Classical justification of the Stern-Gerlach experiment

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September 19, 2007

Abstract

It is shown that contrary to what is thought the classical physics does not predict a uniform distribution for the magnetic dipoles (silver atoms) in a nonuniform magnetic field in the Stern-Gerlach experiment. Its prediction for a concentrated beam is obtained in the form of a circular surface such that the density of the dipoles is much more near the edge than near the center. Some experiments are proposed for testing the contents of the article.

1 Introduction

In the Stern-Gerlach experiment, after collimating a beam of silver atoms stimulated in a heated furnace it is passed through a slit and then through a strong nonuniform magnetic field. It then hits on a sensitive plate. What is observed on the plate is not a uniform distribution of effects of the silver atoms but is a nonuniform one showing two maximums in intensity: one towards the region of intense field and the other towards the region of weak field.

The world of physics has interpreted this fact as an experimental evidence to prove the quantization of the magnetic dipoles (ie silver atoms) in a magnetic field into only two upward and downward directions invoking that the classical physics predicts a uniform distribution for the magnetic dipoles randomly oriented in such a field.

Since wherever analyzing this experiment this is also pointed that the classical physics predicts a uniform distribution, without presenting any reason that how in the framework of this physics such a distribution is predicted, this article intends to investigate actual prediction of the classical physics for the distribution of the magnetic dipoles randomly oriented in such the above mentioned field.

2 Mathematical analysis of the problem

In order to have a mathematical analysis we benefit from the similarity between the electrostatics and magnetostatics. Nonuniform magnetostatic field produced in the Stern-Gerlach experiment is such as if almost all the magnetic lines of force are spread around (in fact onto the flat pole) from the sharp edge of a magnet (ie the other pole). Points near this edge, in which density of the lines is much, form the region of intense field, while points far from this edge (and near to the flat pole), in which density of the lines is slight, form the region of weak field.

Electrical analog of this situation with a proper approximation is the electrostatic field arising from a charged straight infinite line in which the electric lines of force radially and normal to the charged line have originated from it and have spread throughout the space. Likewise, the electrical analog of the magnetic dipoles are electric dipoles. Then, let's obtain form of the distribution of a great number of equivalent electric dipoles gathered in a point in such an electrostatic field and oriented randomly. Assume that these are point electric dipoles.

To each dipole, a vector of electric dipole moment, $\mathbf{p}$, is attributed. We call the electrostatic field arising from the charged line as $\mathbf{E}$. We know from the electrostatics [eg see the problems of the second chapter of Foundations of Electromagnetic Theory by J. R. Reitz, F. J. Milford, R. W. Christy, Addison-Wesley, 1979] that the net force exerted on the dipole $\mathbf{p}$ positioned in the field $\mathbf{E}$ is obtained from the formula $(\mathbf{p}\cdot\nabla)\mathbf{E}$, and the field arising from a charged line is $\mathbf{E} = (\lambda/(2\pi\epsilon_0))\hat{\rho}/\rho$ in which λ is the linear charge density of the charged line and ρ is the radial distance from the line. Having these two recent expressions and using Cartesian coordinates and assuming that the z-axis is the same charged line, we have

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{\rho}}{\rho} = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{i}x+\hat{j}y}{x^2+y^2}$$

and

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} = (\hat{i}p_x+\hat{j}p_y+\hat{k}p_z)\cdot(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z})\mathbf{E}$$

$$= (p_x\frac{\partial}{\partial x}+p_y\frac{\partial}{\partial y}+p_z\frac{\partial}{\partial z})(\frac{\lambda}{2\pi\epsilon_0}\frac{\hat{i}x+\hat{j}y}{x^2+y^2})$$

$$= \frac{\lambda}{2\pi\epsilon_0}(\hat{i}\frac{p_x(y^2-x^2)-p_y(2xy)}{(x^2+y^2)^2}+\hat{j}\frac{p_y(x^2-y^2)-p_x(2xy)}{(x^2+y^2)^2})$$

where $\mathbf{F}$ is the net force exerted on the dipole $\mathbf{p}$.

Suppose that the point in which the dipoles have gathered has the Cartesian coordinates (x,0,0). In this case the above expression obtained for $\mathbf{F}$ takes the simple form of

$$\mathbf{F} = \frac{\lambda}{2\pi\epsilon_0 x^2}(-\hat{i}p_x+\hat{j}p_y). \quad (1)$$

This relation shows that all the vectors of the forces exerted on the dipoles are in the xy-plane even though the vector **p** is not in this plane. (Even without any further reasoning it is clear that therefore the dipoles will get away from the point (x,0,0) in the xy-plane and we must expect forming a kind of hole at this point.)

How have the vectors of dipole moment, related to the dipoles gathered in the point (x,0,0), been distributed? Since there is no preferred direction, we can divide the surface of a sphere, which its center is the point (x,0,0) and its radius is $|\mathbf{p}|$, into much small equivalent symmetric areas, and imagine that for each of these partial areas there is only one **p** on the point (x,0,0) aiming at it. We want to obtain the distribution of the forces exerted on these dipoles in the above mentioned field.

Since, according to the relation (1), this force is proportional only to the projection of the vector **p** on the xy-plane, we need only to project each of these vectors, **p**, (which, as we said, has aimed at one of the partial symmetric areas of the surface of the sphere) on the xy-plane. The magnitude of this projection, according to the relation (1), is proportional to the force exerted on the dipole, and its direction, assuming that $\lambda > 0$, is the same direction as the image of the projection of **p** (on the xy-plane) in a mirror plane perpendicular to the x-axis in the point (x,0,0).

Therefore, for obtaining the design of the distribution of the forces exerted on the dipoles, it is sufficient to project each of the **p**-vectors (which as we said, have been distributed symmetrically in a sphere centered in (x,0,0)) on the xy-plane. Let's, instead of this act, take a more analytic action. In this action we consider the points of the surface of the sphere at each of which one of the **p**-vectors has aimed, and project these points on the xy-plane, and finally obtain the surface density of these projections on this plane.

Consider a strip from the surface of a hemisphere projected on its base. Considering R as the hemisphere radius and θ as the angle of the position of this strip relative to the axis of the hemisphere, area of this strip is $2\pi(R^2)\sin\theta d\theta$ and area of its projection is $2\pi r dr$ in which r is the radius of this projection. Assuming that the surface density of the above mentioned points on the surface of the hemisphere (at each of which one of the **p**-vectors has aimed) is sigma, the number of these points on the strip is $2\pi R^2 \sin\theta d\theta \cdot \sigma$.Since just this same number of points are projected on the projection of this strip on the base, the density of the projections of these points on the base, which we call it as σ', at distance r from the center, will be $2\pi R^2 \sin\theta d\theta \cdot \sigma/(2\pi r dr)$. With some simple mathematical operations, it can be seen that we have

$$\sigma' = \frac{R}{\sqrt{R^2 - r^2}}\sigma \tag{2}$$

for the density of the projection of the points on the base, at distance r from the center.

If instead of the hemisphere we consider the whole sphere, it will be sufficient to multiply the expression (2) by 2 for obtaining the density of the projections of the points. Important for us is that we understood that this density is proportional to

$$\frac{1}{\sqrt{R^2 - r^2}}.$$

Now giving different values (from zero to R) to the r, we can easily obtain the design of the distribution of the density. A schematic design of such a distribution has been shown in Figure 1.

In this manner we showed that, assuming that the beginnings of the vectors of forces exerted on the dipoles gathered in the point (x,0,0) are all the same point (x,0,0), the ends of these vectors are in the xy-plane and have a distribution like Figure 1 (assuming that the center of this figure is the same point (x,0,0)). It is obvious that each dipole, due to the force exerted on it, starts moving. Assuming that during the exertion of the field on the dipoles they do not rotate due to the torques exerted on them, if the distance (x) between the point (x,0,0) and the yz-plain is sufficiently large and we don't allow that the dipoles get so much distance from one another due to the net forces exerted on them, we can suppose with a good approximation that during the displacement of the dipoles under the influence of the forces exerted on them, the force exerted on each dipole remains constant. Therefore, the displacement of the dipole is obtained from the famous relation $d = (1/2)Ft^2/m$ in which d is the magnitude of the displacement, and t is the time of exertion of the force, and F is the magnitude of the force exerted on the dipole (ie the magnitude of the expression (1)), and m is the mass of the dipole. What is important for us is that, as we see, in a definite time the displacement of the dipole is proportional to the force exerted on it. Since, on supposition, each dipole has the same mass, the design shown in Figure 1 is also the same design of the displacements of the dipoles due to the forces exerted on them, of course assuming that they don't rotate due to the exertion of the torques on them during the exertion of the field.

If the dipoles are under the influence of the field for a sufficiently long time, they will practically find an opportunity to rotate due to the torques exerted on them. It is clear that this rotation will be in such a manner that the dipoles take such orientations that finally only forces towards the region of intense field are exerted on them (and we have no longer a distribution of forces in every direction as in Figure 1). In this state the design of the distribution will be no longer similar to Figure 1.

But if the dipoles are under the influence of the field only for a very short time (ie only in a very short time the field is exerted on them), the dipoles, oriented randomly, won't find any opportunity to rotate due to the torque exerted on them during the time of exertion of the field. Thus, during this time, the orientations of the dipoles and thereby the net forces exerted on them won't change, and then immediately after switching the field off, we shall have a distribution like Figure 1 for the dipoles but the area of the distribution will be practically very small, because the time of exertion of the field and consequently of the net force exerted on each dipole is very short and consequently its displacement due to this force will be also very small.

But what will be the design of the displacements of the dipoles if after switching the field off we let the dipoles continue, for a definite time, their motion with the speed obtained by them during the same short time of exertion of the field? Such a design of distribution will be exactly the same design shown in Figure 1. To prove this, it is sufficient to prove that if two equal masses m_1 and m_2 in Figure 2 coincide with each other in the time t=0 and in the time $t = t_1$ the mass m_1 gets the distance d_1 from the origin due to the exertion of the constant force F_1 on it and the mass m_2 gets the distance d_2 from the origin in the same direction as m_1 due to the exertion of the constant force F_2 on it and just at this time exertion of the forces is switched off, then the ratio of the distance of the mass m_1 from the origin to the distance of the mass m_2 from the origin will be always equal to d_1/d_2 at every time after switching the forces off (after which the masses continue their motion in the same direction with speeds obtained by them during the exertion of the forces). This is a simple mechanical problem solving of which can be done by the reader easily.

In summary, we showed that if the dipoles concentrated in the point (x,0,0) which are oriented randomly are taken under the influence of the field of a charged line for a very short time, they will be distributed only in the plane perpendicular to the charged line at (x,0,0) in a manner showing maximum and minimum in intensity as shown in Figure 1 (not a uniform intensity). As the time elapses (exerting no field), only the area of the distribution increases but its form won't alter.

3 Experimental ways for testing the theory

It is obvious that if the random dipoles concentrated in (x,0,0) have a great speed parallel with the z-axis (eg because of the exit from a hot furnace) such that the time of their passing through the fixed field (of the charged line with a limited length) or in other words such that the time during which the field is exerted on them is very short and afterwards they descend on a sensitive plate perpendicular to the z-axis at a point of the trajectory sufficiently far from the field region, then according to what we have said so far we must expect that they will show a design of the density distribution on the sensitive plate similar to Figure 1. Notice that we have assumed that the beam is concentrated as far as possible, ie the ratio of its thickness to its distance from the z-axis is very, very small. In fact it is better to consider the beam consisting of very much small spherical volumes occupying the whole volume of the beam practically. In this state the random dipoles of each spherical volume will have a distribution like Figure 1 after passing through the field and descending on the sensitive plate, and since all of these spherical volumes are concentrated very near around the beam axis, the distribution related to all of the spherical volumes after their passing through the field is also similar to Figure 1. The only condition is that the width of the distribution is sufficiently larger than the thickness of the beam.

Now, how will the distribution be after passing through the field if instead of the above concentrated beam we have a nonconcentrated one arising from passing of the dipoles through a slit with a non-negligible length? Although analytical obtaining of the distribution in this case won't be certainly as straightforward as one we discussed here (ie one related to a concentrated beam), it is intuitively clear that the distribution will be outwards from an imaginary line drawn parallel to the slit and equal to its length (ie we shall have two maximums in intensity: one above and the other under this line) and will have an extension parallel to the slit, and it is expected that the length of this extension will be more than the slit length.

Anyway, it is quite clear from the analysis presented here that this claim that the classical physics predicts a uniform distribution not a distribution having maximum and minimum in intensity is quite wrong.

By performing an experiment the assertions of this article will be completely either confirmed or rejected. Instead of passing the silver atoms through a slit, pass them through a very small circular aperture and afterwards through the magnetic field. If the form of the distribution will have only two upward and downward maximums, the contents of this article will be certainly wrong and the result of the experiment can be cited as an experimental evidence for quantization of the magnetic spin in a magnetic field into only two cases of 1/2 and -1/2, but if the distribution will be similar to Figure 1, the contents of this article will be decisively confirmed.

Another experimental test is investigating that whether the length of the distribution on the sensitive plate is larger than the slit length or not. For this test the beam must be so collimated beforehand that if there was no magnetic field, the length of the distribution would be equal to the same slit length.

Certainly if the magnetic dipoles have only two (quantized) upward and downward directions in the magnetic field, there won't be any reason that any forces parallel to the slit length are exerted on them causing lengthening of the distribution compared with the length of the slit. Notice that as we saw previously, the forces exerted on the dipoles in the magnetic field are oriented in every direction not in only two directions. (The photographs I have seen in this respect show that for a beam passing through a slit the effect on the sensitive plate has a rather single lengthened elliptical shape (instead of having two separate lines which is more expected as a quantum prediction) (eg see Halliday)).

As we mentioned previously, we expect that with a long-time passing of the beam through the field (eg by lengthening the field) the intensity of the beam in the region of weak field is decreased (because the dipoles find opportunity to rotate gradually due to the torques exerted on them and consequently one by one will be drawn towards the region of intense field). This matter can be another experimental test for the theory.

It is obvious that when each of the separated beams (one towards the region of intense field and the other towards the region of weak field) without elapsing of much time are passed again through a similar field, because of lack of enough time for rotation of the dipoles and serious change of their orientations, we expect, as the experiment shows, that the separated beam show the same behavior in the new field as in the old one, ie the beam turned towards the intense field will now again turn towards the intense field and the other one turned towards the weak field will now again turn towards the weak field. But if we let the separated beams continue their motion in a region lacking any field for a sufficiently long time after their passing through the field, then the dipoles, due to the angular speeds they have gained during the exertion of torque on them in the field, will continue their gradual rotation till their orientations will change completely. In this state we expect that if each of the separated beams is directed into a similar field, it won't turn only towards the same direction of deflection undergone in the first field, but we expect that, if not deflected towards the opposite direction, it separates again into two beams (towards the regions of intense and weak fields) at least. This matter can also be another experimental test for the theory.

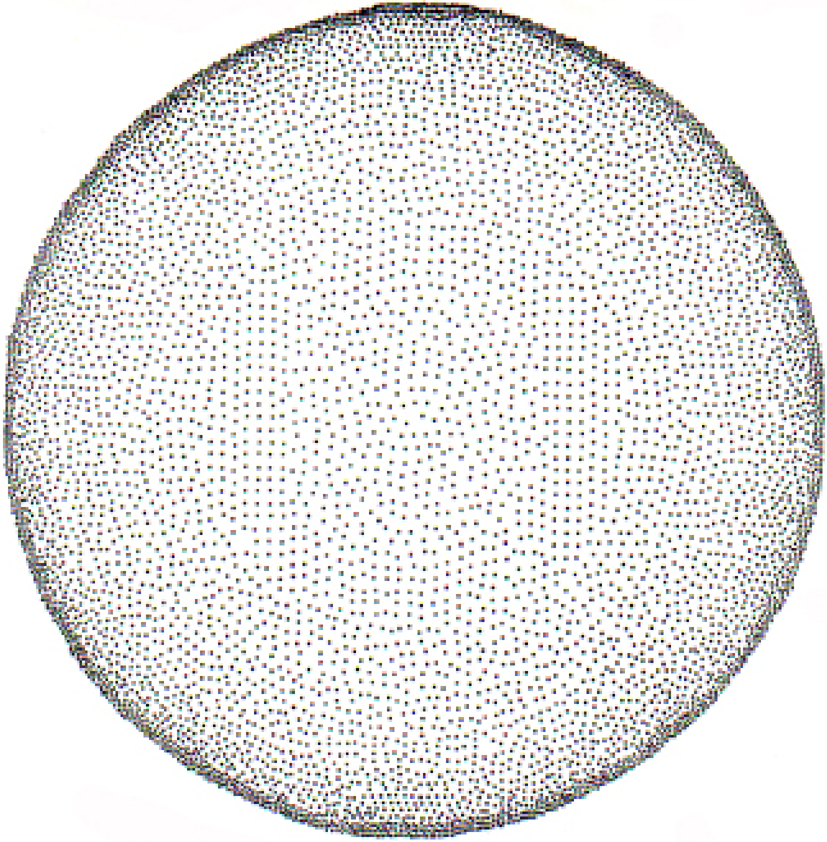

Fig. 1

Fig. 2