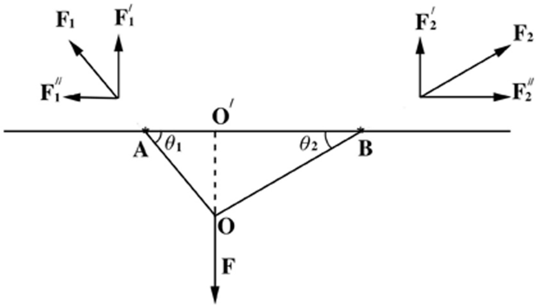

Fig. 2. *A string tied to a fixed support .*

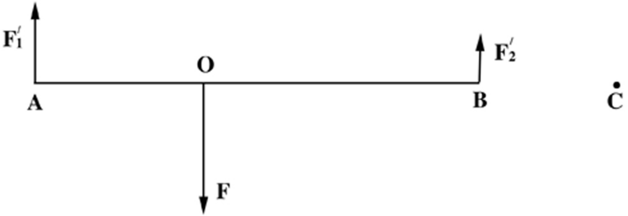

Fig. 3. *The string of Fig. 2 when θ_1 and θ_2 have approached zero .*