3x3 Unitary to Magic Matrix Transformations

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We prove that any 3x3 unitary matrix can be transformed to a magic matrix by multiplying its rows and columns by phase factors. A magic matrix is defined as one for which the sum of the elements in any row or column add to the same value. This result is relevant to recent observations on particle mixing matrices.

1 Introduction

Recently it has been found that magic unitary matrices are relevant in the structure of particle mixing matrices [1-3]. A mathematical question arises as to whether every unitary matrix can be transformed to a magic unitary matrix using simple transformations which just multiply rows and columns by complex phase factors, and if so, how many ways can there be of doing it?

In the case of 2x2 matrices the problem is easily solved. A 2x2 unitary matrix can first be transformed to a real orthogonal matrix by multiplying rows and columns by phase factors. It then has the form

$$\begin{pmatrix} \cos x & -sinx \\ sinx & cosx \end{pmatrix}$$

This can then be transformed to a magic matrix whose rows and columns sum to one.

$$\begin{pmatrix} ie^{-ix}\cos x & e^{-ix}\,sinx \\ e^{-ix}sinx & ie^{-ix}cosx \end{pmatrix}$$

A second solution is given by the complex conjugate and this gives all solutions up to an overall phase factor.

For 3x3 matrices there is no known way of constructing solutions in closed form. In this paper we use topological arguments to prove that solutions always exist. For larger matrices the problem is still unsettled.

2 The 3x3 Case

Problem: *Show that any 3x3 unitary matrix can be transformed to a magic matrix by multiplying its rows and columns by phase factors.*

A magic matrix M is a matrix with the property that the elements in any row or column add up to the same value L. If $n = (1,1,1)$ is a row vector with all elements equal to one then the magic matrix property for 3x3 matrices can be written as:

$$n\,M = L\,n \quad \text{and} \quad M\,n^{T} = L\,n^{T}$$

To transform the unitary matrix U we can multiply on the left by a unitary diagonal matrix D_L and on the right by a unitary diagonal matrix D_R. So we require,

$$D_L\, U\, D_R = M$$

Define two column vectors a and b whose components are all phases by $a = D_R\, n^T$ and $b = D^\dagger_L\, n^T$. Then,

$$M\, n^T = L\, n^T \iff U\, a = L\, b$$

$$n\, M = L\, n \iff b^\dagger\, U = L\, a^\dagger$$

Since U is unitary and $|a| = |b| = 3$ it follows that $|L| = 1$. Furthermore the two equations are equivalent and we can fix $L = 1$ without changing the problem because its phase can be absorbed into b. In conclusion, the original problem is equivalent to showing that for any 3x3 unitary matrix U we can find vectors a and b whose components are all pure phases such that $U\, a = b$. Although this is a system of three equations, the unitary property of U tells us that for any vectors satisfying this equation we will have $|a| = |b|$, So if any two of equations are satisfied then the third can be too.

To show that these equations always have solutions we will consider two distinct cases depending on the size of the components. For each row of the unitary matrix U take x, y and z to be the components in the row ordered so that $|x| \geq |y| \geq |z|$. The value of $|x| + |y| - |z|$ will be between $1/\sqrt{3}$ and $\sqrt{2}$ for each row. The first case we consider is when this value is larger than 1 for all three rows. The second case is where is it less than or equal to 1 for at least one row.

First Case: Since some elements must be bigger than others in this case, it is not possible that all three of the smallest elements z are in the same column. Select two rows in which it is in different columns, then fixing one of the phases, the equations we need to solve can always be written in this form,

$$u_1\, p_1 + u_2\, p_2 + u_3 = q_1 \quad \text{with } |u_3| + |u_2| - |u_1| > 1,\ |u_1| \leq |u_2|,\ |u_1| \leq |u_3|$$
$$v_1\, p_1 + v_2\, p_2 + v_3 = q_2 \quad \text{with } |v_3| - |v_2| + |v_1| > 1,\ |v_2| \leq |v_1|,\ |v_2| \leq |v_3|$$

$$\text{where } |p_1| = |p_2| = |q_1| = |q_2| = 1$$

Consider solutions of the first equation when we fix a value for the phase p_1 while varying q_1 and p_2.

$$q_1 - u_2\, p_2 = u_3 + u_1\, p_1$$

Solving this equation is equivalent to forming a triangle with sides of fixed length $a = |q1| = 1$, $b = |u_2\, p_2| = |u_2|$, $c = |u_3 + u_1\, p_1|$, $|u_3| - |u_1| \leq c \leq |u_3| + |u_1|$. This is always possible in exactly two ways (one the reflection of the other) provided the triangle inequalities are satisfied which is always true if,

$$1 + |u_2| > |u_3| + |u_1|$$
$$1 + (|u_3| - |u_1|) > |u_2|$$
$$|u_2| + (|u_3| - |u_1|) > 1$$

These are indeed true in the case being considered. It follows that for any choice of the phase p_1 we can find two solutions in q_1 and p_2. Furthermore the two solutions form two disjoint but continuous curves. On the toroidal phase space of p_1 and p_2 the locus of solutions must be two separate curves that wrap round the torus in the direction generated by varying p_1. This direction was determined by the fact that $|u_1| < |u_2|$

When a similar argument is applied to the second equation the locus of solutions in p_1 and p_2 is also found to form two curves wrapping round the same torus, but because $|v_2| < |v_1|$ the curves wrap round the torus in the direction generated by variation of p_2. Two curves that wrap round a torus in different directions will always intersect. In this case there are two curves winding in each direction so we get four different intersections which provide four distinct solutions to the equations. This completes the first case.

Second Case: At least one of the rows does not satisfy the inequality so we have to show that there is a solution to these equations.

$$u_1\, p_1 + u_2\, p_2 + u_3 = q_1 \qquad \text{with } |u_3| + |u_2| - |u_1| \leq 1,\ |u_1| \leq |u_2|,\ |u_1| \leq |u_3|$$
$$v_1\, p_1 + v_2\, p_2 + v_3 = q_2$$

$$\text{where } |p_1| = |p_2| = |q_1| = |q_2| = 1$$

The first equation can be analysed as before but in this case we find that there are two values for p_1 at which the triangle becomes degenerate and there is only one solution instead of two. We can conclude that the locus of solutions of the first equation on the torus form a single connected set of points in this case.

The equations can be rewritten in another form.

$$|u_1\, p_1 + u_2\, p_2 + u_3|^2 = 1$$

$$\iff \mathrm{Re}(\, u_1\, u_2^*\, p_1\, p_2^* + u_3\, u_1^*\, p_1^* + u_2\, u_3^*\, p_2) = 0$$

To simplify the algebra we can assume without loss of generality that u_1, u_2 and u_3 are non-negative real numbers. This is because we can make an initial transformation by applying phase factors to the columns so that they take this form. With $p_1 = exp(i\ t_1)$ and $p_2 = exp(i\ t_2)$ this becomes,

$$u_1\, u_2\, cos(t_1 - t_2) + u_3\, u_1\, cos(t_1) + u_2\, u_3\, cos(t_2) = 0$$

We know that all solutions lie on a single connected curve on the t_1,t_2 torus, but consider four specific solution (not necessarily distinct) given as follows:

$$\begin{aligned}(t_1,t_2) &= (\pi/2,\ T)\\ &= (\pi/2,\ T + \pi)\\ &= (-\pi/2,\ -T)\\ &= (-\pi/2,\ -T + \pi)\end{aligned}$$

$$\text{where } T = tan^{-1}(u_1/u_3)$$

The second equation can also be rewritten,

$$|v_1\, p_1 + v_2\, p_2 + v_3|^2 = 1$$

$$\Leftrightarrow \;\; A(t_1,t_2) = 0$$
$$\text{where } A(t_1,t_2) = \;\; \mathrm{Re}(\, v_1\, v_2{*}\, exp(i(t_1\text{-}t_2)) + v_3\, v_1{*}\, exp(\text{-}i\, t_1) + v_2\, v_3{*}\, exp(i\, t_2)\,)$$

Sum this expression over the four specific solutions to the first equation,

$$S = A(\pi/2,\, T) + A(\pi/2,\, T + \pi) + A(\text{-}\pi/2,\, \text{-}T) + A(\text{-}\pi/2,\, \text{-}T + \pi)$$

This expression is identically zero because each term in the formula for $A(t_1,t_2)$ cancels when summed using $exp(it) = \text{-}exp(it + \pi)$

This tells us that the four values of $A(t_1,t_2)$ must either all be zero, or at least one must be negative and at least one must be positive. Since the solution curve for the first equation is connected this means that there must be a point on the curve where $A(t_1,t_2) = 0$. This gives the required simultaneous solution to both equations and we are done.

Some Notes

- This problem was suggested by Carl Brannen and Marni Dee Sheppeard in relation to the physics of the CKM matrix.

- It may be possible to simplify this proof. In particular the first case of the proof may never arise and there might be a simple proof of that.

- The analogous problem for 2x2 matrices is easy to solve but it remains an open problem to prove that there are always similar transformations to a magic matrix for 4x4 unitary matrices or larger.

- The transformation to a 3x3 magic matrix whose rows and columns sum to one is not unique except in special cases such as a diagonal matrix or a permutation matrix. The question of what is the maximum number of distinct solutions is open but there can be as many as six, for example when the unitary matrix is the matrix for a discrete Fourier transform.

- For larger sizes of matrices the maximum number of distinct solutions increases exponentially. If you have an NxN matrix with S distinct solutions and an MxM matrix with T solutions, then it is easy to see that the $(N+M)$x$(N+M)$ matrix formed with those two as separate blocks will have ST solutions. In fact it is also possible to use them to construct an $(N+M\text{-}1)$x$(N+M\text{-}1)$ matrix with ST solutions by extending them with ones on the diagonal and multiplying together with just one row overlapping. E.g. this means that there are 4x4 matrices with 12 solutions and 5x5 matrices with 36 solutions.

References

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