Koide mass equations for hadrons

Carl Brannen*[1]

[1]8500 148th Ave. NE, T-1064, Redmond, WA, USA

Email: Carl Brannen*- carl@brannenworks.com;

*Corresponding author

Abstract

Koide's mass formula relates the masses of the charged leptons. It is related to the discrete Fourier transform. We analyze bound states of colored particles and show that they come in triplets also related by the discrete Fourier transform. Mutually unbiased bases are used in quantum information theory to generalize the Heisenberg uncertainty principle to finite Hilbert spaces. The simplest complete set of mutually unbiased bases is that of 2 dimensional Hilbert space. This set is compactly described using the Pauli $SU(2)$ spin matrices. We propose that the six mutually unbiased basis states be used to represent the six color states R, G, B, $\bar{R}$, $\bar{G}$, and $\bar{B}$. Interactions between the colors are defined by the transition amplitudes between the corresponding Pauli spin states. We solve this model and show that we obtain two different results depending on the Berry-Pancharatnam (topological) phase that, in turn, depends on whether the states involved are singlets or doublets under $SU(2)$. A postdiction of the lepton masses is not convincing, so we apply the same method to hadron excitations and find that their discrete Fourier transforms follow similar mass relations. We give 39 mass fits for 137 hadrons. **PACS Codes:** 12.40.Yx, 12.40.-y, 03.67.-a

Background

Introduction

The standard model of elementary particle physics takes several dozen experimentally measured characteristics of the leptons and quarks as otherwise arbitrary parameters to the theory. There are enough coincidences in these parameters that the idea of "quark-lepton universality" is discussed in phenomenology papers, particularly with the weak mixing angles.

To compare the lepton masses directly with that of the quarks is difficult since quarks have been observed only in bound states. Consequently, quark masses can only be estimated and the estimates are model dependent. In looking for quark-lepton universality a natural place to begin looking is by comparing generations of mesons. For example, one could compare the ω d-dbar, ϕ s-sbar, and Υ b-bbar. However, this would not be an examination of the resonance structure of the quarks so much as an examination of what happens when one substitutes one resonance of a quark with another.

On the other hand, each individual hadron is an example of a color bound state; we will consider the excitations of that bound state. Rather than comparing the ω to the Υ, we will be looking at quantum states with the same quark content and quantum numbers such as the $\omega(782)$, $\omega(1420)$, $\omega(1650)$, $\omega(2145)$, $\omega(2205)$, and the $\omega(2290)$ all of which are d-dbar mesons with quantum numbers $I^G(J^{PC}) = 0^-(1^{--})$.

The standard phenomenal approximation of the force between quarks is that of a flux tube. The potential is Coulomb at short distances but is linear at long distance so that the force becomes constant. This assumption works best for the heavy quarks. Consequently some of their states are conventionally labeled as radial excitations of the Coulomb force, i.e. $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$. The model fails for the hadrons and lighter mesons. The color states of the quark are not treated except to adjust the force; this is over simplified in how it treats color but remains complicated in how it treats distance and momentum.

Coulomb excitations follow a $1/n^2$ law. For example, the excitations of the hydrogen atom have masses:

$$m_n = m - k/n^2, \tag{1}$$

where m is the sum of the proton and electron masses and k is an appropriate constant. Similar mass equations would apply to the hydrogen like excitations of other nuclei. The constant k will be changed according to the nuclear charge and the constant m will change according to the nuclear mass.

Coulomb excitations are infinite in number, and become much closer together as the energy increases. This is not what is seen in the hadrons, which instead have excitations that increase by what appears to be random jumps. In addition, no more than about a half dozen excitations of a hadron appear.

This paper models the color force using the approximations of quantum information theory. Rather than keeping track of the position and momentum of the quark, we will instead model only its color. This is a complementary approach to the standard. Color can be red, green, or blue so this reduces to a 3-dimensional Hilbert space. This implies that solutions to Schroedinger's equation will come with three orthogonal solutions, that is, as triplets. The assumption that the color force is symmetric with respect to color permutations implies that the triplets are related by the discrete Fourier transform.

Koide's mass formula for the charged leptons can be written as a statement about the discrete Fourier transform of the charged lepton masses. The present author used this method to generalize Koide's formula to the neutrinos. These two mass equations are applied in this paper to the hadron excitations.

The mass equations for the leptons are striking in their simplicity but somewhat mysterious. The first subsection of the results section gives a loose derivation of these equations. The derivation assumes that the color states and their interactions can be modeled as a complete set of mutually unbiased bases of the Pauli algebra (i.e. MUBs of qubits). Readers who are interested only in the coincidences in the discrete Fourier transforms of the hadron excitations are invited to skip ahead to those subsections of the results section.

Specialist knowledge

Mutually unbiased bases are a generalization of the idea of complementary operators such as momentum and position, and are a subject of current research interest in quantum information theory. Complementary operators and mutually unbiased bases are related, in finite Hilbert spaces, to the discrete Fourier transform; this paper analyzes the masses of hadrons after they are so transformed.

A complete collection of mutually unbiased bases has the advantage of providing, in one neat package, both the states and their interactions so there is no need to assume a separate force law or symmetry. The MUB states have their interactions defined by transition probabilities. Interactions are noncommutative and sequences of them generate Berry-Pancharatnam or topological phases in the transition amplitudes. Topological phase calculations are easily done using pure density matrices and their extension to non-Hermitian pure density matrices. This gives a derivation of the extension of Koide's mass formula for the charged leptons to the neutrinos. The charged lepton and neutrino mass equations are used as the templates for the hadron mass equations provided in the results section of the paper.

Mutually unbiased bases (MUBs)

Quantum information theory attempts to explore quantum mechanics by exploring the ideas present in the theory in the simplest possible forms. The Heisenberg uncertainty principle specifies that it is impossible to measure the position and momentum of a quantum state to arbitrary accuracy at the same time. This is because position and momentum are "complementary observables". More generally, let us consider a finite Hilbert H space of dimension N with two complementary observables A and B with basis (eigen)states $|a_n\rangle$ and $|b_m\rangle$ for $n, m \in 1, 2, 3, ...N$. We say that two complementary observables are "mutually unbiased."

Since A and B are complementary observables, the eigenstates of A provide no information about the eigenstates of B. Therefore we have:

$$|\langle a_n|b_m\rangle|^2 = \langle a_n|b_m\rangle\langle b_m|a_n\rangle = k, \tag{2}$$

for some real constant k. Thus the matrix elements for the transition probabilities between the states of A and the states of B are all equal (sometimes called "democratic") to k. Summing over m and noting that $|b_m\rangle$ form a complete basis set for the Hilbert space so that $\sum |b_m\rangle\langle b_m| = 1$, we find that $k = 1/N$.

Given a set of M such bases for H, we say that they are a set of "mutually unbiased bases" if they have this property pairwise. It is easy to show that the maximum number of mutually unbiased bases for such a Hilbert space is $N + 1$. A collection of $N + 1$ MUBs for a Hilbert space of dimension N is known as a "complete set of MUBs". Complete sets of MUBs are of "central importance in many discussions about the foundations of quantum mechanics." [1] Making measurements along these bases gives the least number of measurements required to fully characterize an unknown quantum state (quantum tomography). The question of which values of N admit a complete set of MUBs is an as yet unsolved problem of quantum information theory that is attracting current interest. While this paper will approach the problem of describing the Pauli MUBs algebraically, an elegant description using category theory may also be of interest. [2]

It is known that Hilbert spaces whose dimension are the power of a prime do possess complete sets of MUBs. The simplest nontrivial Hilbert space has dimension given by the smallest prime power, that is, 2. The states of this Hilbert space are known as "qubits" in quantum information theory. Physically, they can be thought of as the spin states available to a spin-1/2 fermion such as a quark or lepton. The usual basis set for qubits is that defined by spin-1/2 as measured in the z direction. The two states in this basis represent spin up and spin down.

Rather than representing quantum states with state vectors or kets, we will use pure density matrices. The choice of pure density matrices over state vectors amounts to a choice of one interpretation / formalism of quantum mechanics over another and is motivated primarily by ease of calculation as will become clear. To convert a state represented by a pure density matrix to the same state represented by a ket, take any nonzero column, treat it as a vector, and normalize.

In converting from state vectors to pure density matrices, we will write:

$$\rho_{an} = |a_n\rangle\langle a_n|. \tag{3}$$

The usual basis states for qubits are the Pauli algebra kets:

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4}$$

These represent the same states as the (pure) density matrices:

$$\rho_{+z} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1+\sigma_z)/2, \quad \rho_{-z} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = (1-\sigma_z)/2, \tag{5}$$

where σ_z is the z component of the vector of Pauli spin matrices, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and 1 is the 2×2 unit matrix. Since we will be using pure density matrix states rather than state vectors a few changes are made to the definition of a basis set. Since the basis states $|a_n\rangle$ are normalized so that $\langle a_n|a_n\rangle = 1$, the pure density matrix states are "idempotent":

$$\rho_{an}\,\rho_{an} = \rho_{an}. \tag{6}$$

The requirement that basis kets be orthogonal, $\langle a_n|a_m\rangle = 0$ for $n \neq m$, becomes, for pure density matrices, the requirement that the basis elements "annihilate" or multiply to zero:

$$\rho_{an}\rho_{am} = \rho_{am}\rho_{an} = 0. \tag{7}$$

The requirement that the basis set is complete implies that the sum over all the basis states is the unit $N \times N$ matrix:

$$\sum_n \rho_{an} = 1, \tag{8}$$

Finally, transition probabilities between two states are given by the trace:

$$|\langle a_n|b_m\rangle|^2 = \text{ tr } (\rho_{an}\,\rho_{bm}). \tag{9}$$

Similarly, the expectation value of an operator O is given by the trace of the product of the operator with the state.

Since the dimensionality of qubits is 2, a complete set of MUBs will have three basis sets. For this we need two more bases in addition to the usual spin up and spin down basis set. The obvious choice is spin measurements in the x and y directions – this works. The three basis sets each have two elements. As an abbreviation for the basis states, we will sometimes use R, G, and B for the three positive pure qubit density matrices (that is, the ones oriented respectively in the $+x$, $+y$, and $+z$ directions), and $\bar{R}$, $\bar{G}$, and $\bar{G}$ for the states oriented oppositely. Thus a complete set of MUBs for qubits consists of the three basis sets:

$$\begin{array}{c} \{\ (1+\sigma_x)/2,\ \ (1-\sigma_x)/2\ \} = \{R, \bar{R}\}, \\ \{\ (1+\sigma_y)/2,\ \ (1-\sigma_y)/2\ \} = \{G, \bar{G}\}, \\ \{\ (1+\sigma_z)/2,\ \ (1-\sigma_z)/2\ \} = \{B, \bar{B}\}. \end{array} \tag{10}$$

The transition probabilities are all 1/2. The above notation, i.e. $(1+\sigma_x)/2$, does not depend on the choice of representation of the Pauli spin matrices.

The usual technique for analyzing color would be to assign orthogonal states to the three colors. But for the situation we are examining, meson bound states, we are only considering one quark in the presence of the force field created by the other quark. Since we are considering only one quark, it can have only one color at a time. Consequently, there is no problem with assigning the color states to non-orthogonal states. The advantage of making this assignment is that the transition probabilities between states are automatically defined by the Pauli algebra for us. These probabilities are all equal to 1/2, but the situation is not completely trivial due to the effects of Berry-Pancharatnam phase.

In the usual spinor formalism of quantum mechanics, a "complete set of commuting observables" means a set of observables that define a set of eigenfunctions completely, up to complex phase. Pure density matrices can be thought of as operators and a basis set of pure density matrices can be used to define a complete set of commuting observables. For the case of the Pauli algebra this is trivial. If the basis set of pure density matrices is defined by spin in the $\vec{v}$ direction, that is, the basis is $\{(1 \pm \vec{v}\cdot\vec{\sigma})/2\}$, then the corresponding complete set of commuting observables has only one element, spin in the $\vec{v}$ direction: $\vec{v}\cdot\vec{\sigma}$.

For any finite Hilbert space, it is possible to define a single observable that, by itself, is a complete set of commuting observables. Given N pure density matrices ρ_{ak} that form a basis, one chooses N distinct real numbers c_k. The observable $\sum c_k\ \rho_{ak}$ has N distinct eigenstates $|ak\rangle$ with eigenvalues c_k and so is a complete set.

A "primitive idempotent" is an idempotent that cannot be written as the sum of two idempotents. A primitive idempotent finite matrix that is also Hermitian is a pure density matrix. We will refer to the non-Hermitian primitive idempotent matrices as “non-Hermitian pure density matrices”, though we are likely to forget the “pure”.

Any non-Hermitian pure density matrix can be written uniquely as the complex multiple of a product of two distinct Hermitian pure density matrices. The complex number is the inverse of the transition amplitude. That is for $|aj\rangle$ and $|bk\rangle$ two different normalized ket states,

$$|aj\rangle\langle aj|\;\; |bk\rangle\langle bk|/\langle bk|aj\rangle \tag{11}$$

is such an object and it is a non-Hermitian primitive idempotent (which we call a non-Hermitian pure density matrix).

The MUB bound state problem

The color force is usually modeled using the methods of quantum field theory as opposed to the potential methods of quantum mechanics. The advantage of field theory is that it uses creation and annihilation operators and so allows the number of particles to be non constant. The disadvantage of field theory methods is that they typically assume a perturbation expansion. The strength of the strong force makes it difficult to get perturbation methods to work.

For the color force, perturbation theory works best in the limit of very heavy quarks. Consequently, the heavy mesons, the J/ψ (or c-cbar) and Υ (b-bbar) mesons are typically modeled using an approximate potential between them. This method avoids the difficulty of modeling the color force but has the advantage of being an analogy to the Coulomb force in the hydrogen atom (and positronium) and so is easy to calculate with. The success of these calculations have led to the lowest four Υ and lowest two J/ψ resonances being renamed from the usual meson scheme of $\Upsilon(M)$ where M is the approximate mass in MeV, to $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$, $\Upsilon(4S)$, $J/\psi(1S)$, and $\psi(2S)$. These designate the radial excitations of the bound states of the Coulomb force.

This method of approximation works reasonably well because the color potential between quarks is approximately Coulomb at short distances. From [3] a suitable potential function is of the form:

$$V = -\frac{4}{3}\frac{\alpha_s(r)\hbar c}{r} + k\cdot r. \tag{12}$$

The approximation worsens at long distances, hence the failure of the method in modeling the higher Υ and J/ψ resonances, as well as the hundreds of mesons made from lighter quarks. On the other hand, this approximation ignores the color degrees of freedom.

This paper will also approach the color bound state problem from the point of view of a potential, but rather than ignoring the color degrees of freedom we will follow the quantum information approximation and ignore the position degrees of freedom. That is, we will keep track of the color of the quark but not of its position.

Physical situations exist where qubits form a useful approximation of a quantum state; quantum dots, or electrons in $1S$ states of isolated atoms are particularly good examples. In this paper we will be ignoring the $J = L + S$ quantum numbers of hadrons, we are interested primarily in how a state's mass depends on its color degrees of freedom. Consequently, we can expect the individual analyses to apply to all the various angular momentum cases.

The quark can be found in three colors, red, green, and blue which we will abbreviate R, G, and B. A state vector describing a quark will have three components, one for each color:

$$|abc\rangle = \left(\begin{array}{c} a_R \\ b_G \\ c_B \end{array} \right). \tag{13}$$

In order for the color potential V,

$$V = \left(\begin{array}{ccc} v_{RR} & v_{RG} & v_{RB} \\ v_{GR} & v_{GG} & v_{GB} \\ v_{BR} & v_{BG} & v_{BB} \end{array} \right), \tag{14}$$

to give real eigenvalues we will assume that it is Hermitian. The eigenvector equation,

$$V|abc\rangle = E_{abc}|abc\rangle, \tag{15}$$

is easy to solve; there will be three orthogonal solutions that depend on the symmetry of V. Thus the first prediction of this model of mesons is that their resonances will appear as triplets. The six Υ and six J/ψ resonances is compatible with this, each must consist of a pair of triplets.

The color force is an unbroken symmetry and so we will assume that the V_{jk} are cyclically symmetric, that is, along with Hermiticity, V will be of the form:

$$V = \left(\begin{array}{ccc} v & se^{+i\delta} & se^{-i\delta} \\ se^{-i\delta} & v & se^{+i\delta} \\ se^{+i\delta} & se^{-i\delta} & v \end{array} \right), \tag{16}$$

where $v, s,$ and δ are three real parameters. The above matrix has each consecutive row identical to the one above but shifted one to the right. The mathematicians call this a "1-circulant" matrix.

So long as $s \neq 0$, the eigenvectors of a 1-circulant 3×3 matrix do not depend on the parameters and are of the form

$$|g\rangle = \frac{1}{\sqrt{3}} \left(\begin{array}{c} 1 \\ w^{+g} \\ w^{-g}, \end{array} \right) \tag{17}$$

where $w = \exp(2i\pi/3)$ is the cubed root of unity, and where $g = 1, 2, 3$ stands for the generation in analogy with the generation structure of the elementary fermions. The corresponding eigenvalues are

$$\lambda_g = v + 2s\cos(\delta + 2g\pi/3). \tag{18}$$

The parameter v contributes equally to the energy of the three solutions. If the mass of the quark being modeled (i.e. the valence quark) were changed, this value would change, hence the use of the symbol v. The off diagonal parameters cause colors to be mixed by the potential. Color changes to the quark are caused by the gluon sea hence the use of the symbol s. The magnitude of s defines how effective the gluon sea is at mixing colors. In the limit of no color force at all, $s = 0$, the potential is diagonal and eigenvectors other than the above are admitted such as pure red $(1, 0, 0)$. This would represent a free quark. Finally, the angle δ is a Berry-Pancharatnam phase; it is a phase that appears when the gluon sea modifies a quark through all the colors in sequence.

Experimental measurements on the three states will give the real eigenvalues λ_g. Because of the assumption that the matrix V is 1-circulant, these three real eigenvalues imply a particular Hermitian matrix V. This matrix is characterized by the three real values v, s, and δ.

The process of converting the three parameters v, s, and δ to the three eigenvalues λ_g can be reversed. We can take three experimental measurements and assume that their origin is as the eigenvalues of a matrix V. We can then compute v, s, and δ from the three measured values. This paper is largely concerned with applications of this technique to the masses of hadrons. The central claim is that on transforming triplets of hadrons with identical quantum numbers, the angle δ tends to be quantized.

The assumption that the state can be considered only in terms of color and not at all in terms of position and momentum implies that the three excitations $g = 1, 2, 3$ correspond to the same position and momentum wave functions. This is comparable to how the qubit approximation gives the states available to an electron which has spin-1/2; there are two possible spin states but in the qubit approximation their wave functions are otherwise assumed to be identical. Accordingly, in this model, each radial excitation is split into three "color excitations". We will designate these by adding a number to the radial excitation. So instead of assuming that heavy quarkonium consists of $1S$, $2S$, ... $6S$ states, we will designate them as $1S1$, $1S2$, $1S3$, $2S1$, $2S2$, and $2S3$ but not necessarily in that order.

The discrete Fourier transform

To convert between a position space representation and a momentum space representation, one uses a Fourier transform:

$$\psi(p) = \int \exp(ixp)\psi(x)\, d^3x \tag{19}$$

The transform maps a basis set for the Hilbert space to a new basis set which is a complementary observable. That is, the basis set and the transformed basis sets are mutually unbiased. Thus the Fourier transform is related to the theory of mutually unbiased bases.

For a finite Hilbert space, the Fourier transform becomes the discrete Fourier transform. This paper is concerned with triplets of quantum states and so the Hilbert spaces have three dimensions. Let $\{a_1, a_2, a_3\}$ be three complex numbers. We will define the discrete Fourier transform using a matrix F defined as follows:

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^* \\ 1 & w^* & w \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \tag{20}$$

where w is the cubed root of unity, as before. The inverse transform is given by the complex conjugate. The three columns of the discrete Fourier transform matrix F are the same (up to normalization) as the three solutions to the color bound state problem.

The discrete Fourier transform can be applied to the three solutions to the color bound state eigenvector problem given in Eq. (17). The $|1\rangle$ ket transforms to

$$|\tilde{1}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{21}$$

The other two eigenvectors transform similarly into the other canonical basis vectors for the Hilbert space. In other words, the discrete Fourier transform moves a state back and forth between its representation in color space and its representation in generation space.

One can also apply the discrete Fourier transform to the potential V. The potential is an operator and is bilinear, so to transform it one applies the discrete Fourier transform matrix to both sides of it: $\tilde{V} = F^{-1}VF$. Since the columns of F are the eigenvectors of V, the result is that the $\tilde{V}$ is the diagonal matrix of the eigenvalues. Thus the conversion between the eigenvalues λ_g and v, s, and δ amounts to taking a discrete Fourier transform.

The claim of this paper is that the hadrons have a structure similar to the generation structure of the leptons. The leptons appear three times, with identical quantum numbers but three different masses. This paper claims that when one considers all hadrons with a given set of quantum numbers, there will also be a multiple of three of them, and their structure can be detected using the discrete Fourier transform.

Among the leptons, the discrete Fourier transform underlies Koide's mass formula for the charged leptons and its extension to the neutrinos. The mass equations for the hadrons are similar to these equations in that they share the δ parameter. Marni Sheppeard pointed out that these equations are related to the discrete Fourier transform. This author and Sheppeard are preparing a paper on the applications of the discrete Fourier transform to the quark and lepton mixing matrices, the CKM and MNS matrices.

The paper with Sheppeard will show that these matrices are best explained as the sums of 1-circulant and 2-circulant matrices. For the MNS, this is particularly elegant; in the tribimaximal form, the lepton mixing probabilities can be written as a sum of a 1-circulant and a 2-circulant matrix:

$$\begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/3 & 1/6 & 1/2 \\ 1/3 & 1/6 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/6 & 1/3 & 0 \\ 0 & 1/6 & 1/3 \\ 1/3 & 0 & 1/6 \end{pmatrix} + \begin{pmatrix} 1/6 & 1/3 & 0 \\ 1/3 & 0 & 1/6 \\ 0 & 1/6 & 1/3 \end{pmatrix}. \qquad (22)$$

Also see [4] for a 1-circulant + 2-circulant description of the lepton mixing matrix. A unitary form of the MNS matrix can be obtained from the above decomposition. One takes square roots, choosing signs appropriately, and multiplies by complex phases so that the 1-circulant and 2-circulant portions differ in phase by 90 degrees. Then the matrix becomes unitary, for example:

$$\begin{pmatrix} +1/\sqrt{6} & +1/\sqrt{3} & 0 \\ 0 & +1/\sqrt{6} & +1/\sqrt{3} \\ +1/\sqrt{3} & 0 & +1/\sqrt{6} \end{pmatrix} + i \begin{pmatrix} -1/\sqrt{6} & +1/\sqrt{3} & 0 \\ +1/\sqrt{3} & 0 & -1/\sqrt{6} \\ 0 & -1/\sqrt{6} & +1/\sqrt{3} \end{pmatrix}. \qquad (23)$$

The above is of particular elegance in that in addition to being unitary and giving the tribimaximal probabilities, all rows and columns add to the same number, a complex phase, $\exp(i\ \tan^{-1}(3-\sqrt{8})) \approx \exp(i\ 0.169918454727)$. David Lackey points out that this can also be written as

$$\exp(i\ \tan^{-1}(3-\sqrt{8})) = \exp(i\ \cos^{-1}(2\sqrt{2/9})). \qquad (24)$$

Circulant matrices have appeared before with regard to the generation structure of the leptons. [5, 6]

Products of pure density matrices

The calculations in this paper will be of products of pure density matrices, for example:

$$R\bar{B}GR = (1+\sigma_x)/2\ \ (1-\sigma_z)/2\ \ (1+\sigma_y)/2\ \ (1+\sigma_x)/2, \qquad (25)$$

though we will only need products with no bars. To motivate these sorts of products, we will review some interpretations and applications of products like the above.

Our first interpretation is the “measurement algebra” of Julian Schwinger. Schwinger’s insight was that the measurement result of a complete set of observables is a complete characterization of a quantum state. Consequently, one can think of the state as being defined by that complete set of measurements. Products of pure density matrices can then be thought of as the results of measurements on particles. For example, the basis set $\{(1+\sigma_x)/2,\ (1-\sigma_x)/2\}$ is defined by the measurement of spin in the x direction. The pure density matrix $(1+\sigma_x)/2$ corresponds to a measurement result of spin $+1/2$ on the $+x$ axis, while $(1-\sigma_x)/2$ corresponds to a result of spin $+1/2$ on the $-x$ axis.

Spin can be measured by a Stern-Gerlach device. A beam of particles enters one end of the device. These are placed on a beam of particles; they split the beam into several beams, with each outgoing beam having a specific value of spin measured in some direction. Stern-Gerlach devices can be chained together. That is, we can connect the beam input of a Stern-Gerlach device to one of the beam outputs of another. This creates a compound measuring device. Then the product $R\bar{B}GR$ above, corresponds to a measurement by four Stern-Gerlach experiments; to get through this particular path, the particle has to consecutively have spin measurement results of $+x$, $+y$, $-z$, and finally $+x$. These are the results of measurements of spin along the x, y, z, and x axes, respectively.

Not all the particles on the input beam of the first Stern-Gerlach device would follow this path. For example, the y measurement of the particle might result in spin $-y$ instead of spin $+y$. To account for all the possible measurements, we will need one x oriented device, two z oriented devices (one for each of the results from the first x measurement), four y oriented devices, and finally 8 x oriented devices. There would be 16 possible paths for a particle. These correspond to the sixteen possible measurement results: $\bar{R}\bar{G}\bar{B}\bar{R}$, $\bar{R}\bar{G}\bar{B}R$, $\bar{R}\bar{G}B\bar{R}$, ... $RGBR$.

Schwinger’s elegant measurement algebra has an echo in a modern interpretation of quantum mechanics, Consistent Histories. One of the perceived inadequacies of most interpretations of quantum mechanics is that the probabilities appear to result from classical apparatus making quantum measurements. Consistent Histories is an attempt at interpreting quantum mechanics in such a way that there is no need for a classical measurement. [7] Since the whole universe is quantum, this is sometimes called quantum cosmology. The interpretation describes particles as sequences of projection operators (which we can think of as measurements or products of pure density matrices). The many worlds interpretation can be put into the form of consistent histories using category theory. [8]

A “history” is a sequence of projection operators P_k that act at a sequence of increasing times

t_k. A history is "consistent" if the projection operators at each time t_k are complete and orthogonal, that is, if they form a basis for the Hilbert space. For the case of a finite Hilbert space, we can split each projection operator into a finite number of primitive projection operators and so, for a finite Hilbert space, the consistent histories interpretation of quantum mechanics can be written as products of pure density matrices.

Berry-Panchartnam or topological phase

The phase δ that appears in the off diagonal terms of V in Eq. (16) is a result of Berry-Panchartnam or topological phase. These appear when a quantum state is sent through a closed path that begins and ends with the same state. On returning to the initial state, it is possible for the state to pick up a complex phase. In this case, a red state picks up a phase $\exp(i\delta)$ when it is converted to a blue state, the same for blue converted to green and for green converted to red. In going from red to blue to green and then back to red, the state picks up a phase of $\exp(3i\delta)$. This is the topological phase. For a review using the relatively complicated state vector notation, see [9], for an article using techniques similar to those used here, see [10].

Berry-Pancharatnam phase can be observed in simple optical experiments with Polaroid lenses and diffractions slits, [11] but despite its simplicity it was not discovered until Pancharatnam's 1956 paper, [12]. This paper was largely ignored until after Berry independently discovered and extended the theory in 1984. [13] The difficulty arises from the arbitrary complex phases present in spinors. To avoid this, we will use pure density matrices.

Writing this in ket form, consider a ket that begins in the state $|+x\rangle$, then progresses through the states $|+y\rangle$, $|+z\rangle$, and then returns to $|+x\rangle$. The amplitude for a state converting from $|+x\rangle$ to $|+y\rangle$ is $\langle +y|+x\rangle$, and similarly for the other conversions. Thus the overall factor for this sequence is

$$k = (\langle +x|+z\rangle)(\langle +z|+y\rangle)(\langle +y|+x\rangle). \tag{26}$$

The topological phase is the phase of k.

If a ket is multiplied by a complex phase, its bra is multiplied by the complex conjugate. Each of the states $|+x\rangle$, $|+y\rangle$, and $|+z\rangle$ appears as both a bra and a ket in the above and so the arbitrary complex phases associated with the bras and kets cancel in the equation for k. Thus even though k is in general complex, it's value does not depend on the arbitrary complex phases of the spinors.

Pure density matrices are particularly convenient for topological phase calculations because they avoid the arbitrary complex phases of kets. In density matrix formalism, the states are all represented by pure density matrices. An example of a state that is sent through a sequence and returns to the initial state is $RGBR$. We return to the notation of the previous subsection.

Since the product $RGBR$ begins and ends with the same pure density matrix R, the product is a left and right sided eigenstate of the x measurement. Up to multiple by a complex number which we will call k_{RGBR}, there is only one such eigenstate so the product $RGBR$ has to be a complex multiple of R. The transition probabilities between different basis states in the Pauli MUBs are $1/2$ so each transition RG or RB, etc., reduces the amplitude of a beam of particles by $\sqrt{1/2}$. In the product $RGBR$ there are three such transitions, (RG), (GB), and (BR) so $|k_{RGBR}| = \sqrt{1/2^3} = \sqrt{1/8}$. As can be verified by substituting in the Pauli spin matrices, the complex phase of k turns out to be $\pi/4$:

$$RGBR = \frac{1}{\sqrt{8}}\, e^{i\pi/4}\, R. \tag{27}$$

Returning to the Stern-Gerlach experiment interpretation, the Berry-Pancharatnam phase is the phase picked up by particles that are sent through a sequence of four Stern-Gerlach experiments. Since kets carry arbitrary complex phases, topological phase calculations with them can be difficult.

If a product of pure density matrices begins with a state taken from a different basis than the state it ends with, for example $RG\bar{R}\bar{B}$, the topological phase can be computed as a phase relative to the product of just the initial and final states. For example,

$$RG\bar{R}\bar{B} = \frac{1}{2}\, e^{-i\pi/2}\, R\bar{B}. \tag{28}$$

In the above product, the magnitude is $1/2$ instead of $1/\sqrt{8}$ as there have been only two transitions eliminated from the product. The phase $\pi/2$ can be found by substituting in the Pauli spin matrices, but there is also a trick that makes it easier to calculate.

In the Pauli algebra, topological phase α is equal to half the oriented surface area σ of the path taken by the state over the Bloch sphere, that is, the sphere of possible unit vectors for the orientation of spin:

$$\alpha = \sigma/2 \tag{29}$$

The six states form the six corners of an octahedron at positions $R = (+1, 0, 0)$, $\bar{R} = (-1, 0, 0)$, $G = (0, +1, 0)$, ... $\bar{B} = (0, 0, -1)$. Each of the faces of the octahedron corresponds to an area on the superscribed sphere of $(4\pi)/8 = \pi/2$ and so is associated with a complex phase of $\pi/4$. One counts up how many faces are contained on the left side of the path; each such contributes $\pi/4$. It may be simpler instead to deal with the faces on the right side of the path; these each contribute $-\pi/4$. Note that no path ever goes directly from one corner to the very opposite corner because opposite corners correspond to orthogonal basis states, which annihilate and so the product would be zero. For example, $RGB\bar{B}GR = 0$. In practice, working out products this way is considerably faster and less prone to error than writing out the Pauli algebra representations and doing matrix multiplication by hand.

This method of simplifying products of Pauli MUBs works for any product except those that begin and end with orthogonal states, for example, $RGB\bar{R}$. These states cannot be simplified as products of $R\bar{R}$ because $R\bar{R} = 0$. Instead, one could insert an intermediate state that is annihilated by neither R nor $\bar{R}$, for example G. Then one could write any product of density matrices that begins with R and ends with $\bar{R}$ as a complex multiple of $RG\bar{R}$.

The product $RG\bar{R}$, when applied to a spinor $|-x\rangle$, converts it to a spinor $|+x\rangle$. On the other hand, $RG\bar{R}$ annihilates the spinor $+x\rangle$. Thus $RG\bar{R}$ is a raising operator for spin in the x direction. The choice of G amounts to the choice of complex phase present in raising and lowering operators. Writing the raising and lowering operators in this form is more elegant than the usual because the arbitrary choice of complex phase receives a geometric interpretation; it's an implicit characterization of the path when computing Berry-Pancharatnam phase.

One can also simplify products of MUB basis elements by choosing a state not in any of the basis elements (and so not annihilated by any of them), say V. One would then write any product of Pauli MUBs as a complex multiple of the 36 products RVR, $RV\bar{R}$, RVG, $RV\bar{G}$, ...$\bar{B}V\bar{B}$. In the notation of Schwinger's measurement algebra, V is a "vacuum state"; one defines bras and kets with it from the pure density matrices by $|R\rangle = RV$, and $\langle R| = VR$. Then the usual bra and ket operator calculations can be recast as the complex multiples of V.

The choice of V defines a method of converting pure density matrices to spinors. The Pauli spin matrices implicitly use $V = (1 \pm \sigma_z)/2$. That is, to get a Pauli spinor from a Pauli pure density matrix, one takes either column vector of the pure density matrix. The left column is selected by $(1+\sigma_z)/2$ and the right column is selected by $(1-\sigma_z)/2$.

Suppose that a Hilbert space has two dimensions. If the Hilbert space represents spin-1/2, then it will pick up topological phase. On the other hand, if the two dimensions are two scalar degrees of freedom, for example two isospin singlets, then the states will not pick up topological phase. In the derivation of Koide's mass formula for the neutrinos, we will find that the neutrinos are apparently bound states of particles that have topological phase while the charged leptons are bound states of particles that do not. This is similar to the situation in the elementary fermions with regard to isospin. Elementary fermions appear four at a time with two states in an isospin doublet along with two isospin singlets, for example, $(u_L, d_L), (u_R), (d_R)$. The singlet states, u_R and d_R would be treated as unchanged upon isospin rotations and consequently would not pick up a topological phase, while the isospin doublet states would.

Non-Hermitian pure density matrices

Our interpretation of the product $RGBR$ as a sequence of four Stern-Gerlach devices describes the product of pure density matrices as a sequence of measurements. A particle which traverses these devices will be modified. To look at what is going on from the point of view of how the particles are modified, we can use idempotency to duplicate the inner states and rearrange terms as follows:

$$R\;G\;B\;R = R\;(GG)\;(BB)\;R = (RG)\;(GB)\;(BR). \tag{30}$$

The compound objects (RG), (GB), and (BR) are (nonzero) products of pure density matrices and therefore are complex multiples of non-Hermitian density matrices. From the point of view of Stern-Gerlach devices, each consists of two consecutive devices, hooked up so that particles only get through if they enter with one spin state and exit with another. In this form, any conversion of a particle from one state to another occurs inside a compound device. These compound objects can also be thought of as the generalization of raising and lowering operators to the case of a complete set of MUBs.

The three MUB basis states R, G, and B are a 3 of $SU(3)$, while the $\bar{R}$, $\bar{G}$, and $\bar{B}$ form a $\bar{3}$. Therefore the $6 \times 6 = 36$ states formed from products of these six states with themselves such as RR, RG, RB, $\bar{R}$, ..., can be broken down into irreducible representations as

$$\begin{array}{rcl} (3+\bar{3})(3+\bar{3}) & = & (3\times\bar{3}) + (\bar{3}\times 3) + (3\times 3) + (\bar{3}\times\bar{3}), \\ (8+1)+(8+1)+(6+\bar{3})+(\bar{6}+\bar{3}). & & \end{array} \tag{31}$$

In looking at products of terms of the form (RR), (RG), (RB), ... (BB) we are looking at a 3×3 representation of $SU(3)$ which is given by

$$3 \times 3 = 8 + 1. \tag{32}$$

The 8 is the irrep for the gluons, but in considering all nine products we end up with an extra 1. Considering the $8+1$ together expands the symmetry from the 8 of $SU(3)$ represented by traceless 3×3 unitary matrices to all 3×3 unitary matrices. Thus we are putting the 8 of $SU(3)$ inside the 9 of $U(3)$.

The (RG) product can be thought of as converting a G state to a R state and so is analogous to the action of a $R^\dagger G$ gluon on a green quark. As with the usual quantum information approximation, the gluon itself is ignored. In addition, we only end up with six gluons, (RG), (RB), (GR), (GB), (BR), and (RB); two are missing.

The missing gluons are the diagonal values $(R^\dagger R - G^\dagger G)/\sqrt{2}$ and $(R^\dagger R + G^\dagger G - 2B^\dagger B)/\sqrt{6}$. Representing these in pure density matrices we would use $(RR - GG)/\sqrt{2} = (R - G)/\sqrt{2}$ and $(R + G - 2B)/\sqrt{6}$. However, the pure density matrix language already includes R, G, and B to represent states without interactions and so the action of the diagonal gluons can be combined into the pure density matrices themselves.

Our representation adds the pure density matrices (which have trace = 1) back into $SU(3)$. So the actual symmetry in this approximation is $U(3)$ instead of $SU(3)$. Including the diagonal gluons would be incompatible with also including R, G, and B as it would doubly use the same degrees of freedom in the wave function.

In the density matrix for the bound state, we have various activities going on. The particle can be in a red, green, or blue state. These we will represent as (what turns out to be real) multiples of R, G, and B. The particle can also transition between the color states. These will pick up topological phases and so in general will be complex multiples of RG, RB, GR, GB, BR, and BG. This gives a total of nine activities the particle can undergo, each will have its own real or complex number associated.

Label these generally complex numbers α_{jk}, that is, α_{RR}, α_{RG}, ... α_{BB}. Then the nine activities in the bound state are:

$$\begin{array}{ccc} \alpha_{RR}R, & \alpha_{RG}RG, & \alpha_{RB}RB, \\ \alpha_{GR}GR, & \alpha_{GG}G, & \alpha_{GB}GB, \\ \alpha_{BR}BR, & \alpha_{BG}BG, & \alpha_{BB}B. \end{array} \tag{33}$$

In order for the above to be thought of as a pure density matrix, we must have that it is idempotent. To verify idempotency, we must square this and verify that it is unchanged. We must compute:

$$\left|\begin{array}{ccc} \alpha_{RR}R, & \alpha_{RG}RG, & \alpha_{RB}RB, \\ \alpha_{GR}GR, & \alpha_{GG}G, & \alpha_{GB}GB, \\ \alpha_{BR}BR, & \alpha_{BG}BG, & \alpha_{BB}B. \end{array}\right| \left|\begin{array}{ccc} \alpha_{RR}R, & \alpha_{RG}RG, & \alpha_{RB}RB, \\ \alpha_{GR}GR, & \alpha_{GG}G, & \alpha_{GB}GB, \\ \alpha_{BR}BR, & \alpha_{BG}BG, & \alpha_{BB}B. \end{array}\right| \tag{34}$$

The above product means to take all the elements in the group on the left and multiply them by all the elements in the group on the right. This would be $9 \times 9 = 81$ multiplications. However, in our approximation we assumed that all color modification occurred in what are the off diagonal terms in the above. This means that we can only multiply a term on the left with a term on the right if they have compatible colors. For example, we can multiply R on the right by R, RG, or RB, but not by any of the other states. We can multiply GB on the right by B, BR, and BG and nothing else, etc.

The result of this restriction on what can multiply what will be that we will only have 27 multiplications instead of 81. The first few of these are:

$$\begin{array}{lll} \alpha_{RR}R\alpha_{RR}R + \alpha_{RG}RG\alpha_{GR}GR + \alpha_{RB}RB\alpha_{BR}BR & . & . \\ \alpha_{RR}R\alpha_{RG}RG + \alpha_{RG}RG\alpha_{GG}G + \alpha_{RB}RB\alpha_{BG}BG & . & ., \\ . & . & . \end{array} \tag{35}$$

The multiplication we've defined here turns out to correspond to the usual multiplication of matrices of complex numbers, except that the R, G, and B adds a little non-commutivity to the field of complex numbers. Physically, the multiplication can be interpreted as consecutive measurements. In this interpretation, addition corresponds to taking two Stern-Gerlach outputs and hooking them up together so that their output beams combine. The various products correspond to different paths in the path integral formalism, and addition corresponds to the requirement that we sum over all possible paths.

While this computation uses noncommutative multiplication, it is still finite dimensional and so can be represented by a matrix multiplication of finite matrices. In the results section of this paper we will solve this problem (i.e. find the α_{jk}), subject to the additional restrictions that the elements be Hermitian, have trace 1, and be neutral with respect to color.

Density matrices, such as the R, G, and B states are Hermitian, i.e. $R^\dagger = R$. The off diagonal objects, such as $\alpha_{RG}RG$, will transform under Hermitian conjugacy as follows:

$$(\alpha_{RG}\ RG)^\dagger = \alpha^*_{RG}\ G^\dagger R^\dagger = \alpha^*_{RG}\ GR. \tag{36}$$

Therefore, to ensure that the bound state matrix is Hermitian, we require that

$$\alpha^*_{jk} = \alpha_{kj}. \tag{37}$$

In addition, the observed bound states are color singlets; colors must be treated equally. Permuting the three color states R, G, and B, will induce a permutation on the Pauli spin matrices as well since, for example, $\sigma_x = 2R - 1$. But the product of the three Pauli spin matrices is the imaginary unit matrix:

$$\sigma_x\ \sigma_y\ \sigma_z = i, \tag{38}$$

and because they anticommute, swapping two of these will negate the sign of i. Consequently, to ensure that the colors are treated equally, we will require that the α_{jk} be unchanged when the colors are rotated or complex conjugated when two colors are swapped. This means that the α_{jk} can be characterized by a single complex number for the off diagonal elements and a single real number for the diagonal. We will write the complex number as $s\exp(i\delta)$, and the real number as v. Then we have

$$\begin{pmatrix} \alpha_{RR} & \alpha_{RG} & \alpha_{RB} \\ \alpha_{GR} & \alpha_{GG} & \alpha_{GB} \\ \alpha_{BR} & \alpha_{BG} & \alpha_{BB} \end{pmatrix} = \begin{pmatrix} v & se^{+i\delta}/2 & se^{-i\delta}/2 \\ se^{-i\delta}/2 & v & se^{+i\delta}/2 \\ se^{+i\delta}/2 & se^{-i\delta}/2 & v \end{pmatrix} \tag{39}$$

for the general form of the α_{jk} values.

Koide's mass formula

In 1982, Yoshio Koide [14] proposed the following relation between the masses of the electron, muon and tau:

$$3(m_e + m_\mu + m_\tau) = 2(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2. \tag{40}$$

This is a quadratic equation in the square roots of the masses of the charged leptons. Since the masses of the electron and muon are accurately known, it can be thought of as a prediction for the mass of the tau. 1982 was long before the tau mass was known accurately, but the formula turned out to be quite good; the value it gives for the mass of the tau is well within today's experimental estimates. Koide's second paper on the subject, [15], gives a composite model of the quarks and leptons. This paper gives further justification for these ideas in that we will be comparing the excitations of the hadrons with what we presume are the excitations of the composite leptons.

While there are three massive neutrinos, only two measurements provide accurate information on their masses. The measurements are neutrino oscillations and give the absolute value of the difference of the squares of the masses, that is, $|m_{\nu 1}^2 - m_{\nu_2}^2|$ and $|m_{\nu 2}^2 - m_{\nu_3}^2|$. It was natural to try to apply Koide's mass formula to the problem of determining the three neutrino masses from these two measurements as it applies a third restriction. Unfortunately, these three equations have no solution; this fact ended up in the literature in several papers.

In a 2005 post on Physics Forums [16], this author rewrote Koide's equation in eigenvalue form as follows. Let v, s, and δ be three real constants. Define the following matrix:

$$\sqrt{M} = \left(\begin{array}{ccc} v & se^{+i\delta}/2 & se^{-i\delta}/2 \\ se^{-i\delta}/2 & v & se^{+i\delta}/2 \\ se^{+i\delta}/2 & se^{-i\delta}/2 & v \end{array} \right). \tag{41}$$

Note that this matrix is of the same form as the general color bound state coefficients given in Eq. (39). This is a "1-circulant" matrix, that is, each row is the same as the one above rotated by one position to the right. Such matrices have a simple set of eigenvectors:

$$\left(\begin{array}{c} 1 \\ \exp(+2ig\pi/3) \\ \exp(-2ig\pi/3) \end{array} \right) \tag{42}$$

where $g = 1, 2, 3$ stands for "generation". Then the corresponding eigenvalues:

$$\lambda_g = \sqrt{m_g} = v + s\cos(\delta + 2n\pi/3). \tag{43}$$

are the square roots of three masses m_g that satisfy the equations:

$$\begin{array}{rcl} \Sigma m_g & = & 3(v^2 + s^2/2) = 3v^2 + 1.5s^2, \\ (\Sigma\sqrt{m_g})^2 & = & (3v)^2 = 9v^2. \end{array} \tag{44}$$

If $s = \sqrt{2}v$, then the above two will be in the ratio $6v^2$ to $9v^2$ or 2 to 3, and the m_g will satisfy Koide's equation.

Thus one obtains a mass matrix whose eigenvalues satisfy Koide's equation by squaring Eq. (41) with s replaced by $v\sqrt{2}$, that is, a matrix of form:

$$M = \left(\begin{array}{ccc} 1 & e^{+i\delta}/\sqrt{2} & e^{-i\delta}/\sqrt{2} \\ e^{-i\delta}/\sqrt{2} & v & e^{+i\delta}/\sqrt{2} \\ e^{+i\delta}/\sqrt{2} & e^{-i\delta}/\sqrt{2} & v \end{array} \right)^2, \tag{45}$$

with eigenvalues

$$\sqrt{m_g} = 1 + \sqrt{2}\cos(\delta + 2g\pi/3) \tag{46}$$

This gives the three masses in terms of the angle δ and an overall scale factor which we've left off.

On entering the values for the masses of the electron, muon, and tau, one finds that $\delta = 2/9$, to six decimal places. This may be related to the phase factor in the simple form of the MNS matrix, Eq. (24). This further coincidence lets the charged lepton masses, m_{en}, be very closely approximated by the compact formula:

$$\sqrt{m_{eg}} = \mu_e\ (\sqrt{2} + \cos(2/9 + 2g\pi/3)). \tag{47}$$

where g gives the generation number $g = 1, 2, 3$. The constant μ_e has units of square root of mass. It is chosen to give an exact fit to the latest PDG estimate of the electron mass. In MeV, its value is given by

$$\mu_e = 25.054309435 = \sqrt{627.718421264 \text{ MeV}}. \tag{48}$$

The fit to the charged leptons is as follows:

State	lepton	formula	MeV
$1e1$	e	0.510998918	0.510998918(44)
$1e2$	μ	105.6594077	105.6583692(94)
$1e3$	τ	1776.984857	1776.99(29)

. (49)

The muon and tau numbers are then approximated fairly well. An alternative would be to choose a fit that would slightly improve the muon fit at the expense of the electron. [17]

Synopsis of results

The results section of this paper contains eight subsections. In the first, we solve the color bound state problem for the mutually unbiased bases of the Pauli algebra. The solution gives a hint on how to write the mass formula for the neutrinos which is the second subsection. The neutrino mass equations has been referenced in the literature but the calculation has not previously been published.

The next five sections cover applications of the equations to the mesons, beginning with the $I = 0$ mesons, and ending with the $I = 1/2$ and $I = 1$ cases. The final section covers the baryons.

Results

The first subsection solves the bound state problem for the Pauli MUBs and from this defines a neutrino version of Koide's mass formula. The two mass equations, neutrino and charged lepton, are proposed for hadron excitations by keeping δ constant and letting s and v vary.

The sharpest measured hadron excitation structures are those of the Υ and J/ψ mesons. These have $I = 0$ and $P = C = -1$. Mesons with these quantum numbers are discussed in the second subsection. The third subsection continues with the $I = 0, P = C = +1$ mesons, while the fourth subsection covers $I = 0, P \neq C$. The fifth subsection covers $I = 1/2$ and the sixth $I = 1$. Baryons are covered in the seventh subsection.

Solution to MUB bound state problem

In this subsection we solve the noncommutative idempotency equation $[\alpha]^2 = [\alpha]$ where $[\alpha]$ is the 3×3 matrix of complex multiples of pure density matrices defined in Eq. (33):

$$\begin{pmatrix} \alpha_{RR}R & \alpha_{RG}RG & \alpha_{RB}RB \\ \alpha_{GR}GR & \alpha_{GG}G & \alpha_{GB}GB \\ \alpha_{BR}BR & \alpha_{BG}BG & \alpha_{BB}B \end{pmatrix}. \tag{50}$$

To solve this problem, we will look for a transformation that eliminates the noncommutative parts of the problem. That is, we seek a transformation that converts the multiplication to standard complex matrix multiplication.

Letting $[\alpha]_{RGB}$, $[\beta]_{RGB}$, and $[\gamma]_{RGB}$ represent three collections of nine complex numbers multiplying pure density matrices R, RG, RB, etc., while $[\alpha']$, $[\beta']$, and $[\gamma']$ represent three 3×3 complex matrices, we seek a transformation on the complex coefficients $\alpha \to \alpha'$ that converts the noncommutative matrix multiplication to standard matrix multiplication of the complex coefficients. That is we seek a transformation such that

$$[\alpha]_{RGB}[\beta]_{RGB} = [\gamma]_{RGB} \text{ if and only if } [\alpha'][\beta'] = [\gamma']. \tag{51}$$

With such a transformation, we convert the problem of finding the primitive idempotents of the matrices of noncommutative field elements to a problem of finding the primitive idempotents of complex 3×3 matrices, a problem trivially solved.

To preserve matrix multiplication means that the transformation will have to preserve the multiplication of matrices with only one nonzero element. There are 9 such matrices and 81 such products but only 27 of them are nonzero. These 27 equations must be satisfied individually. The nine that begin with R are:

$$\left(\begin{array}{rcl} \alpha_{RR}R\beta_{RR}R = \gamma_{RR}R & \equiv & \alpha'_{RR}\beta'_{RR} = \gamma'_{RR}, \\ \alpha_{RR}R\beta_{RG}RG = \gamma_{RG}RG & \equiv & \alpha'_{RR}\beta'_{RG} = \gamma'_{RG}, \\ \alpha_{RR}R\beta_{RB}RB = \gamma_{RB}RB & \equiv & \alpha'_{RR}\beta'_{RB} = \gamma'_{RB}, \\ \alpha_{RG}RG\beta_{GR}GR = \gamma_{RR}R & \equiv & \alpha'_{RG}\beta'_{GR} = \gamma'_{RR}, \\ \alpha_{RG}RG\beta_{GG}G = \gamma_{RG}RG & \equiv & \alpha'_{RG}\beta'_{GG} = \gamma'_{RG}, \\ \alpha_{RG}RG\beta_{GB}GB = \gamma_{RB}RB & \equiv & \alpha'_{RG}\beta'_{GB} = \gamma'_{RB}, \\ \alpha_{RB}RB\beta_{BR}BR = \gamma_{RR}R & \equiv & \alpha'_{RB}\beta'_{BR} = \gamma'_{RR}, \\ \alpha_{RB}RB\beta_{BG}BG = \gamma_{RG}RG & \equiv & \alpha'_{RB}\beta'_{BG} = \gamma'_{RG}, \\ \alpha_{RB}RB\beta_{BB}B = \gamma_{RB}RB & \equiv & \alpha'_{RB}\beta'_{BB} = \gamma'_{RB}, \end{array}\right) \tag{52}$$

and the others can be obtained by cyclic permutations of the letters. We will look for a transformation of the form:

$$\alpha'_{jk} = a_{jk}\alpha_{jk}, \tag{53}$$

where a_{jk} are nine complex constants. If the transformation works for $[\alpha]$ and $[\beta]$ matrices with single one coefficients and eight zero coefficients, then, by linearity, it will work for all matrices. Accordingly we will set $\alpha_{jk} = \beta_{jk} = 1$. This let's us simplify the above requirements to

$$\left(\begin{array}{rcl} (R)(R) = \gamma_{RR}R & \equiv & a_{RR}a_{RR} = a_{RR}\gamma_{RR}, \\ (R)(RG) = \gamma_{RG}RG & \equiv & a_{RR}a_{RG} = a_{RG}\gamma_{RG}, \\ (R)(RB) = \gamma_{RB}RB & \equiv & a_{RR}a_{RB} = a_{RB}\gamma_{RB}, \\ (RG)(GR) = \gamma_{RR}R & \equiv & a_{RG}a_{GR} = a_{RR}\gamma_{RR}, \\ (RG)(G) = \gamma_{RG}RG & \equiv & a_{RG}a_{GG} = a_{RG}\gamma_{RG}, \\ (RG)(GB) = \gamma_{RB}RB & \equiv & a_{RG}a_{GB} = a_{RB}\gamma_{RB}, \\ (RB)(BR) = \gamma_{RR}R & \equiv & a_{RB}a_{BR} = a_{RR}\gamma_{RR}, \\ (RB)(BG) = \gamma_{RG}RG & \equiv & a_{RB}a_{BG} = a_{RG}\gamma_{RG}, \\ (RB)(B) = \gamma_{RB}RB & \equiv & a_{RB}a_{BB} = a_{RB}\gamma_{RB}, \end{array}\right) \tag{54}$$

where it is understood that only one of the γ_{jk} is going to be nonzero at a time, and there are another 18 similar equations obtained by cyclic permutation of R, G, and B. To simplify this further, we need to evaluate the γ_{jk}. To do this, we will compute the following nine products:

$$\begin{array}{ccc} (R)(R) = R, & (RG)(GR) = RGR, & (RB)(BR) = RBR, \\ (R)(RG) = RG, & (RG)(G) = RG, & (RB)(BG) = RBG, \\ (R)(RB) = RB, & (RG)(GB) = RGB, & (RB)(B) = RB, \end{array} \tag{55}$$

which have been reduced using the idempotency equation that pure density matrices or projection operators satisfy.

The five equations $(R)(R) = R$, $(R)(RG) = RG$, $(R)(RB) = RB$, $(RG)(G) = RG$, and $(RB)(B) = RB$ are already reduced and so their respective γ_{jk} are 1 at least on the row where these products apply. This reduces five of the nine equations Eq. (54) to:

$$\left(\begin{array}{ccc} a_{RR}a_{RR} & = & a_{RR}, \\ a_{RR}a_{RG} & = & a_{RG}, \\ a_{RR}a_{RB} & = & a_{RB}, \\ a_{RG}a_{GG} & = & a_{RG}, \\ a_{RB}a_{BB} & = & a_{RB}. \end{array}\right) \tag{56}$$

We immediately find that $a_{RR} = a_{GG} = a_{BB} = 1$ and these five equations are then satisfied no matter what other choices for a_{jk} are made.

It is possible to compute the four remaining γ values by substituting the Pauli spin matrices for R, G, and B using the definition of the complete set of MUBs of the Pauli algebra given in Eq. (10), but it is easier to count the number of transitions to figure out the complex magnitudes, and to use Berry-Pancharatnam phase theory to compute the complex phases. The products RGR and RBR have two transitions which are reduced to zero so their complex magnitudes are 1/2. The products RGB and RBG also have two transitions but they are reduced to products that still have one transition, that is, RB and RG. Therefore the complex magnitudes will be $1/\sqrt{2}$.

Using the method of counting the number of octahedral faces given in Eq. (29), the complex phases for RG and RBR are zero, while the phases for RGB and RBG are $+\pi/4$ and $-\pi/4$:

$$\begin{array}{rcr} (RG)(GR) & = & (1/2)R, \\ (RB)(BR) & = & (1/2)R, \\ (RG)(GB) & = & (e^{+i\pi/4}/\sqrt{2})RB, \\ (RB)(BG) & = & (e^{-i\pi/4}/\sqrt{2})RG. \end{array} \tag{57}$$

These give γ_{jk} values for Eq. (54) that result in

$$\left(\begin{array}{ccl} a_{RG}a_{GR} & = & (1/2), \\ a_{RB}a_{BR} & = & (1/2), \\ a_{RG}a_{GB} & = & (e^{+i\pi/4}/\sqrt{2})a_{RB}, \\ a_{RB}a_{BG} & = & (e^{-i\pi/4}/\sqrt{2})a_{RG}. \end{array}\right) \tag{58}$$

Ignoring the complex phases, the above equations are satisfied if we put all the remaining a_{jk} equal to $1/\sqrt{2}$. To get the phases correct, assume that the a_{jk} are Hermitian and put $a_{RG} = a_{GB} = a_{BR} = \exp(+i\kappa)/\sqrt{2}$ and the complex conjugate for a_{GR}, a_{BG}, and a_{RB}.

Computing phases we find that $\kappa = \pi/12 + 2n\pi/3$.

$$\begin{pmatrix} a_{RR} & a_{RG} & a_{RB} \\ a_{GR} & a_{GG} & a_{GB} \\ a_{BR} & a_{BG} & a_{BB} \end{pmatrix} = \begin{pmatrix} 1 & \exp(+i\pi/12)/\sqrt{2} & \exp(-i\pi/12)/\sqrt{2} \\ \exp(-i\pi/12)/\sqrt{2} & 1 & \exp(+i\pi/12)/\sqrt{2} \\ \exp(+i\pi/12)/\sqrt{2} & \exp(-i\pi/12)/\sqrt{2} & 1 \end{pmatrix}. \quad (59)$$

The above matrix is similar to Eq. (45), the matrix whose eigenvalues gives the square roots of masses satisfying the Koide mass equation. The difference between the above and the mass matrix for the charged leptons is the overall scale factor μ_e, and nature's choice of $\delta = 2/9$ instead of, in the above, $\delta = \pi/12$. Our answer to this puzzle requires an examination of the neutrino masses.

Neutrino mass equation

Neutrino masses can be measured by examining oscillation experiments. Neutrino oscillation results are measurements of the difference in the squares of the masses, that is, $|\Delta m^2_{ij} = |m_i^2 - m_j^2|$. Latest data are

$$\begin{array}{rcl} |m^2_{\nu 2} - m^2_{\nu 1}| & = & 8.0(0.3) \times 10^{-5} \text{ eV}^2, \\ |m^2_{\nu 3} - m^2_{\nu 2}| & = & 2.6(0.4) \times 10^{-3} \text{ eV}^2. \end{array} \quad (60)$$

The ratio of these two measurements is a useful dimensionless number:

$$r = |m^2_{\nu 3} - m^2_{\nu 2}|/|m^2_{\nu 2} - m^2_{\nu 1}| = 32.5(6.5). \quad (61)$$

This ratio is incompatible with a Koide equation of the original form

$$3(m_{\nu 1} + m_{\nu 2} + m_{\nu 3}) = 2(\sqrt{m_{\nu 1}} + \sqrt{m_{\nu 2}} + \sqrt{m_{\nu 3}})^2. \quad (62)$$

The incompatibility can be shown by examining the corresponding ratios of the difference in squares of masses from the eigenvalue version of Koide's equation given in Eq. (46). For a paper showing the incompatibility without the insight of the eigenvalue relation see [18].

Letting the angle δ range from 0 to $2\pi/3$ will give all possible solutions to Koide's equation in eigenvalue form but not all possible ways of assigning the three masses to $m_{\nu 1}$, $m_{\nu 2}$, and $m_{\nu 3}$. Letting δ range over the whole circle arranges for all the permutations to be explored. Thus we solve the equation

$$r(\delta_\nu) = 32.5(6.5) = \frac{|(1+\sqrt{2}\cos(\delta_\nu))^2 - (1+\sqrt{2}\cos(\delta_\nu + 4\pi/3))^2|}{|(1+\sqrt{2}\cos(\delta_\nu + 4\pi/3))^2 - (1+\sqrt{2}\cos(\delta_\nu + 2\pi/3))^2|}. \quad (63)$$

We find a value for δ_ν

$$\delta_\nu = 0.4795^{+.304}_{-0.240} \quad (64)$$

but with this value, the square root of the smallest mass has to be negative. This is natural in the eigenvalue version of Koide's equation but is incompatible with the original version.

With the value $\delta_\nu = 0.4795$, the predicted neutrino masses are approximately

$$\begin{array}{rcl} m_{\nu 1} & \approx & 3.8 \times 10^{-4}, \\ m_{\nu 2} & \approx & 9.0 \times 10^{-3}, \\ m_{\nu 3} & \approx & 5.2 \times 10^{-2}, \end{array} \tag{65}$$

but for this paper, understanding the angle δ is more important.

The central question in this analysis of the neutrino masses is "why should the neutrinos have a different angle δ than the charged leptons?" The calculation from the previous section showed that the value of δ due to Berry-Pancharatnam phase picked up by a doublet state moving through three Pauli MUB basis elements was $\pi/12$ while the angle picked up by a singlet state is 0. The difference between these is approximately the difference between δ_e and δ_ν. Thus the difference in δ angle between the charged and neutral leptons could be due to their being bound states of different subparticles, with the electron subparticles one being singlets under the Berry-Pancharatnam phase while the neutrino subparticles are doublets.

With this assumption, we propose that neutrino angle is exactly $2/9 + \pi/12$:

$$\delta_\nu = 2/9 + \pi/12. \tag{66}$$

This gives a theoretical prediction for the ratio of the squared mass differences:

$$|m_{\nu 3}^2 - m_{\nu 2}^2| / |m_{\nu 2}^2 - m_{\nu 1}^2| = 31.4266, \tag{67}$$

a little smaller than the experimental measurement of 32.5(6.5) but well within the error estimates. Most of the experimental error is in the Δm_{32}^2 number; adjusting the estimates accordingly, we would have $\Delta m_{21}^2 = 8.05(0.3) \times 10^{-5}$ eV^2, and $\Delta m_{32}^2 = 2.53(0.3) \times 10^{-3}$ eV^2. These arguments date to 2005 but appeared only in an unpublished note by this author. [19]

In making the leap into qubit calculations we have avoided having to make a guess for the coupling constants of the theory. Instead, we've compared experimental results with sums over all possible paths. Since the Berry-Pancharatnam phase does not depend on the choice of path, it will contribute identically to all paths that pass through a given sequence of colors. Other QFT contributions to the amplitudes will not act this way.

Suppose we assume that the neutrino and charged lepton mass equations arise from subparticles or preons having an interaction with a color-like force "precolor" that is carried by a massless gluon-like boson. The color mixing potential V amounts to very large changes in color but not all color changes will be this hard. The amplitude also must include contributions from small changes to the color. These all carry the same net Berry-Pancharatnam phase (presumably 0 or $\pi/4$ for charged and neutral leptons respectively), because that phase depends only on the path, but there are other contributions to the phase of an amplitude.

Suppose that precolor is changed only by soft gluons. Given an amplitude $i\alpha$ for each soft gluon emission, the amplitude for emitting n of them is $(i\alpha)^n$. There are $n!$ ways of choosing which boson is associated with which boson propagator but they are all equivalent. So in summing over diagrams with all possible numbers of bosons one must divide the nth contribution by $n!$. This gives a total contribution

$$\sum_{n=0}^{\infty}(i\alpha)^n/n! = \exp(i\alpha). \tag{68}$$

If we assume that this is the source of the common factor $\exp(2i/9)$ in the neutrino and charged lepton mass equations, then we have that the amplitude for each soft gluon emission is $2i/9$.

In a theory involving preons, the natural source of quark-lepton universality is for the quarks and leptons to be composed of the same preons. Quarks have electric charges of $\pm1/3$ and $\pm2/3$ while the leptons have charges ±1 or 0. The ratios of these numbers make it natural to imagine that the quarks and leptons have three internal preons, with each preon having charge $\pm1/3$ or 0. Since both the charged and neutral preons have the pregluon interaction, so will the quarks. And the argument for the amplitude factor $\exp(2i/9)$ will also apply not only to quarks, but also to complex states involving them. Thus we can expect to see this angle in the hadron mass spectra.

The preons making up the charged leptons we presume to be Berry-Pancharatnam phase singlets, while those making up the neutrinos are doublets. Since the quarks have charges in between, they must then be made of mixtures of preons, some carrying the Berry-Pancharatnam phase and some not.

The quantum states we are considering here are bound states; they are stationary states and have no time dependence. For such states, Berry-Pancharatnam phase arises due to paths on the surface of a sphere (such as the Bloch sphere for pure density matrices in the Pauli algebra). For stationary states this phase must be quantized as the paths begin and end at the same state, whose phase does not depend on time. The whole surface of the sphere has an area of 4π. In order for the phase calculation to give the same result when counting the area to the left or right of the path, the whose surface area must contribute no net phase. That is, the whole surface must contribute a multiple of 2π so

$$\theta_o = 2n\pi \tag{69}$$

where θ_o is the phase contribution from the whole surface and n is an integer.

If $n = 1$, which is the neutrino case, then the Berry-Pancharatnam phase for one octant is $(2\pi)/8 = \pi/4$, and the third of such an angle, i.e. $\pi/12$, appears in the mass equations. More generally, we could end up with a contribution $n\pi/12$. Along with the 2/9, this gives a general mass formula of

$$\sqrt{m_g} = v + s\cos(2/9 + 2g\pi/3 + n\pi/12). \tag{70}$$

The above formula fits the charged leptons when $n = 0$, and the neutrinos for $n = 1$. We will use only these values of n, and will call the respective equations, "electron-like":

$$\sqrt{m_g} = v + s\cos(2/9 + 2g\pi/3), \tag{71}$$

and "neutrino-like":

$$\sqrt{m_g} = v + s\cos(2/9 + 2g\pi/3 + \pi/12). \tag{72}$$

Regarding the possibility of other values of n, it is perhaps of interest to note that both spin-1/2 electrons and spin-1 photons use $n = 1$, while spin-0 particles must use $n = 0$ for the topological phase picked up changes to their polarization vectors. However, changes to the direction in which the light is traveling give topological phase with $n = 2$. [9]

Mesons with $I = 0$, and $P = C = -1$

Throughout these sections, data is from the 2008 Particle Data Group's compilation of mesons and baryons. [20]

We will begin with the isospin scalar mesons with $P = C = -1$. This includes heavy quarkonium with quantum numbers $I^G(J^{PC}) = 0^-(1^{--})$. There are 12 states, six b-bbar and six c-cbar, plus more states for s and u/d quarks. The twelve heavy quarkonium states can be divided into four triplets with the following equations:

$$\begin{aligned}
\sqrt{m_{\psi 1g}} &= \mu_e(2.44247 - 0.25002\cos(2/9 + 2g\pi/3 + \pi/12)),\\
\sqrt{m_{\psi 2g}} &= \mu_e(2.51049 - 0.08943\cos(2/9 + 2g\pi/3)),\\
\sqrt{m_{\Upsilon 1g}} &= \mu_e(3.99332 - 0.12667\cos(2/9 + 2g\pi/3 + \pi/12)),\\
\sqrt{m_{\Upsilon 2g}} &= \mu_e(4.13723 - 0.07754\cos(2/9 + 2g\pi/3)),
\end{aligned} \tag{73}$$

where $\mu_e = 25.054309435$ is the same scale factor used in the electron mass formula, Eq. (47) with units of square root MeV. Note the presence of apparent rational numbers in the s and v coefficients (which have been chosen to minimize the least squares error in the computed masses).

The fits to the measured masses are good

State	Meson	m_g	exp. meas.
$\Upsilon(1S1)$	$\Upsilon(1S)$	9456	9460.30(.26)
$\Upsilon(1S2)$	$\Upsilon(2S)$	10035	10023.26(.31)
$\Upsilon(1S3)$	$\Upsilon(4S)$	10554	10579.4(1.2)
$\Upsilon(2S1)$	$\Upsilon(3S)$	10355.2	10355.2(.5)
$\Upsilon(2S2)$	$\Upsilon(10860)$	10864.4	10865(8.0)
$\Upsilon(2S3)$	$\Upsilon(11020)$	11019.5	11019(8.0)
$\psi(1S1)$	$J/\psi(1S)$	3096.916	3096.916(.011)
$\psi(1S2)$	$\psi(3770)$	3775.154	3775.2(1.7)
$\psi(1S3)$	$\psi(4415)$	4421.063	4421.0(4.0)
$\psi(2S1)$	$\psi(2S)$	3686.083	3686.093(.034)
$\psi(2S2)$	$\psi(4040)$	4040.356	4039.0(1.0)
$\psi(2S3)$	$\psi(4160)$	4149.827	4153.0(3.0)

. (74)

The worst fit is in the lowest lying Koide triplet for the b-bbar, the $\Upsilon(1S)$, $\Upsilon(2S)$, and $\Upsilon(4S)$. Not surprisingly, these are states that are best modeled in the usual manner. The method used here complements the older method. The other fits are excellent, especially compared with the usual expectations of mass fits in the mesons.

The quantum states of lighter quarks are more dominated by the color force. These mesons are more difficult to model the usual way but in the previous section we found that among heavy quarkonium, the Koide equations worked best for the color dominated states. In this section we apply the technique to the rest of the mesons. We will organize the effort by subsection based on the parity reversal and charge conjugation quantum numbers.

The probability of creating quarkonium in an experiment depends strongly on the mass of the quarks. For this reason, the t-tbar is thought to be unlikely to be seen. [21] This leaves quarkonium made from the d, u, and s quarks. The s-sbar is observed at the ϕ. The masses of the u and d are quite similar and they are treated together as an isospin singlet, the ω.

There are seven ω states but two of them, the $\omega(2290)$ and the $\omega(2330)$ are close in mass and we will suppose that they are the same state. This gives us six states, just as with heavy quarkonium, and they fit into a similar pattern. The two equations are:

$$\begin{aligned} \sqrt{m_{\omega 1g}} &= \mu_e(1.45834 - 0.38599\cos(2/9 + 2g\pi/3 + \pi/12)), \\ \sqrt{m_{\omega 2g}} &= \mu_e(1.80738 - 0.18818\cos(2/9 + 2g\pi/3)). \end{aligned} \quad (75)$$

The fits resulting from these parameters are:

State	Meson	m_g	exp. meas.
$\omega(1S1)$	$\omega(782)$	782.75	782.65(.12)
$\omega(1S2)$	$\omega(1420)$	1363	1425(25)
$\omega(1S3)$	$\omega(1960)$	1999	1960(25)
$\omega(2S1)$	$\omega(1650)$	1655	1670(30)
$\omega(2S2)$	$\omega(2205)$	2179	2205(30)
$\omega(2S3)$	$\omega(2330)$	2350	2330(30)

. (76)

The $\omega(1420)$ value is a little low but the state is fairly wide at 180 to 250 MeV and estimates of its mass are widespread. The $\omega(1960)$ has been seen in only a 2002 experiment.

The PDG names only three s-sbar quarkonium ϕ states, the $\phi(1020)$, $\phi(1680)$, and $\phi(2170)$. As with the ω, J/ψ, and Υ states, these have quantum numbers $I^G(J^{PC}) = 0^-(1^{--})$. In addition to these states, the PDG lists several X meson states with compatible quantum numbers. The $X(1575)$ has quantum numbers $?^?(1^{--})$. However it is quite wide, around 818 MeV and its mass is not well measured and was seen in only one 2006 paper so we will ignore it.

A 2002 paper mentions a sharp meson called the $X(1750)$ in the PDG, also with $?^?(1^{--})$. It is seen as $\gamma p \to K^+K^-p$. The data for the $X(1750)$ are also referenced in a section of the PDG data for the $\phi(1680)$. And the $K^\pm$ are strange mesons so we will treat it as a ϕ.

The state now listed as the $\omega(2290)$ was, in the 2006 PDG, listed as the $X(2290)$. In our discussion of the ω, we combined this state with the $\omega(2330)$. The $\omega(2290)$ was seen in a partial wave analysis of $p\bar{p} \to \bar{\Lambda}\Lambda$. In terms of quarks, this is $uud\;\bar{u}\bar{u}\bar{d} \to uds\;\bar{u}\bar{d}\bar{s}$, so it is just as natural to assign it to ϕ as ω. To avoid confusion, we will use the 2006 nomenclature and call it the $X(2290)$.

Looking among these five states, the $\phi(1750)$, $\phi(2170)$, and $X(2290)$ form a electron-like Koide triplet. This suggests that the other two, the $\phi(1020)$ and $X(1660)$, might form the bottom two legs of a neutrino-like Koide triplet. The third state would be around 2310 MeV. Thus we can write the ϕ in the same form as the other quarkonium states if we assume that the $X(2290)$ is a double resonance. The equations are:

$$\begin{aligned}\sqrt{m_{\phi 1g}} &= \mu_e(1.60414 - 0.37253\cos(2/9 + 2g\pi/3 + \pi/12)),\\ \sqrt{m_{\phi 1g}} &= \mu_e(1.81437 - 0.14640\cos(2/9 + 2g\pi/3)),\end{aligned} \tag{77}$$

and the fits are

State	Meson	m_g	exp. meas.
$\phi(1S1)$	$\phi(1020)$	1019.47	1019.455(20)
$\phi(1S2)$	$\phi(1680)$	1645	1680(20)
$\phi(1S3)$	$X(2290)$	2312	2290(20)
$\phi(2S1)$	$X(1750)$	1754	1753.5(3.8)
$\phi(2S2)$	$\phi(2170)$	2167	2175(15)
$\phi(2S3)$	$X(2290)$	2299	2290(20)

. (78)

Looking at the trend in the s and v values for the other q-qbar mesons, the first case appears most likely. This gives the following table of Koide parameters for the q-qbar mesons:

q		v_1	s_1	v_2	s_2
u, d	ω	1.45834	−0.38599	1.80738	−0.18818
s	ϕ	1.60414	−0.37253	1.81437	−0.14640
c	J/ψ	2.44247	−0.25002	2.51049	−0.08943
b	Υ	3.99332	−0.12667	4.13723	−0.07754
t					

, (79)

where blanks indicate no data.

The remaining b-bbar and c-cbar mesons do not have enough with the same quantum numbers to see if a Koide equation can be applied to them. This does not stop us from making a few speculations.

The latest unknown c-cbar states are the $X(3872)$, $X(3940)$, $X(3945)$, $X(4260)$, and $X(4360)$. The last two states are said to have $I^G(J^{PC}) = ?^?(1^{--})$. They are both beautifully sharp resonances and are close together in mass so we can suspect them of being two elements of a Koide triplet. This raises the question, where is the 3rd element of the triplet?

There are several c-cbar series that have two of the three elements of a Koide triplet present but these have quantum numbers incompatible with $?^?(1^{--})$. Looking at the other unknown states, we find a natural Koide triplet with the $X(3945)$ or $X(3940)$. The equation is:

$$\sqrt{m_{X1g}} \quad = \quad \mu_e(2.58271 - 0.07832\cos(2/9 + 2g\pi/3)). \tag{80}$$

and the fit is quite good:

State	Meson	m_g	exp. meas.
$X(1)$	$X(3945)$	3943.1	3943(12)/3943(24)
$X(2)$	$X(4260)$	4262.9	4263(9)
$X(3)$	$X(4360)$	4361.2	4361(18)

. (81)

Therefore we predict that the $X(3940)/X(3945)$ will share the same quantum numbers as the $X(4260)$ and $X(4360)$.

The ω_2 also appears in only two states but the ω_3 comes in four: $\omega_3(1670)$, $\omega_3(1945)$, $\omega_3(2255)$, and $\omega_3(2285)$. The masses of the 3rd and 4th are quite close to each other. A fit suggests these two are actually one state with a mass in between. The formula is:

$$\sqrt{m_{\omega_3 1g}} \quad = \quad \mu_e(1.76482 + 0.15967\cos(2/9 + 2g\pi/3)), \tag{82}$$

which gives fits:

State	Meson	m_g	exp. meas.
$\omega_3(1)$	$\omega_3(1670)$	1667	1667(4)
$\omega_3(2)$	$\omega_3(1945)$	1941	1945(20+)
$\omega_3(3)$	$\omega_3(2285)$	2281	2278(28+)

(83)

This concludes the $I = 0, P = C = -1$ mesons.

Mesons with $I = 0$, and $P = C = +1$

The next mesons we will take up are the f_J, χ_{cJ}, and χ_{bJ} with quantum numbers $I^G(J^{PC}) = 0^+(J^{++})$. The f_J are especially prolific with 35 states including 10 f_0s and 14 f_2s.

The Particle Data Group lists the mass of the $f_0(600)/\sigma$ as 400-1200 MeV. Its width is listed as 600-1000 MeV. With such a wide width, our method of approximating a meson as a bound state is problematic. And the wide estimate of its mass renders pointless a mass formula that includes it. A similar problem applies to the $f_0(1370)$. We will avoid attempting to fit these to a Koide formula. Minkowski and Ochs [22] suggest that these two resonances are evidence of a glueball rather than individual mesons.

Leaving off the two f_0 glueball states $f_0(600)$ and $f_0(1370)$, we are left with $f_0(980)$, $f_0(1500)$, $f_0(1710)$, $f_0(2020)$, $f_0(2060)$, $f_0(2100)$, $f_0(2200)$, $f_0(2060)$, and $f_0(2330)$. The 1st, 2nd, and 5th of these give a good Koide formula:

$$\sqrt{m_{f_0 1g}} \quad = \quad \mu_e(1.53567 - 0.32338\cos(2/9 + \pi/12 + 2g\pi/3)), \tag{84}$$

with fit:

State	Meson	m_g	exp. meas.
$f_0(1)$	$f_0(980)$	980	980(10)
$f_0(2)$	$f_0(1500)$	1505	1505(6)
$f_0(3)$	$f_0(2060)$	2054	2055(50+)

(85)

But almost as good a formula can be taken from the 1st, 3rd and 4th:

$$\sqrt{m_{f_0 1g}} \quad = \quad \mu_e(1.56250 - 0.32110\cos(2/9 + 2g\pi/3)), \tag{86}$$

with fit:

State	Meson	m_g	exp. meas.
$f_0(1)$	$f_0(980)$	980	980(10)
$f_0(2)$	$f_0(1710)$	1725	1724(7)
$f_0(3)$	$f_0(2020)$	1990	1992(16)

. (87)

Of these two fits, we think the first is more believable.

There are five f_1 states listed, $f_1(1285)$, $f_1(1420)$, $f_1(1510)$, $f_1(1970)$, and $f_1(2310)$. The only good fit that includes the sharp lowest resonance $f_1(1285)$ is the 1st, 2nd, and 4th. The formula is:

$$\sqrt{m_{f_1 1g}} = \mu_e(1.56911 + 0.20675\cos(2/9 + 2g\pi/3)), \quad (88)$$

with fit:

State	Meson	m_g	exp. meas.
$f_1(1)$	$f_1(1285)$	1281.5	1281.8(0.6)
$f_1(2)$	$f_1(1420)$	1427.0	1426.4(0.9)
$f_1(3)$	$f_1(1970)$	1968.3	1971(15)

. (89)

The f_2 is particularly rich in fairly sharp states. This indicates that there will be a lot of overlap, as in the heavy quarkonium states. The lowest six f_2 are $f_2(1270)$, $f_2(1430)$, $f_2'(1525)$, $f_2(1565)$, $f_2(1640)$, and $f_2(1810)$. Of these, the 1st, 2nd, and 5th give the formula:

$$\sqrt{m_{f_2 1g}} = \mu_e(1.51869 + 0.11037\cos(2/9 + \pi/12 + 2g\pi/3)), \quad (90)$$

with fit:

State	Meson	m_g	exp. meas.
$f_2(1)$	$f_2(1270)$	1275.3	1275.1(1.2)
$f_2(2)$	$f_2(1430)$	1439	1450(30+)
$f_2(3)$	$f_2(1640)$	1640	1639(6)

. (91)

In addition to the above, various other combinations of f_2s give Koide fits. The large number of resonances makes too many possibilities for coincidences to justify further speculation. Hopefully we will find a theory for the values of s and v which removes the other degrees of freedom, and this will define the f_2 as Koide triplets.

The larger values of J for f_J have no more than 2 states each and cannot support a Koide fit. However, the $X(2260)$ has quantum numbers $I^G(J^{PC}) = 0^+(4^{+?})$ and so could possibly be an f_4. If so, then, along with the $f_4(2050)$ and $f_4(2300)$ there are three f_4s and they give a Koide formula as:

$$\sqrt{m_{f_4 1g}} = \mu_e(1.87087 - 0.07945\cos(2/9 + 2g\pi/3)), \quad (92)$$

with fit:

State	Meson	m_g	exp. meas.
$f_4(1)$	$f_4(2050)$	2019	2018(11)
$f_4(2)$	$X(2260)$	2253	2260(20+)
$f_4(3)$	$f_4(2300)$	2326	2320(20+)

. (93)

Given the success of Regge trajectories, it may be useful to assume that all the f_J have a similar form for their lowest energy Koide fit. That is, they should all have the same sign for s and should be either electron-like or neutrino-like. Using this as a guiding principle, and assuming we want to use the f_4 formula given above, we end up with the following equations for the f_J with even J:

$$\begin{aligned}\sqrt{m_{f01g}} &= \mu_e(1.56250 - 0.32110\cos(2/9 + 2g\pi/3)),\\ \sqrt{m_{f21g}} &= \mu_e(1.67273 - 0.25378\cos(2/9 + 2g\pi/3)),\\ \sqrt{m_{f41g}} &= \mu_e(1.87087 - 0.07945\cos(2/9 + 2g\pi/3)),\end{aligned} \tag{94}$$

with fit:

State	Meson	m_g	exp. meas.
$f_0(1)$	$f_0(980)$	980	980(10)
$f_0(2)$	$f_0(1710)$	1725	1724(7)
$f_0(3)$	$f_0(2020)$	1990	1992(16)
$f_2(1)$	$f_2(1270)$	1275	1275.1(1.2)
$f_2(2)$	$f_2(1910)$	1918	1915(7)
$f_2(3)$	$f_2(2140)$	2137	2141(12)
$f_4(1)$	$f_4(2050)$	2019	2018(11)
$f_4(2)$	$X(2260)$	2253	2260(20+)
$f_4(3)$	$f_4(2300)$	2326	2320(20+)

. (95)

The f_0 and f_4 were given above while the f_2 formula was chosen to interpolate between them.

Mesons with $I = 0$, and $P \neq C$

Eleven η states, with quantum numbers $I^G(J^{PC}) = 0^+(0^{-+})$ have been found. The 1st, 2nd, and 4th give a Koide formula:

$$\sqrt{m_{\eta 1g}} = \mu_e(1.22243 - 0.32562\cos(2/9 + 2g\pi/3 + \pi/12)), \tag{96}$$

with fit:

State	Meson	m_g	exp. meas.
$\eta(1S1)$	η	547.844	547.853(.024)
$\eta(1S2)$	η'	957.90	957.66(.24)
$\eta(1S3)$	$\eta(1405)$	1408.2	1409.8(2.5)

. (97)

This is similar to the $1Sg$ states for heavy quarkonium. Searching the remaining eight η states for the analog to the $2Sg$ heavy quarkonium states, we find that the 3rd, 7th and 9th give a formula:

$$\sqrt{m_{\eta 2g}} = \mu_e(1.69605 - 0.26661\cos(2/9 + 2g\pi/3)), \tag{98}$$

with an unconvincing fit:

State	Meson	m_g	exp. meas.
$\eta(2S1)$	$\eta(1295)$	1294	1294(4)
$\eta(2S2)$	$\eta(2010)$	1978	2010(60)
$\eta(2S3)$	$\eta(2190)$	2211	2190(50)

. (99)

With so many ηs, and with relatively unconstrained mass measurements, there are many other fits.

The three lowest η_2 states have a Koide formula:

$$\sqrt{m_{\eta 2g}} = \mu_e(1.7060 - 0.51212\cos(2/9 + 2g\pi/3 + \pi/12)), \tag{100}$$

with fit:

State	Meson	m_g	exp. meas.
$\eta_2(1S1)$	$\eta_2(1645)$	1618	1617(5)
$\eta_2(1S2)$	$\eta_2(1870)$	1836	1842(8)
$\eta_2(1S3)$	$\eta_2(2030)$	2038	2030(20)

. (101)

There is a single remaining η_2 and no other η states come in more than one state.

There are five h_1 states with quantum numbers $I^G(J^{PC}) = 0^-(1^{+-})$. The 1st, 2nd, and 3rd give a Koide formula of

$$\sqrt{m_{h1g}} = \mu_e(1.48128 - 0.13140\cos(2/9 + 2g\pi/3 + \pi/12)), \tag{102}$$

with fit:

State	Meson	m_g	exp. meas.
$h_1(1S1)$	$h_1(1170)$	1170	1170(20)
$h_1(1S2)$	$h_1(1380)$	1387	1386(19)
$h_1(1S3)$	$h_1(1595)$	1592	1594(75)

. (103)

The rest of the h have too few states to fit.

Mesons with $I = 1/2$

These mesons have one up or down quark, and one heavier anti-quark (or the anti-meson). Since they have isospin $I = 1/2$, they come in pairs. The higher order resonances are typically reported as average values for the two isospin cases and consequently we will treat the up and down quark cases together. In the Particle Data Group tables, these are relatively uncommon mesons. To get enough states to analyze for a Koide triplet, the anti-quark has to be strange; these are the K mesons.

The K^0 with $I(J^P) = 1/2(0^-)$, has mass measured at 497.614(22) MeV while the $K^\pm$ has 493.677(13). The weighted average of an isospin doublet of these states is 495.646(22). Two other states are known, the $K(1460)$ and $K(1830)$. The $K(1460)$ is known from two measurements with masses of 1460 and 1400 for an average of 1430. The $K(1830)$ has only a single measurement at 1830. These two resonances have widths of around 250 MeV. Taken with the average of the K, the three states form a Koide triplet with the following formula:

$$\sqrt{m_{Kg}} \quad = \quad \mu_e(1.36802 - 0.49152\cos(2/9 + 2g\pi/3)). \tag{104}$$

The fit is good:

State	Meson	m_g	exp. meas.
$K(1S1)$	K	495.645	495.646(22)
$K(1S2)$	$K(1460)$	1439	1430(60)
$K(1S3)$	$K(1830)$	1817	1830(100)

(105)

but the loose mass measurements are not tight constraints.

The K_1 states have $I(J^P) = 1/2(1^+)$ and there are three known. They give a mass formula:

$$\sqrt{m_{K_1g}} \quad = \quad \mu_e(1.50634 + 0.10255\cos(2/9 + 2g\pi/3 + \pi/12)). \tag{106}$$

The fit is as follows:

State	Meson	m_g	exp. meas.
$K_1(1S1)$	$K_1(1270)$	1265	1272(7)
$K_1(1S2)$	$K_1(1400)$	1417	1403(7)
$K_1(1S3)$	$K_1(1650)$	1601	1650(50)

. (107)

The $K_1(1400)$ fit is slightly on the high side.

The $I(J^P) = 1/2(2^-)$ states have 4 listed, $K_2(1580)$, $K_2(1770)$, $K_2(1820)$, and $K_2(2250)$. Of the four possible ways of picking a Koide triplet out of these states, only the first three give masses within the measurements. The resulting formula is:

$$\sqrt{m_{K_2g}} \quad = \quad \mu_e(1.50634 + 0.10255\cos(2/9 + 2g\pi/3 + \pi/12)). \tag{108}$$

The fit is as follows:

State	Meson	m_g	exp. meas.
$K_2(1S1)$	$K_2(1270)$	1265	1272(7)
$K_2(1S2)$	$K_2(1400)$	1417	1403(7)
$K_2(1S3)$	$K_2(1650)$	1601	1650(50)

. (109)

Of the four K_2 states, the lowest, the $K(1580)$, is the least established, seen only in one experiment dating back to 1979. The third has been seen twice. The second and fourth states are more firmly established.

There are three K_0^* states. These have $I(J^P) = 1/2(0^+)$ and form a Koide triplet:

$$\sqrt{m_{K_0^* g}} \;\; = \;\; \mu_e(1.44114 - 0.42838\cos(2/9 + 2g\pi/3)). \tag{110}$$

The fit is as follows:

State	Meson	m_g	exp. meas.
$K_0^*(1S1)$	$K_0^*(800)$	657	672(40)
$K_0^*(1S2)$	$K_0^*(1430)$	1544	1425(50)
$K_0^*(1S3)$	$K_0^*(1950)$	1883	1945(30)

. (111)

The $K_0^*(800)$ has a rather wide width of around 550 MeV and measurements of its mass have been widely scattered. The Particle Data Group uses 1988 data for the $K_0^*(1950)$ mass. The more recent data would improve the fit somewhat.

Three K^* states, with $I(J^P) = 1/2(1^-)$ have been measured. Their masses are fairly well measured and do not follow a Koide fit. The remaining K mesons have too few observed states to fit Koide equations.

Mesons with $I = 1$

Mesons with isospin $I = 1$ come in threes, with charges +1, 0, and −1. With most of these states, the Particle Data Group gives one mass for all three particles in the isospin triplet. For the others, we will use the average mass, i.e., $m_\pi = (2m_{\pi^+} + m_{\pi^0})/3 = 138.0390$ MeV.

The π has quantum numbers of $I^G(J^{PC}) = 1^-(0^{-+})$. There are five resonances observed. The lowest three form a Koide triplet with equation:

$$\sqrt{m_{\pi g}} \;\; = \;\; \mu_e(1.19678 - 0.74619\cos(2/9 + 2g\pi/3)). \tag{112}$$

The fit is within the error estimates:

State	Meson	m_g	exp. meas.
$\pi(1S1)$	π	138.0396	138.0390(6)
$\pi(1S2)$	$\pi(1300)$	1262	1300(100)
$\pi(1S3)$	$\pi(1800)$	1821	1812(14)

. (113)

The $\pi(1300)$ has a particularly large width, 200 to 600 MeV, and its loose mass measurement leaves considerable latitude in fitting.

There are three observed $I^G(J^{PC}) = 1^-(1^{-+})$ π_1 mesons. They form a Koide triplet with equation:

$$\sqrt{m_{\pi_1 g}} \;\; = \;\; \mu_e(1.63300 - 0.17936\cos(2/9 + 2g\pi/3 + \pi/12)). \tag{114}$$

The fit is:

State	Meson	m_g	exp. meas.
$\pi_1(1S1)$	$\pi_1(1400)$	1377	1376(17)
$\pi_1(1S2)$	$\pi_1(1600)$	1659	1662(15)
$\pi_1(1S3)$	$\pi_1(2015)$	2015	2013(25)

, (115)

which is close to the mass estimates.

The π_2 mesons with $I^G(J^{PC}) = 1^-(1^{-+})$ appear five times. The lowest state, $\pi_2(1670)$ has a mass given quite sharply, compared to its width of 250 MeV. The lowest three π_2s form a sloppy Koide triplet:

$$\sqrt{m_{\pi 2g}} \quad = \quad \mu_e(1.71417 - 0.09183\cos(2/9 + 2g\pi/3 + \pi/12)). \tag{116}$$

The fit is:

State	Meson	m_g	exp. meas.
$\pi_2(1S1)$	$\pi_2(1670)$	1673.7	1672.4(3.2)
$\pi_2(1S2)$	$\pi_2(1880)$	1852	1880(30)
$\pi_2(1S3)$	$\pi_2(2005)$	2015	2000(30)

. (117)

A better fit is obtained by swapping the $\pi_2(2100)$ for the $\pi_2(2005)$. The equation is:

$$\sqrt{m_{\pi 2g}} \quad = \quad \mu_e(1.72968 - 0.11026\cos(2/9 + 2g\pi/3 + \pi/12)), \tag{118}$$

which gives:

State	Meson	m_g	exp. meas.
$\pi_2(1S1)$	$\pi_2(1670)$	1672.0	1672.4(3.2)
$\pi_2(1S2)$	$\pi_2(1880)$	1887	1880(30)
$\pi_2(1S3)$	$\pi_2(2090)$	2086	2090(29)

. (119)

The remaining π resonances have too few observed states for any further triplets.

The ρ mesons have $I^G(J^{PC}) = 1^+(1^{--})$. The situation in the 2008 Particle Data Group is considerably improved since 2006. Then there were nine of these states, now they've been consolidated down to six. These six divide into two Koide triplets with the first three states making up the first triplet and the second three making up the second triplet. The first triplet is:

$$\sqrt{m_{\rho 1g}} \quad = \quad \mu_e(1.43166 - 0.32830\cos(2/9 + 2g\pi/3)), \tag{120}$$

which gives a good fit:

State	Meson	m_g	exp. meas.
$\rho(1S1)$	$\rho(770)$	775.41	775.43(1.1)
$\rho(1S2)$	$\rho(1450)$	1468	1465(25)
$\rho(1S3)$	$\rho(1700)$	1718	1720(20)

. (121)

The second triplet has an equation:

$$\sqrt{m_{\rho 2g}} \quad = \quad \mu_e(1.84615 - 0.06778\cos(2/9 + 2g\pi/3 + \pi/12)), \tag{122}$$

also with a satisfactory fit:

State	Meson	m_g	exp. meas.
$\rho(2S1)$	$\rho(2000)$	2002	2000(30)
$\rho(2S2)$	$\rho(2150)$	2146	2149(17)
$\rho(2S3)$	$\rho(2270)$	2274	2270(40)

(123)

The form of these two triplets is similar to the heavy quarkonium triplets.

There are not enough ρ_1 or ρ_2 states to make any triplets but there are four ρ_3 states which have $I^G(J^{PC}) = 1^+(3^{--})$. Of these four, the two with the highest masses are suspiciously similar, 2250(50) and 2300(80) MeV. The lowest three ρ_3 have a Koide equation:

$$\sqrt{m_{\rho 3g}} = \mu_e(1.77038 - 0.14697\cos(2/9 + 2g\pi/3 + \pi/12)), \tag{124}$$

with a satisfactory fit:

State	Meson	m_g	exp. meas.
$\rho_3(1S1)$	$\rho_3(1690)$	1688.9	1688.8(2.1)
$\rho_3(1S2)$	$\rho_3(1990)$	1980	1982(14)
$\rho_3(1S3)$	$\rho_3(2250)$	2253	2250(50)

(125)

The remaining ρ resonances are too sparse to fit Koide triplets.

The a_0 mesons have $I^G(J^{PC}) = 1^-(0^{++})$. There are three observed states and their Koide equation is:

$$\sqrt{m_{a_0 g}} = \mu_e(1.52729 - 0.31072\cos(2/9 + 2g\pi/3 + \pi/12)), \tag{126}$$

with fit:

State	Meson	m_g	exp. meas.
$a_0(1S1)$	$a_0(980)$	984.4	984.7(1.2)
$a_0(1S2)$	$a_0(1450)$	1488	1474(19)
$a_0(1S3)$	$a_0(2020)$	2011	2025(30)

(127)

Note that the above equation uses $\pi/12$.

The next entry in the a series, the a_1 with quantum numbers $I^G(J^{PC}) = 1^-(1^{++})$, has six entries. The first, second and fourth form the lower Koide triplet:

$$\sqrt{m_{a_1 1g}} = \mu_e(1.61283 - 0.24319\cos(2/9 + 2g\pi/3 + \pi/12)), \tag{128}$$

the fit:

State	Meson	m_g	exp. meas.
$a_1(1S1)$	$a_1(1260)$	1226	1230(40)
$a_1(1S2)$	$a_1(1640)$	1652	1647(22)
$a_1(1S3)$	$a_1(2095)$	2076	2096(138)

(129)

As with a_0, the above equation uses $\pi/12$. The Koide equation for the remaining three states:

$$\sqrt{m_{a_1 2g}} \;=\; \mu_e(1.86157 - 0.10845\cos(2/9 + 2g\pi/3)), \tag{130}$$

and the fit:

State	Meson	m_g	exp. meas.
$a_1(2S1)$	$a_1(1930)$	1935	1930(70)
$a_1(2S2)$	$a_1(2270)$	2251	2270(55)
$a_1(2S3)$	$a_1(2340)$	2351	2340(40)

. (131)

As with heavy quarkonium, one of the triplets uses $\pi/12$, the other does not.

There are also six observed states for the a_2 with quantum numbers $I^G(J^{PC}) = 1^-(2^{++})$. As with the a_1, the first, second and fourth form a Koide triplet with equation using $pi/12$:

$$\sqrt{m_{a_2 1g}} \;=\; \mu_e(1.64249 - 0.21819\cos(2/9 + 2g\pi/3 + \pi/12)), \tag{132}$$

the fit is close but not perfect:

State	Meson	m_g	exp. meas.
$a_2(1S1)$	$a_2(1320)$	1318.6	1318.3(6)
$a_2(1S2)$	$a_2(1700)$	1711	1732(16)
$a_2(1S3)$	$a_2(2080)$	2095	2080(20)

, (133)

with the equation giving the $a_2(1700)$ a mass a little closer to that originally observed. The remaining three states have an equation:

$$\sqrt{m_{a_2 2g}} \;=\; \mu_e(1.85002 - 0.07067\cos(2/9 + 2g\pi/3)), \tag{134}$$

and the fit:

State	Meson	m_g	exp. meas.
$a_2(2S1)$	$a_2(1990)$	1991	1995(30)
$a_2(2S2)$	$a_2(2175)$	2197	2175(40)
$a_2(2S3)$	$a_2(2270)$	2261	2270(20)

. (135)

As with heavy quarkonium and the other a_χ states, there is one neutrino-like equation using $\pi/12$, and one electron-like equation without such a factor.

There are three a_3 observed. These have quantum numbers $I^G(J^{PC}) = 1^-(3^{++})$. A Koide equation for them requires $s > 0$:

$$\sqrt{m_{a_3 g}} \;=\; \mu_e(1.82055 + 0.11020\cos(2/9 + 2g\pi/3 + \pi/12)), \tag{136}$$

with fit:

State	Meson	m_g	exp. meas.
$a_3(1S1)$	$a_3(1875)$	1873	1874(139)
$a_3(1S2)$	$a_3(2070)$	2071	2070(20)
$a_3(1S3)$	$a_3(2310)$	2309	2310(40)

. (137)

The above is well within the error bars on the mass measurement but it is suspicious that the equation has $s > 0$. This comes about because the third state, $a_3(2310)$, has too large of a mass. The mass for $a_3(1S3)$ predicted by $a_2(1875)$ and $a_2(2070)$ would be around 2250 MeV but the large error bar in $a_2(1875)$ makes the prediction worthless. Another way of fixing this triplet to make $s < 0$ is to adjust the mass of $a_2(1875)$ lower to around 1810 MeV. This is within the error bars of the $a_3(1875)$. The remaining a_χ states are too sparsely populated to give Koide triplets.

Finally, we have three b_1 states with quantum numbers $I^G(J^{PC}) = 1^+(1^{+-})$. Their Koide formula is:

$$\sqrt{m_{b_1 g}} \quad = \quad \mu_e(1.68560 - 0.29340\cos(2/9 + 2g\pi/3)), \tag{138}$$

with fit:

State	Meson	m_g	exp. meas.
$b_1(1S1)$	$b_1(1235)$	1229	1229.5(3.2)
$b_1(1S2)$	$b_1(1960)$	1973	1960(35)
$b_1(1S3)$	$b_1(2240)$	2230	2240(35)

. (139)

These are the only b states with sufficient states to give triplets.

Baryons

While baryons have three valence quarks instead of the quark and antiquark making up mesons, it is possible that the same analysis that led to a Koide formula for mesons will also work for the baryons. In this subsection we will discuss possible applications in the baryons. In doing this, we will assume that the correct equations to use are the same used in the mesons, electron-like Eq. (71) or neutrino-like Eq. (72).

As with the mesons, the heavier baryons have masses that are more exactly known. However, heavy baryons are more difficult to obtain experimentally than heavy mesons and so the data on heavy baryons is more limited than that on heavy mesons. In addition, the extra quark in a baryon greatly increases the number of quantum states. Since the Koide mass formula is intended to apply only to states that share the same quantum numbers, this diversity of states makes baryons more difficult to match up. Some of the baryons come in isospin n-plets. Unless a state is marked with its charge, we have used an average over the charge states (or the Particle Data Group did not differentiate them).

No baryons with top quarks have been seen. Only four bottom baryon states have been observed and no two of them have the same quantum numbers. This leaves us with only baryons containing up, down, strange, and charm quarks. These have an approximate $SU(4)$

symmetry. The symmetry multiplets for three of these quarks is

$$4 \times 4 \times 4 = 20 + 20'_1 + 20'_2 + \bar{4}. \tag{140}$$

In applying Koide equations to the baryons, we need to distinguish between particles taken from different multiplets in the above. For example, the Ξ_c^+ and the Ξ'_c are states from the 20 and $20'$, respectively, and so cannot be in the same Koide triplet.

The experimenter has a problem in distinguishing the different baryon multiplets. Since the experimenter can observe the decay products of the baryon, it is much simpler to determine the quark content. For example, the Ξ_c^+ has quark content csu. Looking at the problem in reverse, the csu quarks have symmetry multiplicity $3! = 6$. These correspond (in multiplicity but not in symmetry) to the six ways of ordering the three quarks, i.e. csu, cus, ... suc. Thus when detecting a csu state, in determining the symmetry class the experimenter has six assignment choices. (One each for the 20 and $\bar{4}$, and two each for the $20'_1$ and $20'_2$.) This is such a large number, compared to the number of detected states, that it is impossible to find Koide triplets in these states.

In terms of quark content instead of symmetry, the multiplicity of the baryons becomes

$$4 \times 4 \times 4 = 4\begin{pmatrix} 3 \\ 3 \end{pmatrix} + 12\begin{pmatrix} & 3 & \\ 2 & & 1 \end{pmatrix} + 4\begin{pmatrix} & 3 & \\ 1 & 1 & 1 \end{pmatrix} = 4(1) + 12(3) + 4(6). \tag{141}$$

That is, there are four ways of choosing among the four light quarks so that all three are identical and these make four states; there are 12 ways of choosing quarks so that one appears twice and the other once and each of these has multiplicity three; and finally, there are four ways of choosing three different quarks and each of these has multiplicity six.

In order to simplify the task of distinguishing different symmetries, we will look only at the states with multiplicity 1. These are the states that have all three quarks the same and they correspond to the "4(1)" in the above equation.

When all three quarks are u or d, the states is a Δ^{++} or Δ^{-}, respectively. These states are the end points of the Δ isospin multiplet. The presence of isospin symmetry allows the intermediate states to be picked out from the alternative symmetries, the $20'_1$ and $20'_2$. These alternative symmetries include the nucleons, the proton and neutron and we will discuss them later.

Three $I(J^P) = 3/2(3/2^+)$ (P33) Δ states are observed. They follow a Koide equation:

$$\sqrt{m_{f_{\Delta 3/2+g}}} \;=\; \mu_e(1.58452 - 0.20727\cos(2/9 + 2g\pi/3 + \pi/12)), \tag{142}$$

with fit:

State	Baryon	m_g	exp. meas.
$\Delta_{3/2+}(1S1)$	$\Delta(1232)$	1232	1232(1)
$\Delta_{3/2+}(1S2)$	$\Delta(1600)$	1592	1625(75)
$\Delta_{3/2+}(1S3)$	$\Delta(1920)$	1944	1935(35)

. (143)

There are also three $I(J^P) = 3/2(1/2^-)$ (S31) states. Their Koide equation is:

$$\sqrt{m_{f_{\Delta 1/2-g}}} = \mu_e(1.73401 - 0.13839\cos(2/9 + 2g\pi/3 + \pi/12)), \tag{144}$$

with fit:

State	Baryon	m_g	exp. meas.
$\Delta_{1/2-}(1S1)$	$\Delta(1620)$	1630	1630(30)
$\Delta_{1/2-}(1S2)$	$\Delta(1900)$	1899	1900(50)
$\Delta_{1/2-}(1S3)$	$\Delta(2150)$	2151	2150(100)

. (145)

The equations for the Δ resonances are of the neutrino type.

When all three quarks are s, the baryon is the Ω^-. There are four Ω^- entries in the PDG but they do not have a convincing Koide triplet formula between them. The best formula takes the first, second, and fourth states and assumes that they are $P33$:

$$\sqrt{m_{f_{\Omega 3/2+g}}} = \mu_e(1.83715 - 0.21009\cos(2/9 + 2g\pi/3)), \tag{146}$$

with fit:

State	Baryon	m_g	exp. meas.
$\Omega^-_{3/2+}(1S1)$	Ω^-	1672.33	1672.43(32)
$\Omega^-_{3/2+}(1S2)$	$\Omega(2250)^-$	2265	2252(9)
$\Omega^-_{3/2+}(1S3)$	$\Omega(2470)^-$	2460	2474(12)

. (147)

This is an electron-like formula. The remaining pure baryon with composition ccc, has not yet been observed.

Among the states with multiplicity 3, the nucleons N (in the $SU(4)$ symmetry $20'_1$) are distinguished in that their multiplicity is shared with the Δ (in the 20). The Δ states are easily distinguished because they are related by isospin symmetry to the Δ^{++} and Δ^- which have quark content multiplicity 1. Consequently, the effective multiplicity of the N is only 2. This makes them the next best hope for a Koide formula.

Among the N, there are four $P11$ states with quantum numbers $I(J^P) = 1/2(1/2^+)$. Other than the familiar neutron and proton, there masses are well determined and do not exhibit any convincing Koide formula. Similarly, four $D13$ exist. Other than the lowest $N(1520)$, their masses are also not tightly defined. Similarly, there are three $S11$ states but no Koide formula. There are hints in the PDG that some of the nucleon states are actually two states with the same quantum number and slightly different masses. Perhaps this has to do with the fact that there are two symmetry groups that apply, the $20'_1$ and $20'_2$. An example of how one can split a state to get a Koide fit is the $S11$ state $N(1650)$. Splitting it into two states with masses 1635 and 1660 gives a Koide triplet with the lightest $S11$, $N(1535)$.

The Σ states have a single s and two u or d quarks, forming an isospin triplet. Their mixed quark composition puts them into three $SU(4)$ symmetry classes, 20, $20'_1$, and $20'_2$. With three symmetry contributions, it's natural that the experimenters would find plenty of complicated states. These are difficult to associate with Koide equations, but it won't stop the author from trying.

The lowest lying Σ state is a $P11$ states with quantum numbers $I(J^P) = 1(1/2^+)$ and a mass of 1193.154(60). The PDG gives three resonances, the $\Sigma(1660)$, $\Sigma(1770)$, and the $\Sigma(1880)$. However, the $\Sigma(1880)$ is a bump collection ranging from 1826 to 1950. If we take the lowest two, the Σ and $\Sigma(1660)$ and assume that these form the lowest two of an electron type Koide triplet (with negative s), the third element is around 1815(40) which is compatible with the 1826(20).

Two $P13$ Σ states with quantum numbers $I(J^P) = 1(3/2^+)$ are listed in the PDG, the accurately measured $Sigma(1385)$ and a bump collection called the $Sigma(1840)$ which includes five badly verified states. These do not possess a convincing split into two Koide triplets but of course with that many wide mass estimates it is possible to fit a single triplet.

The Σ includes four $S11$ states with quantum numbers $1(1/2^-)$ but no Koide triplets. The second lowest state, the $\Sigma(1750)$ is seen over a wide range of masses and could be split to give an unconvincing Koide triplet with the lowest $\Sigma(1620)$ state.

There are three Σ $D13$ states with quantum numbers $1(3/2^-)$. Again, no Koide triplet but the middle state, $\Sigma(1670)$ is said to consist of more than one state, but with the same quantum numbers, on the basis of branching ratios. To get this state to provide two elements of a Koide triplet with the lowest mass state, $\Sigma(1580)$, would require a splitting of around 22 MeV.

The Ξ state is doubly strange. These states come in isospin doublets with charge 0 and +1, and have there are three possible symmetries, one each of 20, $20'_1$, and $20'_2$. There are only eleven entries in the PDG for all quantum numbers and no quantum number appears twice. Seven of the states have completely unknown quantum numbers. This is too sparse to find Koide triplets.

The remaining charmless baryon is the Λ with composition uds. With six possible symmetries for these quarks, the 20 and one each of the $20'_1$ or $20'_2$ should be eliminated by isospin symmetry with the other Σ states. What's left is one state each from the $20'_1$, $20'_2$, and $\bar{4}$.

The lowest mass Λ has symmetry $P01$ and quantum numbers $I(J^P) = 0(1/2^+)$. There are three observed states. The two higher states are listed in the PDG as likely to have multiple masses, but the three states as they are fit a Koide equation as follows:

$$\sqrt{m_{f_{\Lambda 1/2+g}}} \quad = \quad \mu_e(1.54587 - 0.21805\cos(2/9 + 2g\pi/3)), \tag{148}$$

with fit:

State	Baryon	m_g	exp. meas.
$\Lambda_{1/2+}(1S1)$	Λ	1115.683	1115.683(6)
$\Lambda_{1/2+}(1S2)$	$\Lambda(1600)$	1628	1630(70)
$\Lambda_{1/2+}(1S3)$	$\Lambda(1810)$	1801	1800(50)

(149)

The next Λ state is $I(J^P) = 0(1/2^-)$ with $S01$. The three observed states also satisfy a Koide equation:

$$\sqrt{m_{f_{\Lambda 1/2-g}}} \quad = \quad \mu_e(1.60153 - 0.10751\cos(2/9 + 2g\pi/3)), \tag{150}$$

with fit:

State	Baryon	m_g	exp. meas.
$\Lambda_{1/2-}(1S1)$	$\Lambda(1405)$	1406.1	1406.5(40)
$\Lambda_{1/2-}(1S2)$	$\Lambda(1670)$	1675	1670(10)
$\Lambda_{1/2-}(1S3)$	$\Lambda(1800)$	1760	1785(65)

(151)

There are also three Λ $D03$ states with quantum numbers $0(3/2^-)$. They do not form a Koide triplet. If the bottom two states, $\Lambda(1520)$ and $\Lambda(1690)$ are the lowest two elements of a triplet, the third state should be around 1743(20).

Of the charmed baryons, the only states with enough to find a Koide triplet is the Ξ_c. These states are isospin doublets with charge 0 or +1. Most experimental data are for the charged state, the Ξ_c^+ which has quark composition of csu, rather than the Ξ_c^0 with csd. Since the c is a heavy quark, masses are more accurately measured than in the light baryons, but there are six possible symmetry states; the 20 and $\bar{4}$ contribute one each, and the $20'_1$ and $20'_2$ give two more each. Consequently, assigning Koide triplets to the states is difficult.

The lowest mass Ξ_c^+ is at 2467.6(10) MeV. It is expected, on theoretical grounds only, to be the $P11$ symmetry with quantum numbers $I(J^P) = 1/2(1/2^+)$. There is another $P11$ state just above this that is thought to be from the $20'_1$, the Ξ'_c. It's mass is around 100 MeV higher. There are four Ξ_cs with unknown quantum numbers. Two of these fit into a Koide triplet with the Ξ_c:

$$\sqrt{m_{f_{\Xi c1/2+g}}} \quad = \quad \mu_e(2.11912 - 0.13986\cos(2/9 + 2g\pi/3)), \tag{152}$$

with fit:

State	Baryon	m_g	exp. meas.
$\Xi^+_{c1/2+}(1S1)$	Ξ'	2467.62	2467.6(10)
$\Xi^+_{c1/2+}(1S2)$	$\Xi(2930)$	2930.4	2931(8)
$\Xi^+_{c1/2+}(1S3)$	$\Xi(3080)^+$	3077.0	3077.0(4)

(153)

The Ξ_c state $\Xi(3055)$ has unknown quantum numbers and is about 100 MeV above the second of the above triplet. If it were the second element of a triplet with the Ξ'_c as the lowest entry, the third entry would be at around 3206(3) MeV.

There are six Λ_c^+ states in the Particle Data Group book. Four have $J^P = 1/2^-$, $1/2^+$, $3/2^-$, and $5/2^+$. The other two have unknown quantum numbers and so could fit out a Koide triplet with one of the other four but no such exists. There are only three Σ_c states, and two Ω_cs, all of which have mutually incompatible quantum numbers. This leaves only the Ξ_c to look for Koide triplets.

Discussion

Any physics paper purporting to model reality with quantum mechanics is, at best, a choice of formulation of quantum mechanics which is solved with some mathematics and results in a collection of coincidences. In this paper, the formulation of quantum mechanics is a quantum information derived model of the color force and the collection of coincidences is two mass equations for the leptons, 33 mass equations for the mesons, and six for the baryons.

Historically, the justification for new formulations of quantum mechanics has been the fact that the new formulation provides solutions to problems previously unsolved. For a description of nine previous formulations of quantum mechanics see [23]. The unsolved problem solved here is the structure of hadron excitations. The new formulation given here is an approximation of the color bound state based on the principles of quantum information theory. The formulation gives non perturbative calculations.

Formulations of quantum mechanics are not mutually exclusive. For example, when computing the fine and hyperfine structure of the hydrogen atom, one uses quantum field theory (path integrals) as a perturbative correction to a non perturbative calculation using Schroedinger's equation for the bound state of an electron in a Coulomb force. We hope that the non perturbative calculations found here will also receive a perturbative correction.

Mutually unbiased bases are a central concept of quantum information theory. To practitioners of both quantum information theory and elementary particles it may appear strange that MUBs would have an application in the hadrons but would it not be stranger still if the foundations of quantum information theory turned out to be completely useless in understanding the structure of elementary particles? And would it not be surprising that nature would use a completely different approach to excitations of hadrons made from quarks than She used in the excitations of the leptons?

Quantum mechanics has a rich history of making good use of the simplest mathematical examples. The Pauli algebra is the simplest nontrivial Hilbert space. The simplest mutually unbiased basis set is that of the Pauli algebra, so if there are every to be any applications of mutually unbiased bases it seems most likely that it will be with the three mutually unbiased bases of the Pauli algebra.

Of the things one might consider doing with a complete set of mutually unbiased bases, the most obvious is to use their transition probabilities as amplitudes for transitions. A bound state is stationary and these transition amplitudes must be consistent with each other. The idempotency requirement of a pure density matrix gives a consistency relation for the transition amplitudes. The consistency relations can be thought of as a collection of path integrals set up so that amplitudes going into each state are balanced by the amplitudes going out of it. Alternatively, the non-Hermitian pure density matrix products used here, such as (RG) can be thought of as representing pairs of Stern-Gerlach devices. The consistency relation amounts to requiring that number and phase of particles entering each device match the number and (modified) phase exiting. This is the mathematical problem solved in this paper.

This paper provides 41 examples of the application of the theory to the hadrons and leptons. Each equation removes one degree of freedom from three masses. Some of the fits are undoubtedly incorrect; they are the result of random chance. A sad and beautiful fact about mathematics is that it is an infinite subject; it can always provide an equation that fits any amount of data to any desired accuracy. Even a collection of $41 \times 3 = 123$ random data points has an equation that removes 41 degrees of freedom from it. But the equation used here are quite simple, and the degrees of freedom left unspecified do not seem excessively random, and many of the fits are quite good. And in addition to postdictions of 43 known excitations, the paper also includes predictions for a few new states in terms of masses, or identification of observed X states in terms of quantum numbers.

The hadrons provide more information than just their masses to theoreticians. This paper discusses primarily the masses but the decay modes are also of great interest. Among the leptons, a charged lepton l^+ can decay into a neutrino by the weak force, emitting a W^+. The neutrino produced depends on which lepton is decaying and also on probability. The table of probabilities for this is known as the weak mixing angle matrix. It was briefly discussed above in the context of applications of the discrete Fourier transform to the generation structure of the leptons, see Eq. (22).

The model we've been using here ignores the bosons; we've been treating them as if they were part of a force field. The analogy with the weak force would be to ignore the $W^{\pm}$; thus we can imagine a theory of how charged leptons convert into neutrinos and vice versa. Would this behavior have an analogy in the hadrons, say in quarkonium? One would look up the decay rates, correct for phase factors and obtain the amplitude $|M|$ for the process, and then obtain a matrix of these values. Unfortunately, there is not sufficient decay rate data to execute this procedure. Most of the data that is available is in branching ratios and this is fairly sparse with heavy quarkonium.

Despite the dearth of experimental data, there are a few things we can say. The standard models of heavy quarkonium treat them only as radial excitations or apply lattice gauge theory to understanding them [24, 25]. The transition probabilities would not follow the pattern predicted by radial excitation, but instead should exhibit mixing angles, perhaps related to the lepton mixing angles.

Conclusions

Since the strong force is the strongest of the known forces, it would be natural that our models of the hadrons should be based on an accurate representation of the strong force. This has not been done due to difficulties in applying perturbation theory in the strong regime. This paper shows that the strong force can be put into a solvable form if it is simplified by examining only the information content of the theory. This shows that the hadron excitations should appear as triplets.

These triplets are related by the discrete Fourier transform. Using this tool to look at the lepton masses, we find an extension of Koide's mass formula to the neutrinos. Examining the hadron excitations, we find an echo of the generation structure of the leptons; slight modifications of the lepton mass equations allow fits to the hadron excitations. This provides a practical and useful example of quark-lepton universality.

The methods used here for modeling the hadrons can be corrected in the same way that qubit approximations of spin-1/2 particles can be corrected using ordinary perturbation theory. [26] The technique described here opens up a unique new way of modeling the hadrons.

Our understanding of elementary particles has always contributed to our understanding of quantum information theory. This paper shows that the favor can go both ways.

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