Jiang’s function $J_{n+1}(\omega)$ in prime distribution

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Abstract

We define that prime equations

$$f_1(P_1,\cdots,P_n),\cdots,f_k(P_1,\cdots P_n) \quad （5）$$

are polynomials (with integer coefficients) irreducible over integers, where $P_1,\cdots,P_n$ are all prime. If Jiang’s function $J_{n+1}(\omega)=0$ then （5）has finite prime solutions. If $J_{n+1}(\omega)\neq 0$ then there are infinitely many primes $P_1,\cdots,P_n$ such that $f_1,\cdots f_k$ are primes. We obtain a unite prime formula in prime distribution

$$\pi_{k+1}(N,n+1)=\left|\{P_1,\cdots,P_n\le N: f_1,\cdots,f_k \text{ are } k \text{ primes}\}\right|$$

$$=\prod_{i=1}^{k}(\deg f_i)^{-1}\times\frac{J_{n+1}(\omega)\omega^k}{n!\phi^{k+n}(\omega)}\frac{N^n}{\log^{k+n}N}(1+o(1)). \quad （8）$$

Jiang’s function is accurate sieve function. Using Jiang’s function we prove about 600 prime theorems [6]. Jiang’s function provides proofs of the prime theorems which are simple enough to understand and accurate enough to be useful.

Mathematicians have tried in vain to discover some order in the sequence of prime numbers but we have every reason to believe that there are some mysteries which the human mind will never penetrate.

Leonhard Euler

It will be another million years, at least, before we understand the primes.

Paul Erdös

Suppose that Euler totient function

$$\phi(\omega)=\prod_{2\le P}(P-1)=\infty \quad \text{as} \quad \omega\to\infty\,, \tag{1}$$

where $\omega=\prod_{2\le P}P$ is called primorial.

Suppose that $(\omega,h_i)=1$, where $i=1,\cdots,\phi(\omega)$. We have prime equations

$$P_1=\omega n+1,\cdots,P_{\phi(\omega)}=\omega n+h_{\phi(\omega)} \tag{2}$$

where $n=0,1,2,\cdots$.

（2）is called infinitely many prime equations (IMPE). Every equation has infinitely many prime solutions. We have

$$\pi_{h_i}=\sum_{\substack{P_i\le N\\ P_i\equiv h_i \pmod{\omega}}}1=\frac{\pi(N)}{\phi(\omega)}(1+o(1)).\,, \tag{3}$$

where π_{h_i} denotes the number of primes $P_i\le N$ in $P_i=\omega n+h_i$ $n=0,1,2,\cdots$, $\pi(N)$ the number of primes less than or equal to N.

We replace sets of prime numbers by IMPE. (2) is the fundamental tool for proving the prime theorems in prime distribution.

Let $\omega=30$ and $\phi(30)=8$. From (2) we have eight prime equations

$$P_1=30n+1,\ P_2=30n+7,\ P_3=30n+11,\ P_4=30n+13,\ P_5=30n+17,$$
$$P_6=30n+19,\ P_7=30n+23,\ P_8=30n+29,\ n=0,1,2,\cdots \tag{4}$$

Every equation has infinitely many prime solutions.

THEOREM. We define that prime equations

$$f_1(P_1,\cdots,P_n),\cdots,f_k(P_1,\cdots,P_n) \tag{5}$$

are polynomials (with integer coefficients) irreducible over integers, where $P_1,\cdots,P_n$ are primes. If Jiang's function $J_{n+1}(\omega)=0$ then (5) has finite prime solutions. If $J_{n+1}(\omega)\neq 0$ then there exist infinitely many primes $P_1,\cdots,P_n$ such that each f_k is a prime.

PROOF. Firstly, we have Jiang's function [1-11]

$$J_{n+1}(\omega)=\prod_{3\le P}[(P-1)^n-\chi(P)], \qquad （6）$$

where $\chi(P)$ is called sieve constant and denotes the number of solutions for the following congruence

$$\prod_{i=1}^{k} f_i(q_1,\cdots,q_n)\equiv 0 \pmod P, \qquad （7）$$

where $q_1=1,\cdots,P-1,\cdots,q_n=1,\cdots,P-1$.

$J_{n+1}(\omega)$ denotes the number of sets of $P_1,\cdots,P_n$ prime equations such that $f_1(P_1,\cdots,P_n),\cdots,f_k(P_1,\cdots,P_n)$ are prime equations. If $J_{n+1}(\omega)=0$ then (5) has finite prime solutions. If $J_{n+1}(\omega)\neq 0$ using $\chi(P)$ we sift out from (2) prime equations which can not be represented $P_1,\cdots,P_n$, then residual prime equations of (2) are $P_1,\cdots,P_n$ prime equations such that $f_1(P_1,\cdots,P_n),\cdots,$ $f_k(P_1,\cdots,P_n)$ are prime equations. Therefore we prove that there exist infinitely many primes $P_1,\cdots,P_n$ such that $f_1(P_1,\cdots,P_n),\cdots,$ $f_k(P_1,\cdots,P_n)$ are primes.

Secondly, we have the best asymptotic formula [2,3,4,6]

$$\pi_{k+1}(N,n+1)=\left|\{P_1,\cdots,P_n\le N: f_1,\cdots,f_k \text{ are } k \text{ primes}\}\right|$$

$$=\prod_{i=1}^{k}(\deg f_i)^{-1}\times\frac{J_{n+1}(\omega)\omega^k}{n!\phi^{k+n}(\omega)}\frac{N^n}{\log^{k+n}N}(1+o(1)). \qquad （8）$$

（8）is called a unite prime formula in prime distribution. Let $n=1,k=0$, $J_2(\omega)=\phi(\omega)$. From (8) we have prime number theorem

$$\pi_1(N,2)=\left|\{P_1\le N: P_1 \text{ is prime}\}\right|=\frac{N}{\log N}(1+o(1)).. \qquad （9）$$

Number theorists believe that there are infinitely many twin primes, but they do not have rigorous proof of this old conjecture by any method. All the prime theorems are conjectures except the prime number theorem, because they do not prove that prime equations have infinitely many prime solutions. We prove the following conjectures by this theorem.

Example 1. Twin primes $P, P+2$ (300BC)**.**

From (6) and (7) we have Jiang's function

$$J_2(\omega) = \prod_{3 \le P}(P-2) \ne 0.$$

Since $J_2(\omega) \ne 0$ in (2) exist infinitely many P prime equations such that $P+2$ is a prime equation. Therefore we prove that there are infinitely many primes P such that $P+2$ is a prime.

Let $\omega = 30$ and $J_2(30) = 3$. From (4) we have three P prime equations

$$P_3 = 30n+11, \quad P_5 = 30n+17, \quad P_8 = 30n+29.$$

From (8) we have the best asymptotic formula

$$\pi_2(N,2) = \left|\{P \le N : P+2 \text{ prime}\}\right| = \frac{J_2(\omega)\omega}{\phi^2(\omega)} \frac{N}{\log^2 N}(1+o(1))$$

$$= 2\prod_{3 \le P}\left(1 - \frac{1}{(P-1)^2}\right)\frac{N}{\log^2 N}(1+o(1)).$$

In 1996 we proved twin primes conjecture [1]

Remark. $J_2(\omega)$ denotes the number of P prime equations, $\frac{\omega}{\phi^2(\omega)} \frac{N}{\log^2 N}(1+o(1))$ the number of solutions of primes for every P prime equation.

Example 2. Even Goldbach's conjecture $N = P_1 + P_2$**.** Every even number $N \ge 6$ is the sum of two primes**.**

From (6) and (7) we have Jiang's function

$$J_2(\omega)=\prod_{3\le P}(P-2)\prod_{P|N}\frac{P-1}{P-2}\ne 0\,.$$

Since $J_2(\omega)\ne 0$ as $N\to\infty$ in (2) exist infinitely many P_1 prime equations such that $N-P_1$ is a prime equation. Therefore we prove that every even number $N\ge 6$ is the sum of two primes.

From (8) we have the best asymptotic formula

$$\pi_2(N,2)=\left|\{P_1\le N, N-P_1\ \text{prime}\}\right|=\frac{J_2(\omega)\omega}{\phi^2(\omega)}\frac{N}{\log^2 N}(1+o(1)).$$

$$=2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\prod_{P|N}\frac{P-1}{P-2}\frac{N}{\log^2 N}(1+o(1))\,.$$

In 1996 we proved even Goldbach's conjecture [1]

Example 3. Prime equations $P, P+2, P+6$**.**

From (6) and (7) we have Jiang's function

$$J_2(\omega)=\prod_{5\le P}(P-3)\ne 0\,,$$

$J_2(\omega)$ is denotes the number of P prime equations such that $P+2$ and $P+6$ are prime equations. Since $J_2(\omega)\ne 0$ in (2) exist infinitely many P prime equations such that $P+2$ and $P+6$ are prime equations. Therefore we prove that there are infinitely many primes P such that $P+2$ and $P+6$ are primes.

Let $\omega=30,\ J_2(30)=2$. From (4) we have two P prime equations

$$P_3=30n+11,\quad P_5=30n+17\,.$$

From (8) we have the best asymptotic formula

$$\pi_3(N,2)=\left|\{P\le N: P+2, P+6\,\text{are primes}\}\right|=\frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}(1+o(1)).$$

Example 4. Odd Goldbach's conjecture $N=P_1+P_2+P_3$. Every odd number $N\ge 9$ is the sum of three primes.

From (6) and (7) we have Jiang's function

$$J_3(\omega)=\prod_{3\le P}\left(P^2-3P+3)\right)\prod_{P|N}\left(1-\frac{1}{P^2-3P+3}\right)\ne 0 .$$

Since $J_3(\omega)\ne 0$ as $N\to\infty$ in (2) exist infinitely many pairs of P_1 and P_2 prime equations such that $N-P_1-P_2$ is a prime equation. Therefore we prove that every odd number $N\ge 9$ is the sum of three primes.
From (8) we have the best asymptotic formula

$$\pi_2(N,3)=\left|\left\{P_1,P_2\le N:N-P_1-P_2 \text{ prime}\right\}\right|=\frac{J_3(\omega)\omega}{2\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+o(1)) .$$

$$=\prod_{3\le P}\left(1+\frac{1}{(P-1)^3}\right)\prod_{P|N}\left(1-\frac{1}{P^3-3P+3}\right)\frac{N^2}{\log^3 N}(1+o(1)) .$$

Example 5. Prime equation $P_3=P_1P_2+2$.
From (6) and (7) we have Jiang's function

$$J_3(\omega)=\prod_{3\le P}\left(P^2-3P+2\right)\ne 0$$

$J_3(\omega)$ denotes the number of pairs of P_1 and P_2 prime equations such that P_3 is a prime equation. Since $J_3(\omega)\ne 0$ in (2) exist infinitely many pairs of P_1 and P_2 prime equations such that P_3 is a prime equation. Therefore we prove that there are infinitely many pairs of primes P_1 and P_2 such that P_3 is a prime.
From (8) we have the best asymptotic formula

$$\pi_2(N,3)=\left|\left\{P_1,P_2\le N:P_1P_2+2 \text{ prime}\right\}\right|=\frac{J_3(\omega)\omega}{4\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+o(1)).$$

Note. $\deg(P_1P_2)=2$.

Example 6 [12]. Prime equation $P_3=P_1^3+2P_2^3$.
From (6) and (7) we have Jiang's function

$$J_3(\omega)=\prod_{3\le P}\left[(P-1)^2-\chi(P)\right]\ne 0 ,$$

where $\chi(P)=3(P-1)$ if $2^{\frac{P-1}{3}}\equiv 1(\bmod P)$; $\chi(P)=0$ if $2^{\frac{P-1}{3}}\not\equiv 1(\bmod P)$; $\chi(P)=P-1$ otherwise.

Since $J_3(\omega)\neq 0$ in (2) there are infinitely many pairs of P_1 and P_2 prime equations such that P_3 is a prime equation. Therefore we prove that there are infinitely many pairs of primes P_1 and P_2 such that P_3 is a prime.
From (8) we have the best asymptotic formula

$$\pi_2(N,3)=\left|\{P_1,P_2\leq N:P_1^3+2P_2^3 \text{ prime}\}\right|=\frac{J_3(\omega)\omega}{6\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+o(1)).$$

Example 7 [13]. Prime equation $P_3=P_1^4+(P_2+1)^2$.
From (6) and (7) we have Jiang's function

$$J_3(\omega)=\prod_{3\leq P}\left[(P-1)^2-\chi(P)\right]\neq 0$$

where $\chi(P)=2(P-1)$ if $P\equiv 1(\bmod 4)$; $\chi(P)=2(P-3)$ if $P\equiv 1(\bmod 8)$; $\chi(P)=0$ otherwise.
Since $J_3(\omega)\neq 0$ in (2) there are infinitely many pairs of P_1 and P_2 prime equations such that P_3 is a prime equation. Therefore we prove that there are infinitely many pairs of primes P_1 and P_2 such that P_3 is a prime.
From (8) we have the best asymptotic formula

$$\pi_2(N,3)=\left|\left\{P_1,P_2\leq N:P_3 \text{ prime}\right\}\right|=\frac{J_3(\omega)\omega}{8\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+o(1)).$$

Example 8 [14-20]. Arithmetic progressions consisting only of primes. We define the arithmetic progressions of length k.

$$P_1,P_2=P_1+d,P_3=P_1+2d,\cdots,P_k=P_1+(k-1)d,(P_1,d)=1. \qquad (10)$$

From (8) we have the best asymptotic formula

$$\pi_2(N,2)=\left|\{P_1\leq N:P_1,P_1+d,\cdots,P_1+(k-1)d \text{ are primes}\}\right|$$

$$=\frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)}\frac{N}{\log^k N}(1+o(1))..$$

If $J_2(\omega)=0$ then (10) has finite prime solutions. If $J_2(\omega)\neq 0$ then there are infinitely many primes P_1 such that $P_2,\cdots,P_k$ are primes.
To eliminate d from (10) we have

$$P_3=2P_2-P_1,\quad P_j=(j-1)P_2-(j-2)P_1, 3\le j\le k\ .$$

From (6) and (7) we have Jiang's function

$$J_3(\omega)=\prod_{3\le P<k}(P-1)\prod_{k\le P}(P-1)(P-k+1)\neq 0$$

Since $J_3(\omega)\neq 0$ in (2) there are infinitely many pairs of P_1 and P_2 prime equations such that $P_3,\cdots,P_k$ are prime equations. Therefore we prove that there are infinitely many pairs of primes P_1 and P_2 such that $P_3,\cdots,P_k$ are primes.
From (8) we have the best asymptotic formula

$$\pi_{k-1}(N,3)=\left|\left\{P_1,P_2\le N:(j-1)P_2-(j-2)P_1 \text{ prime}, 3\le j\le k\right\}\right|$$

$$=\frac{J_3(\omega)\omega^{k-2}}{2\phi^k(\omega)}\frac{N^2}{\log^k N}(1+o(1))$$

$$=\frac{1}{2}\prod_{2\le P<k}\frac{P^{k-2}}{(P-1)^{k-1}}\prod_{k\le P}\frac{P^{k-2}(P-k+1)}{(P-1)^{k-1}}\frac{N^2}{\log^k N}(1+o(1))\ .$$

Example 9. It is a well-known conjecture that one of $P,P+2,P+2^2$ is always divisible by 3. To generalize above to the $k-$primes, we prove the following conjectures. Let n be a square-free even number.

1. $P,P+n,P+n^2$,

where $3|(n+1)$.

From (6) and (7) we have $J_2(3)=0$, hence one of $P,P+n,P+n^2$ is always divisible by 3.

2. $P,P+n,P+n^2,\cdots,P+n^4$,

where $5|(n+b), b=2,3.$

From (6) and (7) we have $J_2(5)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^4$ is always divisible by 5.

3. $P, P+n, P+n^2, \cdots, P+n^6$,

where $7|(n+b), b=2,4.$

From (6) and (7) we have $J_2(7)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^6$ is always divisible by 7.

4. $P, P+n, P+n^2, \cdots, P+n^{10}$,

where $11|(n+b), b=3,4,5,9.$

From (6) and (7) we have $J_2(11)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^{10}$ is always divisible by 11.

5. $P, P+n, P+n^2, \cdots, P+n^{12}$,

where $13|(n+b), b=2,6,7,11.$

From (6) and (7) we have $J_2(13)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^{12}$ is always divisible by 13.

6. $P, P+n, P+n^2, \cdots, P+n^{16}$,

where $17|(n+b), b=3,5,6,7,10,11,12,14,15.$

From (6) and (7) we have $J_2(17)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^{16}$ is always divisible by 17.

7. $P, P+n, P+n^2, \cdots, P+n^{18}$,

where $19|(n+b), b=4,5,6,9,16.17.$

From (6) and (7) we have $J_2(19)=0$, hence one of $P, P+n, P+n^2, \cdots, P+n^{18}$ is always divisible by 19.

Example 10. Let n be an even number.

1. $P, P+n^i, i=1,3,5,\cdots,2k+1$,

From (6) and (7) we have $J_2(\omega)\neq 0$. Therefore we prove that there exist infinitely many primes P such that $P, P+n^i$ are primes for any k.

2. $P, P+n^i, i=2,4,6,\cdots,2k$.

From (6) and (7) we have $J_2(\omega)\neq 0$. Therefore we prove that there exist infinitely many primes P such that $P, P+n^i$ are primes for any k.

Example 11. Prime equation $2P_2 = P_1 + P_3$

From (6) and (7) we have Jiang's function

$$J_3(\omega) = \prod_{3 \le P} (P^2 - 3P + 2) \ne 0 .$$

Since $J_3(\omega) \ne 0$ in (2) there are infinitely many pairs of P_1 and P_2 prime equations such that P_3 is prime equations. Therefore we prove that there are infinitely many pairs of primes P_1 and P_2 such that P_3 is a prime.

From (8) we have the best asymptotic formula

$$\pi_2(N,3) = \left|\left\{P_1, P_2 \le N : P_3 \text{ prime}\right\}\right| = \frac{J_3(\omega)\omega}{2\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+o(1)).$$

In the same way we can prove $2P_2^2 = P_3 + P_1$ which has the same Jiang's function.

Jiang's function is accurate sieve function. Using it we can prove any irreducible prime equations in prime distribution. There are infinitely many twin primes but we do not have rigorous proof of this old conjecture by any method [20]. As strong as the numerical evidence may be, we still do not even know whether there are infinitely many pairs of twin primes [21]. All the prime theorems are conjectures except the prime number theorem, because they do not prove the simplest twin primes. They conjecture that the prime distribution is randomness [12-25], because they do not understand theory of prime numbers.

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