New evidence for colored leptons

M. Pitkänen
Email: matpitka@mappi.helsinki.fi.
http://www.helsinki.fi/∼matpitka/.

November 28, 2008

Contents

Abstract

The recent discovery of CDF anomaly suggest the existence of a new long-lived particle which means a dramatic deviation from standard model. This article summarizes the quantum model of CDF anomaly. The anomaly is interpreted in terms of production of τ-pions which can be regarded as pion like bound states of color octet excitations of τ-leptons and corresponding neutrinos. Colored leptons are one of the basic predictions of TGD distinguishing it from standard model and for 18 years ago were applied to explain the anomalous production of electron-positron pairs in heavy ion collisions near Coulomb wall. First it is shown that the model explains the basic characteristics of the anomaly. Then various alternatives generalizing the earlier model for electro-pion production are discussed and a general formula for differential cross section is deduced. Three alternatives inspired by eikonal approximation generalizing the earlier model inspired by Born approximation to a perturbation series in the Coulomb interaction potential of the colliding charges. The requirement of manifest relativistic invariance for the formula of differential cross section leaves only two options, call them I and II. The production cross section for τ-pion is estimated and found to be consistent with the reported cross section of order 100 pb for option I using same energy cutoff for lepto-pions as in the model for electro-pion production. For option II the production cross section is by several orders of magnitude too small under these assumptions. Since the model involves only fundamental coupling constants, the result can be regarded as a further success of the τ-pion model of CDF anomaly. Analytic expressions for the production amplitude are deduced in the Appendix as a Fourier transform for the inner product of the non-orthogonal magnetic and electric fields of the colliding charges in various kinematical situations. This allows to reduce numerical integrations to an integral over the phase space of lepto-pion and gives a tight analytic control over the numerics.

1 Introduction

Colored leptons represent one of the basic predictions of TGD distinguishing it from standard model [2]. For about eighteen years ago [8] I applied the notion of color octet electron to explain the anomalous production of electron-positron pairs in heavy ion collisions near Coulomb wall [20, 21, 14, 15]. The older version of the model can be found in [7] and the recent version in [6]. There is a lot of literature about this anomaly dating back to seventies [9, 10, 11, 12, 13], which for some reason has been forgotten more or less completely. In fact, the article about the first version of the model published in International Journal of Theoretical Physics [8] remained one of the last of my publications. After my lifework was doomed to be complete nonsense by two young professors from Finland, a censorship was established preventing publishing in both journals and in arXiv. Only the articles based on talks in some conferences such as CASYS and the emergence of web made it possible for some information to leak through the communication wall.

High energy limit for lepto-hadron physics predicts leptonic analogs of hadronic jets and the old evidence [16, 17, 18, 19] for the surplus of electrons and positrons in hadronic reactions might relate to this. Quite recently, similar surplus have been discovered in cosmic ray spectrum [28, 27].

Electro-pions could explain also the orto-positronium decay rate anomaly [25, 26]. Lepto-pions could relate also to the anomalous value of the anomalous magnetic moment of muon [34] and the magnetic moments of leptons give strong constraints on the parameters of the the possibly existing lepto-hadron physics.

It is quite possible that lepto-hadrons exists only in the dark sectors of quantum TGD characterized by non-standard values of Planck constant. Since different values of Planck constant correspond to different pages of the book like structure formed by the generalized 8-D imbedding space, the absence of direct couplings of ordinary intermediate gauge bosons to colored leptons is predicted for dark leptonic color. This would explain why colored states of leptons do not contribute to the decay widths of intermediate gauge bosons. More generally, TGD allows a fractal hierarchy of dark copies of the standard model physics consistent with the decay rates of intermediate gauge bosons. In particular, nuclear string model [4, 5] assumes that scaled variants of quarks play a key role in nuclear physics. Nucleons are assumed to be connected by color bonds having exotic light quark and antiquark at their ends. The model predicts the existence of exotic nuclei with excitation energies in X ray range [4, 5] and there are anomalies supporting this prediction.

TGD predicts that also μ and τ should possess colored excitations. Karmen anomaly [23, 24] provides less direct evidence for μ-pions and for about year ago evidence for colored excitations of muons emerged [32, 33]. At the end of October 2008, CDF collaboration published an e-print [39] describing its experimental findings suggesting the existence of a new long-lived particle producing muons in their decays. Few days later CDF collaboration published an e-print proposing that the anomaly involves a cascade like decay of at least three new particles [40] with masses coming in good approximation as powers of two. In TGD framework this suggests strongly that p-adically scaled up variants of same particle are in question.

It soon became clear that colored τ-leptons could explain CDF anomaly [6]. The reported lifetime of the long-lived state is consistent with the prediction for the lifetime of charged τ-pion in weak decays. The masses coming as powers of two could be interpreted in terms of p-adically scaled variants of neutral τ-pion. The observed muon jets result from the peculiar decay kinematics in which neutral τ-pion decays to three τ-pions with p-adically halved mass scale and therefore almost at rest (leptonic variants of quark jets are not in question). That muons are favored follows from the fact that the allowed phase space volume is much smaller for electrons and τ-leptons. If τ and its neutrino and their colored variants have nearly the same masses as required by the jet structure, rather precise information about the mass spectrum of τ-mesons can be deduced and a rich spectroscopy would be waiting to be discovered.

Besides dezcribing the basic model, this article summarizes the quantum model for lepto-pion production and applies it to the recent situation. Various alternatives generalizing the earlier model for electro-pion production are discussed and a general formula for the differential cross section is deduced. Three alternatives inspired by eikonal approximation generalizing the earlier model inspired by Born approximation to a perturbation series in the Coulombic interaction potential of the colliding charges. The requirement of manifest relativistic invariance for the formula of differential cross section leaves only two options, call them I and II. The production cross section for τ-pion is estimated and found to be consistent with the reported cross section of order 100 pb for option I using same energy cutoff for lepto-pions as in the model for electropion production. For option II the production cross section is by several orders of magnitude too small under these assumptions. Since the model involves only fundamental coupling constants, the result can be regarded as a further success of the τ-pion model of CDF anomaly. Analytic expressions for the production amplitude are deduced in the Appendix as a Fourier transform for the inner product of the non-orthogonal magnetic and electric fields of the colliding charges in various kinematical situations. This allows to reduce numerical integrations to an integral over the phase space of lepto-pion and gives a tight analytic control over the numerics.

The updated model predicts also the production cross section for the production of electro-pions in heavy ion collisions correctly if maximal impact parameter for heavy ion collisions is about 1 Angstrom [6]. Hence the model possesses remarkable internal consistency and there are reasons for certain optimism after 31 years of futile attempts to communicate TGD to the physics community: perhaps Nature itself forces the colleagues to take TGD seriously when nothing else helps.

2 Evidence for τ-hadrons

The evidence for τ-leptons came in somewhat funny but very pleasant manner. During my friday morning blog walk, the day next to my birthday October 30, I found that Peter Woit had told in his blog about a possible discovery of a new long-lived particle by CDF experiment [35] emphasizing how revolutionary finding is if it is real. There is a detailed paper [39] with title *Study of multi-muon events produced in p-pbar collisions at* $\sqrt(s) = 1.96$ *TeV* by CDF collaboration added to the ArXiv October 29 - the eve of my birthday. I got even second gift posted to arXiv the very same day and reporting an anomalously high abundance of positrons in cosmic ray radiation [27]. Both of these articles give support for basic predictions of TGD differentiating between TGD and standard model and its generalizations.

2.1 The first gift

A brief summary of Peter Woit about the finding gives a good idea about what is involved.

The article originates in studies designed to determine the b-bbar cross-section by looking for events, where a b-bbar pair is produced, each component of the pair decaying into a muon. The b-quark lifetime is of order a picosecond, so b-quarks travel a millimeter or so before decaying. The tracks from these decays can be reconstructed using the inner silicon detectors surrounding the beam-pipe, which has a radius of 1.5 cm. They can be characterized by their impact parameter, the closest distance between the extrapolated track and the primary interaction vertex, in the plane transverse to the beam.

If one looks at events where the b-quark vertices are directly reconstructed, fitting a secondary vertex, the cross-section for b-bbar production comes out about as expected. On the other hand, if one just tries to identify b-quarks by their semi-leptonic decays, one gets a value for the b-bbar cross-section that is too large by a factor of two. In the second case, presumably there is some background being misidentified as b-bbar production.

The new result is based on a study of this background using a sample of events containing two muons, varying the tightness of the requirements on observed tracks in the layers of the silicon detector. The background being searched for should appear as the requirements are loosened. It turns out that such events seem to contain an anomalous component with unexpected properties that disagree with those of the known possible sources of background. The number of these anomalous events is large (tens of thousands), so this cannot just be a statistical fluctuation.

One of the anomalous properties of these events is that they contain tracks with large impact parameters, of order a centimeter rather than the hundreds of microns characteristic of b-quark decays. Fitting this tail by an exponential, one gets what one would expect to see from the decay of a new, unknown particle with a lifetime of about 20 picoseconds. These events have further unusual properties, including an anomalously high number of additional muons in small angular cones about the primary ones.

The lifetime is estimated to be considerably longer than b quark life time and below the lifetime 89.5 ps of $K_{0,s}$ mesons. The fit to the tail of "ghost" muons gives the estimate of 20 picoseconds.

2.2 The second gift

In October 29 also another remarkable paper [27] had appeared in arXiv. It was titled *Observation of an anomalous positron abundance in the cosmic radiation.* PAMELA collaboration finds an excess of cosmic ray positron at energies $10 \rightarrow 50$ GeV. PAMELA anomaly is discussed in Resonaances blog [36]. ATIC collaboration in turn sees an excess of electrons and positrons going all the way up to energies of order 500-800 GeV [28].

Also Peter Woit refers to these cosmic ray anomalies and also to the article *LHC Signals for a SuperUnified Theory of Dark Matter* by Nima Arkadi-Hamed and Neal Weiner [29], where a model of dark matter inspired by these anomalies is proposed together with a prediction of lepton jets with invariant masses with mass scale of order GeV. The model assumes a new gauge interaction for dark matter particles with Higgs and gauge boson masses around GeV. The prediction is that LHC should detect "lepton jets" with smaller angular separations and GeV scale invariant masses.

2.3 Explanation of the CDF anomaly

Consider first the CDF anomaly. TGD predicts a fractal hierarchy of QCD type physics. In particular, colored excitations of leptons are predicted to exist. Neutral lepto-pions would have mass only slightly above two times the charged lepton mass. Also charged lepto-pions are predicts and their masses depend on what is the p-adic mass scale of neutrino and it is not clear whether it is much longer than that for charge colored lepton as in the case of ordinary leptons.

1. There exists a considerable evidence for colored electrons [6]. The anomalous production of electron positron pairs discovered in heavy ion collisions can be understood in terms of decays of electro-pions produced in the strong non-orthogonal electric and magnetic fields created in these collisions. The action determining the production rate would be proportional to the product of the lepto-pion field and highly unique "instanton" action for electromagnetic field determined by anomaly arguments so that the model is highly predictive.

2. Also the .511 MeV emission line [30, 31] from the galactic center can be understood in terms of decays of neutral electro-pions to photon pairs. Electro-pions would reside at magnetic flux tubes of strong galactic magnetic fields. It is also possible that these particles are dark in TGD sense.

3. There is also evidence for colored excitations of muon and muo-pion [32, 33]. Muo-pions could be produced by the same mechanism as electro-pions in high energy collisions of charged particles when strong non-orthogonal magnetic and electric fields are generated.

Also τ-hadrons are possible and CDF anomaly can be understood in terms of a production of higher energy τ-hadrons as the following argument demonstrates.

1. τ-QCD at high energies would produce "lepton jets" just as ordinary QCD. In particular, muon pairs with invariant energy below $2m(\tau) \sim 3.6$ GeV would be produced by the decays of neutral τ-pions. The production of monochromatic gamma ray pairs is predicted to dominate the decays. Note that the space-time sheet associated with both ordinary hadrons and τ lepton correspond to the p-adic prime $M_{107} = 2^{107} - 1$.

2. The model for the production of electro-pions in heavy ion collisions suggests that the production of τ-pions could take place in higher energy collisions of protons generating very strong non-orthogonal magnetic and electric fields. This This would reduce the model to the quantum model for electro-pion production.

3. One can imagine several options for the detailed production mechanism.

   (a) The decay of *virtual* τ-pions created in these fields to pairs of lepto-baryons generates lepton jets. Since colored leptons correspond to color octets, lepto-baryons could correspond to states of form LLL or $L\overline{L}L$.

   (b) The option inspired by a blog discussion with Ervin Goldfein is that a coherent state of τ-pions is created first and is then heated to QCD plasma like state producing the lepton jets like in QCD. The linear coupling to $E \cdot B$ defined by em fields of colliding nucleons would be analogous to the coupling of harmonic oscillator to constant force and generate the coherent state.

   (c) The option inspired by CDF model [40] is that a p-adically scaled up variant of *on mass shell* neutral τ-pion having $k = 103$ and 4 times larger mass than $k = 107$ τ-pion is produced and decays to three $k = 105$ τ-pions with $k = 105$ neutral τ-pion in turn decaying to three $k = 107$ τ-pions.

4. The basic characteristics of the anomalous muon pair prediction seems to fit with what one would expect from a jet generating a cascade of τ-pions. Muons with both charges would be produced democratically from neutral τ-pions; the number of muons would be anomalously high; and the invariant masses of muon pairs would be below 3.6 GeV for neutral τ-pions and below 1.8 GeV for charged τ-pions if colored neutrinos are light.

5. The lifetime of 20 ps can be assigned with charged τ-pion decaying weakly only into muon and neutrino. This provides a killer test for the hypothesis. In absence of CKM mixing for colored neutrinos, the decay rate to lepton and its antineutrino is given by

$$\Gamma(\pi_\tau \rightarrow L + \overline{\nu}_L) = \frac{G^2 m(L)^2 f^2(\pi)(m(\pi_\tau)^2 - m(L)^2)^2}{4\pi m^3(\pi_\tau)} . \tag{1}$$

The parameter $f(\pi_\tau)$ characterizing the coupling of pion to the axial current can be written as $f(\pi_\tau) = r(\pi_\tau)m(\pi_\tau)$. For ordinary pion one has $f(\pi) = 93$ MeV and $r(\pi) = .67$. The decay rate for charged τ-pion is obtained by simple scaling giving

$$\begin{aligned} \Gamma(\pi_\tau \rightarrow L + \overline{\nu}_L) &= 8x^2u^2y^3(1-z^2)\frac{1}{cos^2(\theta_c)}\Gamma(\pi \rightarrow \mu + \overline{\nu_\mu}) \ , \\ x &= \frac{m(L)}{m(\mu)} \ , \ y = \frac{m(\tau)}{m(\pi)} \ , \ z = \frac{m(L)}{2m(\tau)} \ , \ u = \frac{r(\pi_\tau)}{r(\pi)} \ . \end{aligned} \tag{2}$$

If the p-adic mass scale of the colored neutrino is same as for ordinary neutrinos, the mass of charged lepto-pion is in good approximation equal to the mass of τ and the decay rates to τ and electron are for the lack of phase space much slower than to muons so that muons are produced preferentially.

6. For $m(\tau) = 1.8$ GeV and $m(\pi) = .14$ GeV and the same value for f_π as for ordinary pion the lifetime is obtained by scaling from the lifetime of charged pion about 2.6×10^{-8} s. The prediction is 3.31×10^{-12} s to be compared with the experimental estimate about 20×10^{-12} s. $r(\pi_\tau) = .41 r_\pi$ gives a correct prediction. Hence the explanation in terms of τ-pions seems to be rather convincing unless one is willing to believe in really nasty miracles.

7. Neutral τ-pion would decay dominantly to monochromatic pairs of gamma rays. The decay rate is dictated by the product of τ-pion field and "instanton" action, essentially the inner product of electric and magnetic fields and reducing to total divergence of instanton current locally. The rate is given by

$$\begin{aligned} \Gamma(\pi_\tau \rightarrow \gamma + \gamma) &= \frac{\alpha_{em}^2 m^3(\pi_\tau)}{64\pi^3 f(\pi_\tau)^2} = 2x^{-2}y \times \Gamma(\pi \rightarrow \gamma + \gamma) \ , \\ x &= \frac{f(\pi_\tau)}{m(\pi_\tau)} \ , \ y = \frac{m(\tau)}{m(\pi)} \ .\Gamma(\pi \rightarrow \gamma + \gamma) = 7.37 \ eV \ . \end{aligned} \tag{3}$$

The predicted lifetime is 1.17×10^{-17} seconds.

8. Second decay channel is to lepton pairs, with muon pair production dominating for kinematical reasons. The invariant mass of the pairs is 3.6 GeV of no other particles are produced. Whether the mass of colored neutrino is essentially the same as that of charged lepton or corresponds to the same p-adic scale as the mass of the ordinary neutrino remains an open question. If colored neutrino is light, the invariant mass of muon-neutrino pair is below 1.78 GeV.

2.4 PAMELA and ATIC anomalies

TGD predicts also a hierarchy of hadron physics assignable to Mersenne primes. The mass scale of M_{89} hadron physics is by a factor 512 higher than that of ordinary hadron physics. Therefore a very rough estimate for the nucleons of this physics is 512 GeV. This suggest that the decays of M_{89} hadrons are responsible for the anomalous positrons and electrons up to energies 500-800 GeV reported by ATIC collaboration. An equally naive scaling for the mass of pion predicts that M_{89} pion has mass 72 GeV. This could relate to the anomalous cosmic ray positrons in the energy interval 10-50 GeV reported by PAMELA collaboration. Be as it may, the prediction is that M_{89} hadron physics exists and could make itself visible in LHC.

The surprising finding is that positron fraction (the ratio of flux of positrons to the sum of electron and positron fluxes) increases above 10 GeV. If positrons emerge from secondary production during the propagation of cosmic ray-nuclei, this ratio should decrease if only standard physics is be involved with the collisions. This is taken as evidence for the production of electron-positron pairs, possibly in the decays of dark matter particles.

Lepto-hadron hypothesis predicts that in high energy collisions of charged nuclei with charged particles of matter it is possible to produce also charged electro-pions, which decay to electrons or positrons depending on their charge and produce the electronic counterparts of the jets discovered in CDF. This proposal - and more generally lepto-hadron hypothesis - could be tested by trying to find whether also electronic jets can be found in proton-proton collisions. They should be present at considerably lower energies than muon jets. I decided to check whether I have said something about this earlier and found that I have noticed years ago that there is evidence for the production of anomalous electron-positron pairs in hadronic reactions [16, 17, 18, 19]: some of it dates back to seventies.

The first guess is that the center of mass energy at which the jet formation begins to make itself visible is in a constant ratio to the mass of charged lepton. From CDF data this ratio satisfies $\sqrt{s}/m_\tau = x < 10^3$. For electro-pions the threshold energy would be around $10^{-3}x \times .5$ GeV and for muo-pions around $10^{-3}x \times 100$ GeV.

2.5 Comparison of TGD model with the model of CDF collaboration

Few days after the experimental a theoretical paper by CDF collaboration proposing a phenomenological model for the CDF anomaly appeared in the arXiv [40], and it is interesting to compare the model with TGD based model (or rather, one of them corresponding to the third option mentioned above).

The paper proposes that three new particles are involved. The masses for the particles - christened h_3, h_2, and h_1 - are assumed to be 3.6 GeV, 7.3 GeV, and 15 GeV. h_1 is assumed to be pair produced and decay to h_2 pair decaying to h_3 pair decaying to a τ pair.

h_3 is assumed to have mass 3.6 GeV and life-time of 20×10^{-12} seconds. The mass is same as the TGD based prediction for neutral τ-pion mass, whose lifetime however equals to 1.12×10^{-17} seconds ($\gamma + \gamma$ decay dominates). The correct prediction for the lifetime provides a strong support for the identification of long-lived state as charged τ-pion with mass near τ mass so that the decay to μ and its antineutrino dominates. Hence the model is not consistent with lepto-hadronic model.

p-Adic length scale hypothesis predicts that allowed mass scales come as powers of $\sqrt{2}$ and these masses indeed come in good approximation as powers of 2. Several p-adic scales appear in low energy hadron physics for quarks and this replaces Gell-Mann formula for low-lying hadron masses. Therefore one can ask whether the proposed masses correspond to neutral tau-pion with $p = M_k = 2^k - 1$, $k = 107$, and its p-adically scaled up variants with $p \simeq 2^k$, $k = 105$, and $k = 103$ (also prime). The prediction for masses would be 3.6 GeV, 7.2 GeV, 14.4 GeV.

This co-incidence cannot of course be taken too seriously since the powers of two in CDF model have a rather mundane origin: they follow from the assumed production mechanism producing 8

τ-leptons from h_1. One can however spend some time by looking whether it could be realized somehow allowing p-adically scaled up variants of τ-pion.

1. The proposed model for the production of muon jets is based on production of k=103 neutral τ-pion (or several of them) having 8 times larger mass than k=107 τ-pion in strong EB background of the colliding proton and antiproton and decaying via strong interactions to k=105 and k=107 τ-pions.

2. The first step would be

$$\pi_\tau^0(103) \rightarrow \pi_\tau^0(105) + \pi_\tau^+(105) + \pi_\tau^-(105) \ .$$

   This step is not kinematically possible if masses are obtained by exact scaling and if $m(\pi_\tau^0) < m(pi_\tau^\pm)$ holds true as for ordinary pion. p-Adic mass formulas do not however predict exact scaling. In the case that reaction is not kinematically possible, it must be replaced with a reaction in which second charged k=105 pion is virtual and decays weakly. This option however reduces the rate of the process dramatically and might be excluded.

3. Second step would consist of a scaled variant of the first step

$$\pi_\tau^0(105) \rightarrow \pi_\tau^0(107) + \pi_\tau^+(107) + \pi_\tau^-(107) \ ,$$

   where second charged pion also can be virtual and decay weakly, and the weak decays of the $\pi_\tau^\pm(105)$ with mass $2m(\tau)$ to lepton pairs. The rates for these are obtained from previous formulas by scaling.

4. The last step would involve the decays of both charged and neutral $\pi_\tau(107)$. The signature of the mechanism would be anomalous γ pairs with invariant masses $2^k \times m(\tau)$, $k = 1, 2, 3$ coming from the decays of neutral τ-pions.

5. Dimensionless four-pion coupling λ determines the decay rates for neutral τ-pions appearing in the cascade. Rates are proportional to phase space-volumes, which are rather small by kinetic reasons.

The total cross section for producing single lepto-pion can be estimated by using the quantum model for lepto-pion production. Production amplitude is essentially Coulomb scattering amplitude for a given value of the impact parameter b for colliding proton and anti-proton multiplied by the amplitude $U(b,p)$ for producing on mass shell $k = 103$ lepto-pion with given four-momentum in the fields E and B and given essentially by the Fourier transform of $E \cdot B$. The replacement of the motion with free motion should be a good approximation.

UV and IR cutoffs for the impact parameter appear in the model and are identifiable as appropriate p-adic length scales. UV cutoff could correspond to the Compton size of nucleon ($k = 107$) and IR cutoff to the size of the space-time sheets representing topologically quantized electromagnetic fields of colliding nucleons (perhaps $k = 113$ corresponding to nuclear p-adic length scale and size for color magnetic body of constituent quarks or $k = 127$ for the magnetic body of current quarks with mass scale of order MeV). If one has $\hbar/\hbar_0 = 2^7$ one could also guess that the IR cutoff corresponds to the size of dark em space-time sheet equal to $2^7L(113) = L(127)$ (or $2^7L(127) = L(141)$), which corresponds to electron's p-adic length scale. These are of course rough guesses.

Quantitatively the jet-likeness of muons means that the additional muons are contained in the cone $\theta < 36.8$ degrees around the initial muon direction. If the decay of $\pi_\tau^0(k)$ can occur to on mass shell $\pi_\tau^0(k+2)$, $k = 103, 105$, it is possible to understand jets as a consequence of the decay kinematics forcing the pions resulting as decay products to be almost at rest.

1. Suppose that the decays to three pions can take place as on mass shell decays so that pions are very nearly at rest. The distribution of decay products $\mu\overline{\nu}$ in the decays of $\pi^{\pm}(105)$ is spherically symmetric in the rest frame and the energy and momentum of the muon are given by

$$[E,p] = [m(\tau) + \frac{m^2(\mu)}{4m(\tau)}, m(\tau) - \frac{m^2(\mu)}{4m(\tau)}] \ .$$

The boost factor $\gamma = 1/\sqrt{1-v^2}$ to the rest system of muon is $\gamma = \frac{m(\tau)}{m(\mu)} + \frac{m(\mu)}{4m(\tau})) \sim 18$.

2. The momentum distribution for μ^+ coming from π^+_τ is spherically symmetric in the rest system of π^+ . In the rest system of μ^- the momentum distribution is non-vanishing only for when the angle θ between the direction of velocity of μ^- is below a maximum value of given by $tan(\theta_{max}) = 1$ corresponding to a situation in which the momentum μ^+ is orthogonal to the momentum of μ^- (the maximum transverse momentum equals to $m(\mu)v\gamma$ and longitudinal momentum becomes $m(\mu)v\gamma$ in the boost). This angle corresponds to 45 degrees and is not too far from 36.8 degrees.

3. At the next step the energy of muons resulting in the decays of $\pi^{\pm}(103)$

$$[E,p] = [\frac{m(\tau)}{2} + \frac{m^2(\mu)}{2m(\tau)}, \frac{m(\tau)}{2} - \frac{m^2(\mu)}{2m(\tau)}] \ ,$$

and the boost factor is $\gamma_1 = \frac{m(\tau)}{2m(\mu)} + \frac{m(\mu)}{2m(\tau)} \sim 9$. θ_{max} satisfies the condition $tan(\theta_{max}) = \gamma_1 v_1/\gamma v \simeq 1/2$ giving $\theta_{max} \simeq 26.6$ degrees.

If on mass shell decays are not allowed the situation changes since either of the charged pions is off mass shell. In order to obtain similar result the virtual should occur dominantly via states near to on mass shell pion. Since four-pion coupling is just constant, this option does not seem to be realized.

Quantitatively the jet-likeness of muons means that the additional muons are contained in the cone $\theta < 36.8$ degrees around the initial muon direction. If the decay of $\pi^0_\tau(k)$ can occur to on mass shell $\pi^0_\tau(k+2)$, $k = 103, 105$, it is possible to understand jets as a consequence of the decay kinematics forcing the pions resulting as decay products to be almost at rest.

1. Suppose that the decays to three pions can take place as on mass shell decays so that pions are very nearly at rest. The distribution of decay products $\mu\overline{\nu}$ in the decays of $\pi^{\pm}(105)$ is spherically symmetric in the rest frame and the energy and momentum of the muon are given by

$$[E,p] = [m(\tau) + \frac{m^2(\mu)}{4m(\tau)}, m(\tau) - \frac{m^2(\mu)}{4m(\tau)}] \ .$$

The boost factor $\gamma = 1/\sqrt{1-v^2}$ to the rest system of muon is $\gamma = \frac{m(\tau)}{m(\mu)} + \frac{m(\mu)}{4m(\tau} \sim 18$.

2. The momentum distribution for μ^+ coming from π^+_τ is spherically symmetric in the rest system of π^+ . In the rest system of μ^- the momentum distribution is non-vanishing only for when the angle θ between the direction of velocity of μ^- is below a maximum value of given by $tan(\theta_{max}) = 1$ corresponding to a situation in which the momentum μ^+ is orthogonal to the momentum of μ^- (the maximum transverse momentum equals to $m(\mu)v\gamma$ and longitudinal momentum becomes $m(\mu)v\gamma$ in the boost). This angle corresponds to 45 degrees and is not too far from 36.8 degrees.

3. At the next step the energy of muons resulting in the decays of $\pi^{\pm}(103)$

$$[E,p] = [\frac{m(\tau)}{2} + \frac{m^2(\mu)}{2m(\tau)}, \frac{m(\tau)}{2} - \frac{m^2(\mu)}{2m(\tau)}] \ ,$$

   and the boost factor is $\gamma_1 = \frac{m(\tau)}{2m(\mu)} + \frac{m(\mu)}{2m(\tau)} \sim 9$. θ_{max} satisfies the condition $tan(\theta_{max}) = \gamma_1 v_1/\gamma v \simeq 1/2$ giving $\theta_{max} \simeq 26.6$ degrees.

If on mass shell decays are not possible, the situation changes since either of the charged pions is off mass shell. In order to obtain similar result the virtual should occur dominantly via states near to on mass shell pion. Since four-pion coupling is just constant, this option does not seem to be realized.

2.6 Does the production of lepto-pions involve a phase transition increasing Planck constant?

The critical argument of Tommaso Dorigo in his blog inspired an attempt to formulate more precisely the hypothesis $\sqrt{s}/m_\tau > x < 10^3$. This led to the realization that a phase transition increasing Planck constant might happen in the production process as also the model for the production of electro-pions requires.

Suppose that the instanton coupling gives rise to *virtual* neutral lepto-pions which ultimately produce the jets (this is first of the three models that one can imagine). E and B could be associated with the colliding proton and antiproton or quarks.

1. The amplitude for lepto-pion production is essentially Fourier transform of $E \cdot B$, where E and B are the non-orthogonal electric and magnetic fields of the colliding charges. At the level of scales one has $\tau \sim \hbar/E$, where τ is the time during which $E \cdot B$ is large enough during collision and E is the energy scale of the virtual lepto-pion giving rise to the jet.

2. In order to have jets one must have $m(\pi_\tau) << E$. If the scaling law $E \propto \sqrt{s}$ hold true, one indeed has $\sqrt{s}/m(\pi_\tau) > x < 10^3$.

3. If proton and antiproton would move freely, τ would be of the order of the time for proton to move through a distance, which is 2 times the Lorentz contracted radius of proton: $\tau_{free} = 2 \times \sqrt{1-v^2}R_p/v = 2\hbar/E_p$. This would give for the energy scale of virtual τ-pion the estimate $E = \hbar/\tau_{free} = \sqrt{s}/4$. $x = 4$ is certainly quite too small value. Actually $\tau > \tau_{free}$ holds true but one can argue that without new physics the time for the preservation of $E \cdot B$ cannot be by a factor of order 2^8 longer than for free collision.

4. For a colliding quark pair one would have $\tau_{free} = 4\hbar/\sqrt{s_{pair}(s)}$, where $\sqrt{s_{pair}(s)}$ would be the typical invariant energy of the pair which is exponentially smaller than $\sqrt{s}$. Somewhat paradoxically from classical physics point of view, the time scale would be much longer for the collision of quarks than that for proton and antiproton.

The possible new physics relates to the possibility that lepto-pions are dark matter in the sense that they have Planck constant larger than the standard value.

1. Suppose that the produced lepto-pions have Planck constant larger than its standard value $\hbar_0$. Originally the idea was that larger value of $\hbar$ would scale up the production cross section. IT turned out that this is not the case. For $exp(iS)$ option the lowest order contribution is not affected by the scaling of $\hbar$ and for $exp(iS)-1$ option the lowest order contribution scales down as $1/hbar^2$. The improved formulation of the model however led to a correct order of magnitude estimates for the production cross section.

2. Assume that a phase transition increasing Planck constant occurs during the collision. Hence τ is scaled up by a factor $y = \hbar/\hbar_0$. The inverse of the lepto-pion mass scale is a natural candidate for the scaled up dark time scale. $\tau(\hbar_0) \sim \tau_{free}$, one obtains $y \sim \sqrt{s_{min}}/4m(\pi_\tau) \leq 2^8$ giving for proton-antiproton option the first guess $\sqrt{s}/m(\pi_\tau) > x < 2^{10}$. If the value of y does not depend on the type of lepto-pion, the proposed estimates for muo- and electro-pion follow.

3. If the fields E and B are associated with colliding quarks, only colliding quark pairs with $\sqrt{s_{pair}(s)} > (>)m(\pi_\tau)$ contribute giving $y_q(s) = \sqrt{s_{pair}(s)/s} \times y$.

If the τ-pions produced in the magnetic field are on-mass shell τ-pions with $k = 113$, the value of $\hbar$ would satisfy $\hbar/\hbar_0 < 2^5$ and $\sqrt{s}/m(\pi_\tau) > x < 2^7$.

3 Quantum model for lepto-pion production

The basic question is what quantum model means. The most natural thing is to start from Coulomb scattering and multiply Coulomb scattering amplitude for a given impact parameter value b with the amplitude for lepto-pion production. This because the classical differential cross section given by $2\pi bdb$ in Coulomb scattering equals to the quantum cross section. One might however argue that on basis of $S = 1 + T$ decomposition of S-matrix the lowest order contribution to lepto-pion production in quantum situation corresponds to the absence of any scattering. The lepto-pion production amplitude is indeed non-vanishing also for the free motion of nuclei. The resolution of what looks like a paradox could come from many-sheeted space-time concept: if no scattering occurs, the space-time sheets representing colliding nuclei do not touch and all and there is no interference of em fields so that there is no lepto-pion production. It turns however that lowest order contribution indeed corresponds to the absence of scattering in the model that works.

3.1 Two possible approaches

One can imagine two approaches to the construction of the model for production amplitude in quantum case.

The first approach is based on eikonal approximation [41]. Eikonal approximation applies at high energy limit when the scattering angle is small and one can approximate the orbit of the projectile with a straight orbit.

The expression for the scattering amplitude in eikonal approximation reads as

$$\begin{aligned} f(\theta,\phi) &= \frac{k}{2\pi i}\int d^2b exp(-ik\cdot b)exp(i\xi(b))-1) \ , \\ \xi(b) &= \frac{-m}{k\hbar^2}\int_{z=-\infty}^{z=\infty} dzV(z,b) \ , \\ \frac{d\sigma}{d\Omega} &= |f^2| \ . \end{aligned} \tag{4}$$

as one expands the exponential in lowest in spherically symmetric potential order one obtains the

$$f(\theta,\phi) \simeq -\frac{m}{2\pi\hbar^2}\int J_0(k_Tb)\xi(b)bdb \ . \tag{5}$$

The challenge is to find whether it is possible to generalize this expression so that it applies to the production of lepto-pions.

1. The simplest guess is that one should multiply the eikonal amplitude with the dimensionless amplitude $A(b)$:

$$\begin{aligned} f(\theta,\phi) & \rightarrow & f(\theta,\phi,p) = \frac{k}{2\pi i}\int d^2b exp(-ik\cdot b)exp(i\xi(b)) - 1)A(b,p) \\ & \simeq & -\frac{m}{2\pi\hbar^2}\int J_0(k_Tb)\xi(b)A(b,p)bdb \ . \end{aligned} \tag{6}$$

2. Amplitude squared must give differential cross section for lepto-pion production and scattering

$$\begin{aligned} d\sigma & = & |f(\theta,\phi,p)|^2 d\Omega d^3n \ , \\ d^3n & = & Vd^3p \ . \end{aligned} \tag{7}$$

   This requires an explicit introduction of a volume factor V via a spatial cutoff. This cutoff is necessary for the coordinate z in the case of Coulomb potential, and would have interpretation in terms of a finite spatio-temporal volume in which the space-time sheets of the colliding particles are in contact and fields interfere.

3. There are several objections against this approach. The loss of a manifest relativistic invariance in the density of states factor for lepto-pion does not look nice. One must keep count about the scattering of the projectile which means a considerable complication from the point of view of numerical calculations. In classical picture for orbits the scattering angle in principle is fixed once impact parameter is known so that the introduction of scattering angles does not look logical.

Second approach starts from the classical picture in which each impact parameter corresponds to a definite scattering angle so that the resulting amplitude describes lepto-pion production amplitude and says nothing about the scattering of the projectile. This approach is more in spirit with TGD since classical physics is exact part of quantum TGD and classical orbit is absolutely real from the point of view of lepto-pion production amplitude.

1. The counterpart of the eikonal exponent has interpretation as the exponent of classical action associated with the Coulomb interaction

$$S(b) = \int_\gamma V ds \tag{8}$$

   along the orbit γ of the particle, which can be taken also as a real classical orbit but will be approximated with rectilinear orbit in sequel.

2. The first guess for the production amplitude is

$$\begin{aligned} f(p) & = & \int d^2b exp(-i\Delta k(b)\cdot b)exp[\frac{i}{\hbar}S(b)]A(b,p) \\ & = & \int J_0(k_T(b)b)(1+\frac{i}{\hbar}\int_{z=-a}^{z=a} dzV(z,b) + ..)A(b,p) \ . \end{aligned} \tag{9}$$

Δk is the change of the momentum in the classical scattering and in the scattering plane. The cutoff $|z| \leq a$ in the longitudinal direction corresponds to a finite imbedding space volume inside which the space-time sheets of target and projectile are in contact.

3. The production amplitude is non-trivial even if the interaction potential vanishes being given by

$$f(p) = \int d^2bexp(-ik\cdot b))A(b,p) = 2\pi int J_0(k_T(b)b) \times A(b,p)bdb \ . \quad (10)$$

This formula can be seen as a generalization of quantum formula in the sense that incoherent integral over production probabilities at various values of b is replaced by an integral over production amplitude over b so that interference effects become possible.

4. This result could be seen as a problem. On basis of $S = 1 + iT$ decomposition corresponding to free motion and genuine interaction, one could argue that since the exponent of action corresponds to S, $A(p,b)$ vanishes when the space-time sheets are not in contact. The improved guess for the amplitude is

$$\begin{aligned} f(p) &= \int d^2bexp(-ik\cdot b)exp(\frac{i}{\hbar}S(b))A(b,p) \\ &= \int J_0(k_T(b)b)(\frac{i}{\hbar}\int_{z=-a}^{z=a} V(z,b) + ..)A(b,p) \ . \end{aligned} \quad (11)$$

This would mean that there would be no classical limit when coherence is assumed to be lost. At this stage one must keep mind open for both options.

5. The dimension of $f(p)$ is $L^2/\hbar$

$$d\sigma = |f(p)|^2 \frac{d^3p}{2E_p(2\pi)^3} \ . \quad (12)$$

has correct dimension. This model will be considered in sequel. The earlier work in [7] was however based on the first option.

3.2 Formula for the production cross section

The expression for the differential cross section for lepto-pion production is given by

$$d\sigma = KF^2|\int (CUT_1 + CUT_2)bdb|^2 \frac{d^3p}{2E} \ , \quad (13)$$

This expression and also the expressions of the integrals of CUT_1 and CUT_2 are calculated explicitly as powers series of the impact parameter in the APPENDIX.

1. For $exp(iS)$ option one has

$$K = (Z_1Z_2)^2\alpha_{em}^4 N_c^2[\frac{m(\pi_L)}{f(\pi_L)}]^2\frac{1}{(2\pi)^{13}} ,$$
$$F = 1 . \quad (14)$$

2. For $exp(iS)-1$ option one has

$$K = (Z_1Z_2)^4\alpha_{em}^6 N_c^2[\frac{m(\pi_L)}{f(\pi_L)}]^2\frac{1}{(2\pi)^{13}} ,$$
$$F = 2\langle\langle ln(\frac{\sqrt{a^2-b^2}+a}{b})\rangle\rangle . \quad (15)$$

In the approximation that F is constant the two lowest order predictions are related by a scaling factor

$$R = (Z_1Z_2\alpha_{em})^2F^2 . \quad (16)$$

3.3 Numerical estimate for the production cross section

The numerical estimate of the cross section involves some delicacies. The model has purely physical cutoffs which must be formulated in a precise manner.

1. Since energy conservation is not coded into the model, some assumption about the maximal τ-pion energy in cm system expressed as a fraction ϵ of proton's center of mass energy is necessary. Maximal fraction corresponds to the condition $m(\pi_\tau)\leq m(\pi_\tau)\gamma_1\leq \epsilon m_p\gamma_{cm}$ in cm system giving $[m(\pi_\tau)/(m_p\gamma_{cm})\leq\epsilon\leq 1$. γ_{cm} can be deduced from the center of mass energy of proton as $\gamma_{cm}=\sqrt{s}2m_p$, $\sqrt{s}=1.96$ TeV. This gives $1.6\times 10^{-2}<\epsilon<1$ in a reasonable approximation. It is convenient to parameterize ϵ as

   $$\epsilon = (1+\delta)\times\frac{m(\pi_\tau)}{m_p}\times\frac{1}{\gamma_{cm}} .$$

   The coordinate system in which the calculations are carried out is taken to be the rest system of (say) antiproton so that one must perform a Lorentz boost to obtain upper and lower limits for the velocity of τ-pion in this system. In this system the range of γ_1 is fixed by the maximal cm velocity fixed by ϵ and the upper/lower limit of γ_1 corresponds to a direction parallel/opposite to the velocity of proton.

2. By Lorentz invariance the value of the impact parameter cutoff b_{max} should be expressible in terms τ-pion Compton length and the center of mass energy of the colliding proton and the assumption is that $b_{max}=\gamma_{cm}\times\hbar/m(\pi_\tau)$, where it is assumed $m(\pi_\tau)=8m(\tau)$. The production cross section does not depend much on the precise choice of the impact parameter cutoff b_{max} unless it is un-physically large in which case b_{max}^2 proportionality is predicted.

The numerical estimate for the production cross section involves some delicacies.

1. The power series expansion of the integral of CUT_1 using partial fraction representation does not converge since that roots $c_\pm$ are very large in the entire integration region. Instead the approximation $A_1 \simeq iBcos(\psi)/D$ simplifying considerably the calculations can be used. Also the value of b_1L is rather small and one can use stationary phase approximation for CUT_2. It turns out that the contribution of CUT_2 is negligible as compared to that of CUT_1.

2. Since the situation is singular for $\theta = 0$ and $\phi = 0$ and $\phi = \pi/2$ (by symmetry it is enough to calculate the cross section only for this kinematical region), cutoffs

$$\theta \in [\epsilon_1, (1-\epsilon_1)] \times \pi \ , \quad \phi \in [\epsilon_1, (1-\epsilon_1)] \times \pi/2 \ , \quad \epsilon_1 = 10^{-3} \ .$$

   The result of the calculation is not very sensitive to the value of the cutoff.

3. Since the available numerical environment was rather primitive (MATLAB in personal computer), the requirement of a reasonable calculation time restricted the number of intervals in the discretization for the three kinematical variables γ, θ, ϕ to be below $N_{max} = 80$. The result of calculation did not depend appreciably on the number of intervals above $N = 40$ for γ_1 integral and for θ and ϕ integrals even $N = 10$ gave a good estimate.

The calculations were carried for the $exp(iS)$ option since in good approximation the estimate for $exp(iS) - 1$ model is obtained by a simple scaling. $exp(iS)$ model produces a correct order of magnitude for the cross section whereas $exp(iS) - 1$ variant predicts a cross section, which is by several orders of magnitude smaller by downwards α_{em}^2 scaling. As I asked Tommaso Dorigo for an estimate for the production cross section in his first blog posting [38], he mentioned that authors refer to a production cross section is 100 nb which looks to me suspiciously large (too large by three orders of magnitude), when compared with the production rate of muon pairs from b-bbar. $\delta = 1.5$ which corresponds to τ-pion energy 36 GeV gives the estimate $\sigma = 351$ nb. The energy is suspiciously high.

In fact, in the recent blog posting of Tommaso Dorigo [37] a value of order .1 nb for the production cross section was mentioned. Electro-pions in heavy ion collisions are produced almost at rest and one has $\Delta v/v \simeq .2$ giving $\delta = \Delta E/m(\pi) \simeq 2 \times 10^{-3}$. If one believes in fractal scaling, this should be at least the order of magnitude also in the case of τ-pion. This would give the estimate $\sigma = 1$ nb. For $\delta = \Delta E/m(\pi) \simeq 10^{-3}$ a cross section $\sigma = .16$ nb would result.

One must of course take the estimate cautiously but there are reasons to hope that large systematic errors are not present anymore. In any case, the model can explain also the order of magnitude of the production cross section under reasonable assumptions about cutoffs.

3.4 Could lepto-hadrons correspond to dark matter?

It has been proposed that the particles produced in CDF anomaly might be decay products of dark matter particles. In TGD framework lepto-hadron hypothesis explains successfully the basic quantitative and qualitative factors about CDF anomaly and relates it to a bundle of other anomalies (as previous postings should demonstrate). The question is whether there are compelling reasons for identifying lepto-hadrons as dark matter in TGD sense.

Consider first the experimental side. The proposed identification of cosmic strings (in TGD sense) as the ultimate source of both visible and dark matter discussed in [3] does not exclude the possibility that a considerable portion of topologically condensed cosmic strings have decayed to some light particles. In particular, this could be the situation in the galactic nuclei.

The idea that lepto-hadrons might have something to do with the dark matter has popped up now and then during the last decade but for some reason I have not taken it seriously. Situation changed towards the end of the year 2003. There exist now detailed maps of the dark matter in

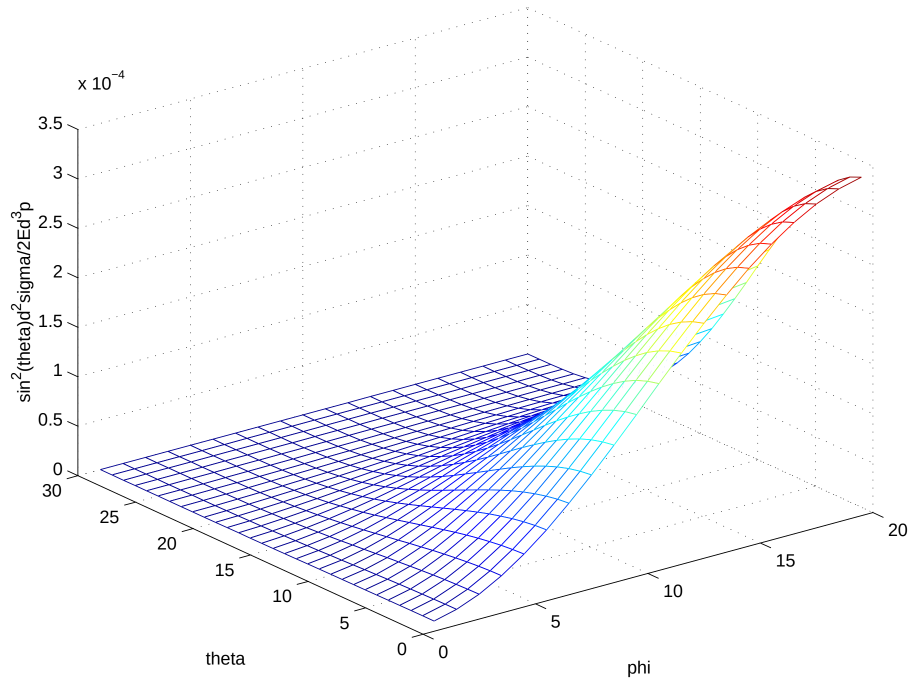

Figure 1: Differential cross section $sin^2(\theta) \times \frac{d^2\sigma}{2Ed^3p}$ for τ-pion production for $\gamma_1 = 1.090 \times 10^3$ in the rest system of antiproton for $\delta = 1.5$. $m(\pi_\tau)$ defines the unit of energy and nb is the unit for cross section. The ranges of θ and ϕ are $(0, \pi)$ and $(0, \pi/2)$.

the center of galaxy and it has been found that the density of dark matter correlates strongly with the intensity of monochromatic photons with energy equal to the rest mass of electron [42].

The only explanation for the radiation is that some yet unidentified particle of mass very nearly equal to $2m_e$ decays to an electron positron pair. Electron and positron are almost at rest and this implies a high rate for the annihilation to a pair of gamma rays. A natural identification for the particle in question would be as a lepto-pion (or rather, electro-pion). By their low mass lepto-pions, just like ordinary pions, would be produced in high abundance, in lepto-hadronic strong reactions and therefore the intensity of the monochromatic photons resulting in their decays would serve as a measure for the density of the lepto-hadronic matter. Also the presence of lepto-pionic condensates can be considered.

These findings force to take seriously the identification of the dark matter as lepto-hadrons. This is however not the only possibility. The TGD based model for tetra-neutrons discussed in [4] is based on the hypothesis that mesons made of scaled down versions of quarks corresponding to Mersenne prime M_{127} (ordinary quarks correspond to $k = 107$) and having masses around one MeV could correspond to the color electric flux tubes binding the neutrons to form a tetra-neutron. The same force would be also relevant for the understanding of alpha particles.

There are also good theoretical arguments for why lepto-hadrons should be dark matter in the sense of having a non-standard value of Planck constant.

1. Since particles with different Planck constant correspond to different pages of the book like structure defining the generalization of the imbedding space, the decays of intermediate gauge

bosons to colored excitations of leptons would not occur and would thus not contribute to their decay widths.

2. In the case of electro-pions the large value of the coupling parameter $Z_1Z_2\alpha_{em} > 1$ combined with the hypothesis that a phase transition increasing Planck constant occurs as perturbative QFT like description fails would predict that electro-pions represent dark matter. Indeed, the power series expansion of the $exp(iS)$ term might well fail to converge in this case since S is proportional to Z_1Z_2. For τ-pion production one has $Z_1 = -Z_2 = 1$ and in this case one can consider also the possibility that τ-pions are not dark in the sense of having large Planck constant. Contrary to the original expectations darkness does not affect the lowest order prediction for the production cross section of lepto-pion.

3.5 Could it have been otherwise?

To sum up, the probability that a correct prediction for the lifetime of the new particle using only known lepton masses and standard formulas for weak decay rates follows by accident is extremely low. Throwing billion times coin and getting the same result every time might be something comparable to this. And not only this: masses of the states proposed by CDF collaboration are predicted correctly as masses of neutral lepto-pions. Also the reaction mechanism producing jets has been identified and production cross section is consistent with the experimental estimate. Therefore my sincere hope is that colleagues would be finally mature to take TGD seriously. If TGD based explanation of the anomalous production of electron positron pairs in heavy ion collisions would have been taken seriously for eighteen years ago, particle physics might look quite different now.

4 APPENDIX

4.1 Evaluation of lepto-pion production amplitude

4.1.1 General form of the integral

The amplitude for lepto-pion production with four momentum

$$\begin{aligned} p &= (p_0, \bar{p}) = m\gamma_1(1, vsin(\theta)cos(\phi), vsin(\theta)sin(\phi), vcos(\theta)) \ , \\ \gamma_1 &= 1/(1-v^2)^{1/2} \ , \end{aligned} \tag{17}$$

is essentially the Fourier component of the instanton density

$$U(b,p) = \int e^{ip\cdot x} E\cdot B d^4x \tag{18}$$

associated with the electromagnetic field of the colliding nuclei.

In order to avoid cumbersome numerical factors, it is convenient to introduce the amplitude $A(b,p)$ as

$$\begin{aligned} A(b,p) &= N_0 \times \frac{4\pi}{Z_1Z_2\alpha_{em}} \times U(b,p) \ , \\ N_0 &= \frac{(2\pi)^7}{i} \end{aligned} \tag{19}$$

Coordinates are chosen so that target nucleus is at rest at the origin of coordinates and colliding nucleus moves along positive z direction in $y = 0$ plane with velocity β. The orbit is approximated with straight line with impact parameter b.

Instanton density is just the scalar product of the static electric field E of the target nucleus and magnetic field B the magnetic field associated with the colliding nucleus, which is obtained by boosting the Coulomb field of static nucleus to velocity β. The flux lines of the magnetic field rotate around the direction of the velocity of the colliding nucleus so that instanton density is indeed non vanishing.

The Fourier transforms of E and B for nuclear charge 4π (chose for convenience) giving rise to Coulomb potential $1/r$ are given by the expressions

$$\begin{aligned} E_i(k) &= N\delta(k_0)k_i/k^2 \ , \\ B_i(k) &= N\delta(\gamma(k_0 - \beta k_z))k_j\varepsilon_{ijz}e^{ik_xb}/((\frac{k_z}{\gamma})^2 + k_T^2) \ , \\ N &= \frac{1}{(2\pi)^2} \ . \end{aligned} \tag{20}$$

The normalization factor corresponds to momentum space integration measure d^4p. The Fourier transform of the instanton density can be expressed as a convolution of the Fourier transforms of E and B.

$$\begin{aligned} A(b,p) &\equiv = N_0N_1\int E(p-k)\cdot B(k)d^4k \ , \\ N_1 &= \frac{1}{(2\pi)^4} \ . \end{aligned} \tag{21}$$

Where the fields correspond to charges $\pm 4\pi$. In the convolution the presence of two delta functions makes it possible to integrate over k_0 and k_z and the expression for U reduces to a two-fold integral

$$\begin{aligned} A(b,p) &= \beta\gamma\int dk_xdk_yexp(ikxb)(k_xp_y - k_yp_x)/AB \ , \\ A &= (p_z - \frac{p_0}{\beta})^2 + p_T^2 + k_T^2 - 2k_T\cdot p_T \\ B &= k_T^2 + (\frac{p_0}{\beta\gamma})^2 \ , \\ p_T &= (p_x, p_y) \ . \end{aligned} \tag{22}$$

To carry out the remaining integrations one can apply residue calculus.

1. k_y integral is expressed as a sum of two pole contributions

2. k_x integral is expressed as a sum of two pole contributions plus two cut contributions.

4.1.2 k_y-integration

Integration over k_y can be performed by completing the integration contour along real axis to a half circle in upper half plane (see Fig. 4.1.3).

The poles of the integrand come from the two factors A and B in denominator and are given by the expressions

$$
\begin{aligned}
k_y^1 &= i(k_x^2 + (\frac{p_0}{\beta\gamma})^2)^{1/2} \quad , \\
k_y^2 &= p_y + i((p_z - \frac{p_0}{\beta})^2 + p_x^2 + k_x^2 - 2p_x k_x)^{1/2} \quad .
\end{aligned}
\tag{23}
$$

One obtains for the amplitude an expression as a sum of two terms

$$
A(b,p) \quad = \quad 2\pi i \int e^{ik_x b}(U_1 + U_2) dk_x \quad , \tag{24}
$$

corresponding to two poles in upper half plane.

The explicit expression for the first term is given by

$$
\begin{aligned}
U_1 &= RE_1 + iIM_1 \quad , \\
RE_1 &= (k_x \frac{p_0}{\beta} y - p_x re_1/2)/(re_1^2 + im_1^2) \quad , \\
IM_1 &= (-k_x p_y re_1/2K_1^{1/2} - p_x p_y K_1^{1/2})/(re_1^2 + im_1^2) \quad , \\
re_1 &= (p_z - \frac{p_0}{\beta})^2 + p_T^2 - (\frac{p_0}{\beta\gamma})^2 - 2p_x k_x \quad , \\
im_1 &= -2K_1^{1/2} p_y \quad , \\
K_1 &= k_x^2 + (\frac{p_0}{\beta\gamma})^2 \quad .
\end{aligned}
\tag{25}
$$

The expression for the second term is given by

$$
\begin{aligned}
U_2 &= RE_2 + iIM_2 \quad , \\
RE_2 &= -((k_x p_y - p_x p_y)p_y + p_x re_2/2)/(re_2^2 + im_2^2) \quad , \\
IM_2 &= (-(k_x p_y - p_x p_y) re_2/2K_2^{1/2} + p_x p_y K_2^{1/2})/(re_2^2 + im_2^2) \quad , \\
re_2 &= -(p_z - \frac{p_0}{\beta})^2 + (\frac{p_0}{\beta\gamma})^2 + 2p_x k_x + \frac{p_0}{\beta} y - \frac{p_0}{\beta} x \quad , \\
im_2 &= 2p_y K_2^{1/2} \quad , \\
K_2 &= (p_z - \frac{p_0}{\beta})^2 + \frac{p_0}{\beta} x + k_x^2 - 2p_x k_x \quad .
\end{aligned}
\tag{26}
$$

A little inspection shows that the real parts cancel each other:$RE_1 + RE_2 = 0$. A further useful result is the identity $im_1^2 + re_1^2 = re_2^2 + im_2^2$ and the identity $re_2 = -re_1 + 2p_y^2$.

4.1.3 k_x-integration

One cannot perform k_x-integration completely using residue calculus. The reason is that the terms IM_1 and IM_2 have cuts in complex plane. One can however reduce the integral to a sum of pole terms plus integrals over the cuts.

The poles of U_1 and U_2 come from the denominators and are in fact common for the two integrands. The explicit expressions for the pole in upper half plane, where integrand converges exponentially are given by

$$\begin{aligned}
re_i^2 &+ im_i^2 = 0 \ , \ i = 1,2 \ , \\
k_x &= (-b + i(-b^2 + 4ac)^{1/2})/2a \ , \\
a &= 4p_T^2 \ , \\
b &= -4((p_z - \frac{p_0}{\beta})^2 + p_T^2 - (\frac{p_0}{\beta\gamma})^2)p_x \ , \\
c &= ((p_z - \frac{p_0}{\beta})^2 + p_T^2 - (\frac{p_0}{\beta\gamma})^2)^2 + 4(\frac{p_0}{\beta\gamma})^2 p_y^2 \ . \qquad (27)
\end{aligned}$$

A straightforward calculation using the previous identities shows that the contributions of IM_1 and IM_2 at pole have opposite signs and the contribution from poles vanishes identically!

The cuts associated with U_1 and U_2 come from the square root terms K_1 and K_2. The condition for the appearance of the cut is that K_1 (K_2) is real and positive. In case of K_1 this condition gives

$$k_x = it, \ t \in (0, \frac{p_0}{\beta\gamma}) \ . \qquad (28)$$

In case of K_2 the same condition gives

$$k_x = p_x + it, \ t \in (0, \frac{p_0}{\beta} - p_z) \ . \qquad (29)$$

Both cuts are in the direction of imaginary axis.

The integral over real axis can be completed to an integral over semi-circle and this integral in turn can be expressed as a sum of two terms (see Fig. 4.1.3).

$$A(b,p) = 2\pi i(CUT_1 + CUT_2) \ . \qquad (30)$$

The first term corresponds to contour, which avoids the cuts and reduces to a sum of pole contributions. Second term corresponds to the addition of the cut contributions.

In the following we shall give the expressions of various terms in the region $\phi \in [0, \pi/2]$. Using the symmetries

$$\begin{aligned}
A(b, p_x, -p_y) &= -A(b, p_x, p_y) \ , \\
A(b, -p_x, -p_y) &= \bar{A}(b, p_x, p_y) \ . \qquad (31)
\end{aligned}$$

of the amplitude one can calculate the amplitude for other values of ϕ.

The integration variable for cuts is the imaginary part t of complexified k_x. To get a more convenient form for cut integrals one can perform a change of the integration variable

$$\begin{aligned}
cos(\psi) &= \frac{t}{(\frac{p_0}{\beta\gamma})} \ , \\
cos(\psi) &= \frac{t}{(\frac{p_0}{\beta} - p_z)} \ , \\
\psi &\in [0, \pi/2] \ . \qquad (32)
\end{aligned}$$

1. The contribution of the first cut

By a painstaking calculation one verifies that the expression for the contribution of the first cut is given by

$$\begin{aligned}
CUT_1 &= D_1 \times \int_0^{\pi/2} exp(-\frac{b}{b_0}cos(\psi))A_1 d\psi \ , \\
D_1 &= -\frac{1}{2}\frac{sin(\phi)}{sin(\theta)} \ , \ b_0 = \frac{\hbar}{m}\frac{\beta\gamma}{\gamma_1} \ , \\
A_1 &= \frac{A + iBcos(\psi)}{cos^2(\psi) + 2iCcos(\psi) + D} \ , \\
A &= sin(\theta)cos(\phi) \ , \ B = K \ , \\
C &= K\frac{cos(\phi)}{sin(\theta)} \ , \ D = -sin^2(\phi) - \frac{K^2}{sin^2(\theta)} \ , \\
K &= \beta\gamma(1 - \frac{v_{cm}}{\beta}cos(\theta)) \ , \ v_{cm} = \frac{2v}{1+v^2} \ .
\end{aligned} \tag{33}$$

The definitions of the various kinematical variables are given in previous formulas. The notation is tailored to express that A_1 is rational function of $cos(\psi)$.

1. The exponential $exp(-bcos(\psi)/b_0)$ is very small in the condition

$$cos(\psi) \geq cos(\psi_0) \equiv \frac{\hbar}{mb}\frac{\beta\gamma}{\gamma_1 cos(\phi)} \tag{34}$$

   holds true. Here $\hbar = 1$ convention has been given up to make clear that the increase of the Compton length of lepto-pion due to the scaling of $\hbar$ increase the magnitude of the contribution. If the condition $cos(\psi_0) << 1$ holds true, the integral over ψ receives contributions only from narrow range of values near the upper boundary $\psi = \pi/2$ plus the contribution corresponding to the pole of X_1. The practical condition is in terms of critical parameter b_{max} above which exponential approaches zero very rapidly.

2. For $cos(\psi_0) << 1$, that is for $b > b_{max}$ and in the approximation that the function multiplying the exponent is replaced with its value for $\psi = \pi/2$, one obtains for CUT_1 the expression

$$\begin{aligned}
CUT_1 &\simeq D_1 A_1(\psi = \pi/2)\frac{\hbar}{mb} \\
&= \frac{1}{2} \times \frac{\beta\gamma}{\gamma_1} \times \frac{\hbar}{mb} \times \frac{sin^2(\theta)cos(\phi)sin(\phi)}{sin^2(\theta)sin^2(\phi) + K^2} \ .
\end{aligned} \tag{35}$$

3. For $cos(\psi_0) >> 1$ exponential factor can be replaced by unity in good approximation and the integral reduces to an integral of rational function of $cos(\psi)$ having the form

$$D_1\frac{A + iBcos(\psi)}{cos^2(\psi) + 2iC \times cos(\psi) + D} \ . \tag{36}$$

which can be expressed in terms of the roots $c_\pm$ of the denominator as

$$D_1 \times \sum_{\pm} \frac{A \mp iBc_\pm}{cos(\psi) - c_\pm} \quad , \quad c_\pm) \quad = \quad -iC \pm \sqrt{-C^2 - D} \quad . \tag{37}$$

Integral reduces to an integral of rational function over the interval $[0,1]$ by the standard substitution $tan(\psi/2) = t$, $d\psi = 2dt/(1+t^2)$, $cos(\psi) = (1-t^2)/(1+t^2)$, $sin(\psi) = 2t/(1+t^2)$.

$$I \quad = \quad 2D_1 \sum_{\pm} \int_0^1 dt \frac{A \mp iBc_\pm}{1 - c_\pm - (1 + c_\pm)t^2} \tag{38}$$

This gives

$$I \quad = \quad 2D_1 \sum_{\pm} \frac{A \mp iBc_\pm}{s_\pm} \times arctan(\frac{1 + c_\pm}{1 - c_\pm}) \quad . \tag{39}$$

$s_\pm$ is defined as $\sqrt{1 - c_\pm^2}$ and one must be careful with the signs. This gives for CUT_1 the approximate expression

$$\begin{aligned} CUT_1 \quad &= \quad D_1 \sum_{\pm} \frac{sin(\theta)cos(\phi) \mp iKc_\pm}{s_\pm} \times arctan(\frac{1 + c_\pm}{1 - c_\pm}) \quad , \\ c_\pm \quad &= \quad \frac{-iKcos(\phi) \pm sin(\phi)\sqrt{sin^2(\theta) + K^2}}{sin(\theta)} \quad . \end{aligned} \tag{40}$$

Arcus tangent function must be defined in terms of logarithm functions since the argument is complex.

4. In the intermediate region, where the exponential differs from unity one can use expansion in Taylor polynomial to sum over integrals of rational functions of $cos(\psi)$ and one obtains the expression

$$\begin{aligned} CUT_1 \quad &= \quad D_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\frac{b}{b_0})^n I_n \quad , \\ I_n \quad &= \quad \sum_{\pm} (A \mp iBc_\pm I_n(c_\pm) \quad , \\ I_n(c) \quad &= \quad \int_0^{\pi/2} \frac{cos^n(\psi)}{cos(\psi) - c} \quad . \end{aligned} \tag{41}$$

$I_n(c)$ can be calculated explicitly by expanding in the integrand $cos(\psi))^n$ to polynomial with respect to $cos(\psi)) - c$, $c \equiv c_\pm$

$$\frac{cos^n(\psi)}{cos(\psi)-c} = \sum_{m=0}^{n-1} \left(\begin{array}{c} n \\ m \end{array} \right) c^m (cos(\psi)-c)^{n-m-1} + \frac{c^n}{cos(\psi)-c} \quad . \tag{42}$$

After the change of the integration variable the integral reads as

$$\begin{aligned}
I_n(c) &= \sum_{m=0}^{n-1} \sum_{k=0}^{n-m-1} \left(\begin{array}{c} n \\ m \end{array} \right) \left(\begin{array}{c} n-m-1 \\ k \end{array} \right) (-1)^k (1-c)^{n-m-1-k} (1+c)^k c^m I(k,n-m) \\
&+ \frac{c^n}{1-c} \times log[\frac{\sqrt{1-c}+\sqrt{1+c}}{\sqrt{1-c}-\sqrt{1+c}}] \quad , \\
I(k,n) &= 2\int dt \frac{t^{2k}}{(1+t^2)^n} \quad .
\end{aligned} \tag{43}$$

Partial integration for $I(k,n)$ gives the recursion formula

$$I(k,n) = -\frac{2^{-n+1}}{n-1} + \frac{2k-1}{2(n-1)} \times I(k-1,n-1) \quad . \tag{44}$$

The lowest term in the recursion formula corresponds to $I(0,n-k)$, can be calculated by using the expression

$$\begin{aligned}
(1+t^2)^{-n} &= \sum_{k=0}^{n} c(n,k)[(1+it)^{-k} + (1-it)^{-k}] \quad , \\
c(n,k) &= \sum_{l=0}^{n-k-1} c(n-1,k+l)2^{-l-2} + c(n-1,n-1)2^{-n+k-1} \quad .
\end{aligned} \tag{45}$$

The formula is deducible by assuming the expression to be known for n and multiplying the expression with $(1+t^2)^{-1} = [(1+it)^{-1} + (1-it)^{-1}]/2$ and applying this identity to the resulting products of $(1+it)^{-1}$ and $(1-it)^{-1}$. This gives

$$I(0,n) = -2i \sum_{k=2,n} \frac{c(n,k)}{(k-1)} [1 + 2^{(k-1)/2} sin((k-1)\pi/4)] + c(n,1) log(\frac{1+i}{1-i}) \quad . \tag{46}$$

This boils down to the following expression for CUT_1

$$\begin{aligned}
CUT_1 &= D_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\frac{b}{b_0})^n I_n \quad , \\
I_n &= \sum_{\pm} [A \mp iBc_{\pm}] I_n(cos(c_{\pm}) \quad ,
\end{aligned}$$

$$
\begin{aligned}
I_n(c) &= \sum_{m=1}^{n-1}\sum_{k=0}^{n-m-1}\left(\begin{array}{c} n \\ m \end{array}\right)\left(\begin{array}{c} n-m-1 \\ k \end{array}\right)(1-c)^{n-m-1-k}(1+c)^k c^m I(k,n-m-1) \\
&+ \frac{c^n}{1-c}\times log[\frac{\sqrt{1-c}+\sqrt{1+c}}{\sqrt{1-c}-\sqrt{1+c}}] \ , \\
I(k,n) &= -\frac{2^{-n+1}}{n-1}+\frac{2k-1}{2(n-1)}\times I(k-1,n-1) \ , \\
I(0,n) &= -2i\sum_{k=2}^{n}\frac{c(n,k)}{(k-1)}[1+2^{(k-1)/2}sin((k-1)\pi/4)]-c(n,1) \ , \\
c(n,k) &= \sum_{l=0}^{n-k-1}c(n-1,k+l)2^{-l-2}+c(n-1,n-1)2^{-n+k-1} \ . \qquad (47)
\end{aligned}
$$

This expansion in powers of $c_\pm$ fails to converge when their values are very large. This happens in the case of τ-pion production amplitude. In this case one typically has however the situation in which the conditions $A_1 \simeq iBcos(\psi)/D$ holds true in excellent approximation and one can write

$$
\begin{aligned}
CUT_1 &\simeq i\frac{D_1B}{D}\times\sum_{n=0,1,...}\frac{(-1)^n}{n!2^n}(\frac{b}{b_0})^n I_n\times \ , \\
I_n &= \int_0^{\pi/2}cos(\psi)^{n+1}d\psi = \sum_{k=0}^{n+1}\left(\begin{array}{c} n+1 \\ k \end{array}\right)\frac{i^{n-2k}-1}{n+1-2k} \ . \qquad (48)
\end{aligned}
$$

The denominator X_1 vanishes, when the conditions

$$
\begin{aligned}
cos(\theta) &= \frac{\beta}{v_{cm}} \ , \\
sin(\phi) &= cos(\psi) \qquad (49)
\end{aligned}
$$

hold. In forward direction the conditions express the vanishing of the z-component of the lepto-pion velocity in velocity cm frame as one can realize by noticing that condition reduces to the condition $v=\beta/2$ in non-relativistic limit. This corresponds to the production of lepto-pion with momentum in scattering plane and with direction angle $cos(\theta)=\beta/v_{cm}$.

CUT_1 diverges logarithmically for these values of kinematical variables at the limit $\phi\rightarrow 0$ as is easy to see by studying the behavior of the integral near as K approaches zero so that X_1 approaches zero at $sin(\phi)=cos(\Phi)$ and the integral over a small interval of length $\Delta\Psi$ around $cos(\Psi)=sin(\phi)$ gives a contribution proportional to $log(A+B\Delta\Psi))/B$, $A=K[K-2isin(\theta)sin^2(\phi)]$ and $B=2sin(\theta)cos(\phi)[sin(\theta)sin(\phi)-iKcos(\phi)]$. Both A and B vanish at the limit $\phi\rightarrow 0$, $K\rightarrow 0$. The exponential damping reduces the magnitude of the singular contribution for large values of $sin(\phi)$ as is clear form the first formula.

2. The contribution of the second cut

The expression for CUT_2 reads as

$$
\begin{aligned}
CUT_2 &= D_2exp(-\frac{b}{b_2})\times\int_0^{\pi/2}exp(i\frac{b}{b_1}cos(\psi))A_2d\psi \ , \\
D_2 &= -\frac{sin(\frac{\phi}{2})}{usin(\theta)} \ ,
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \frac{\hbar}{m}\frac{\beta}{\gamma_1} \ \ , \ \ b_2 = \frac{\hbar}{mb}\frac{1}{\gamma_1 \times sin(\theta)cos(\phi)} \\
A_2 &= \frac{Acos(\psi)+B}{cos^2(\psi)+2iCcos(\psi)+D} \ \ , \\
A &= sin(\theta)cos(\phi)u \ \ , \ \ B = \frac{w}{v_{cm}} + \frac{v}{\beta}sin^2(\theta)[sin^2(\phi)-cos^2(\phi)] \ \ , \\
C &= \frac{\beta w}{uv_{cm}}\frac{cos(\phi)}{sin(\theta)} \ \ , \ \ D = -\frac{1}{u^2}(\frac{sin^2(\phi)}{\gamma^2} + \beta^2(v^2sin^2(\theta) - \frac{2vw}{v_{cm}})cos^2(\phi)) \\
&+ \frac{w^2}{v_{cm}^2u^2sin^2(\theta)} + 2i\frac{\beta v}{u}sin(\theta)cos(\phi) \ \ , \\
u &= 1-\beta vcos(\theta) \ \ , \ \ w = 1 - \frac{v_{cm}}{\beta}cos(\theta) \ \ . \qquad (50) \\
& \qquad (51)
\end{aligned}
$$

The denominator X_2 has no poles and the contribution of the second cut is therefore always finite.

1. The factor $exp(-b/b_2)$ gives an exponential reduction and the contribution of CUT_2 is large only when the criterion

$$
b < \frac{\hbar}{m} \times \frac{1}{v\gamma_1 sin(\theta)cos(\phi)}
$$

for the impact parameter b is satisfied. Large values of $\hbar$ increase the range of allowed impact parameters since the Compton length of lepto-pion increases.

2. At the limit when the exponent becomes very large the variation of the phase factor implies destructive interference and one can perform stationary phase approximation around $\psi = \pi/2$. This gives

$$
\begin{aligned}
CUT_2 &\simeq \sqrt{\frac{2\pi b_1}{b}} \times D_2 \times exp(\frac{b}{b_2})A_2(\psi=0) \ \ , \\
D_2 &= -\frac{sin(\frac{\phi}{2})}{usin(\theta)} \ \ , \ \ A_2 = \frac{A}{D} \ \ . \qquad (52)
\end{aligned}
$$

3. As for CUT_1, the integral over ψ can be expressed as a finite sum of integrals of rational functions, when the value of $(b/b_1)cos(\psi)$ is so small that $exp(i(b/b_1)cos(\psi))$ can be approximated by a Taylor polynomial. More generally, one obtains the expansion

$$
\begin{aligned}
CUT_2 &= D_2exp(-\frac{b}{b_2}) \times \sum_{n=0}^{\infty}\frac{1}{n!}i^n(\frac{b}{b_1})^n I_n(A,B,C,D) \ \ , \\
I_n(A,B,C,D) &= \int_0^{\pi/2} cos(\psi)^n \frac{A+iBcos(\psi)}{cos^2(\psi)+Ccos(\psi)+D} \ \ . \qquad (53)
\end{aligned}
$$

The integrand of $I_n(A,B,C,D)$ is same rational function as in the case of CUT_1 but the parameters A,B,C,D given in the expression for CUT_2 are different functions of the kinematical variables. The functions appearing in the expression for integrals $I_n(c)$ correspond to the roots of the denominator of A_2 and are given by $c_\pm = -iC \pm \sqrt{-C^2-D}$, where C and D are the function appearing in the general expression for CUT_2 in Eq. 51.

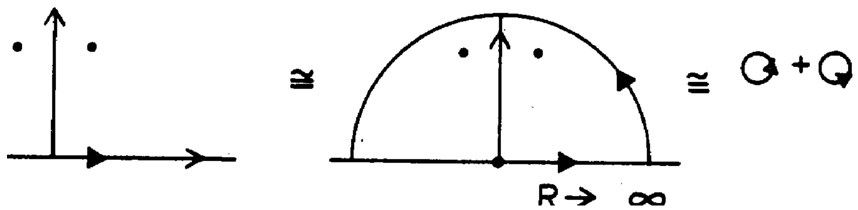

Figure 2: Evaluation of k_y-integral using residue calculus.

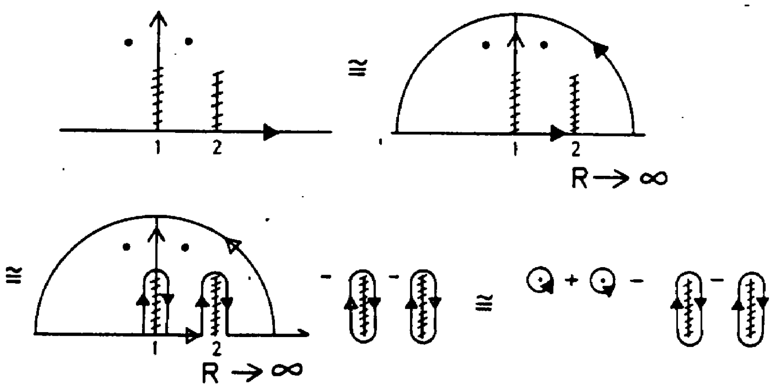

Figure 3: Evaluation of k_x-integral using residue calculus.

4.2 Production amplitude in quantum model

The previous expressions for CUT_1 and CUT_2 as such give the production amplitude for given b in the classical model and the cross section can be calculated by integrating over the values of b. The finite Taylor expansion of the amplitude in powers of b allows explicit formulas when impact parameter cutoff is assumed.

4.2.1 General expression of the production amplitude

In quantum model the production amplitude can be reduced to simpler form by using the defining integral representation of Bessel functions

$$\begin{aligned} f_B &= i\int F(b)J_0(\Delta kb)(CUT_1+CUT_2)bdb \ , \\ F &= 1 \quad \text{for } exp(i(S)) \text{ option} \ , \\ F(b\geq b_{cr}) &= \int dz\frac{1}{\sqrt{z^2+b^2}} = 2ln(\frac{\sqrt{a^2-b^2}+a}{b}) \ \text{ for } exp(i(S))-1 \text{ option} \ , \\ \Delta k &= 2ksin(\frac{\alpha}{2}) \ , \quad k=M_R\beta \quad . \end{aligned} \tag{54}$$

Note that F is a rather slowly varying function of b and in good approximation can be replaced by its average value $A(b,p)$, which has been already explicitly calculated as power series in b. α_{em} corresponds to the value of α_{em} for the standard value of Planck constant.

4.2.2 The limit $\Delta k = 0$

The integral of the contribution of CUT_1 over the impact parameter b involves integrals of the form

$$\begin{aligned} J_{1,n} &= b_0^2\int J_0(\Delta kb)F(b)x^{n+1}dx \ , \\ x &= \frac{b}{b_0} \ . \end{aligned} \tag{55}$$

Here a is the upper impact parameter cutoff. For CUT_2 one has integrals of the form

$$\begin{aligned} J_{2,n} &= b_1^2(\frac{b_2}{b_1})^{n+2}\int J_0(\Delta kb)F(b)exp(-x)x^{n+1}dx \ , \\ x &= \frac{b}{b_2} \ . \end{aligned} \tag{56}$$

Using the following approximations it is possible to estimate the integrals analytically.

1. The logarithmic term is slowly varying function and can be replaced with its average value

$$F(b) \rightarrow \langle F(b)\rangle \equiv F \ . \tag{57}$$

2. Δk is fixed once the value of the impact parameter is known. At the limit $\Delta k = 0$ making sense for very high energy collisions one can but the value of Bessel function to $J_0(0) = 1$. Hence it is advantageous to calculated the integrals of $\int CUT_ibdb$.

Consider first the integral $\int CUT_1 bdb$. If exponential series converges rapidly one can use Taylor polynomial and calculate the integrals explicitly. When this is not the case one can calculate integral approximately and the total integral is sum over two contributions:

$$\int CUT_1 bdb = I_a + I_b \ . \tag{58}$$

1. The region in which Taylor expansion converges rapidly gives rise integrals

$$\begin{aligned} I_{1,n} &\simeq b_0^2 \int x^{n+1}dx = b_0^2 \frac{1}{n+2}[(\frac{b_{max}}{b_0})^{n+2} - (\frac{b_{cr}}{b_0})^{n+2}] \simeq b_0^2 \frac{1}{n+2}(\frac{b_{max}}{b_0})^{n+2} \ , \\ I_{2,n} &\simeq b_1^2 (\frac{b_2}{b_1})^{n+2} \int exp(-x)x^{n+1}dx = b_1^2(\frac{b_2}{b_1})^{n+2}(n+1)! \ . \end{aligned} \tag{59}$$

2. For the perturbative part of CUT_1 one obtains the expression

$$\begin{aligned} I_a &= \int_0^{b_{max}} CUT_1 bdb = D_1 \times b_0^2 \times \sum_{n=0}^{\infty} \frac{1}{n!(n+2)}(\frac{b_{max}}{b_0})^{n+2} I_n(A,B,C,D) \ , \\ D_1 &= -\frac{1}{2}\frac{sin(\phi)}{sin(\theta)} \ , \ b_0 = \frac{\hbar\beta\gamma}{m\gamma_1} \ . \end{aligned} \tag{60}$$

   There b_{max} is the largest value of b for which the series converges sufficiently rapidly.

3. The convergence of the exponential series is poor for large values of b/b_0, that is for $b > b_m$. In this case one can use the approximation in which the multiplier of exponent function in the integrand is replaced with its value at $\psi = \pi/2$ so that amplitude becomes proportional to b_0/b. In this case the integral over b gives a factor proportional to ab_0, where a is the impact parameter cutoff.

$$\begin{aligned} I_b \equiv \int_{b_m}^{a} CUT_1 bdb &\simeq b_0(a-b_m)D_1 \times A_1(\psi=\pi/2) \\ &= \frac{\beta\gamma}{\gamma_1} \times \frac{\hbar}{m} \times \frac{sin^2(\theta)cos(\phi)sin(\phi)}{sin^2(\theta)sin^2(\phi)+K^2} \ , \\ D_1 &= -\frac{1}{2}\frac{sin(\phi)}{sin(\theta)} \ , \ A_1(\psi=\pi/2) = \frac{A}{D} \ . \end{aligned} \tag{61}$$

4. As already explained, the expansion based on partial fractions does not converge, when the roots $c_{\pm}$ have very large values. This indeed occurs in the case of τ-pion production cross section. In this case one has $A_1 \simeq iBcos(\psi)/D$ in excellent approximation and one can calculate CUT_1 in much easier manner. Using the formula of Eq. 48 for CUT_1, one obtains

$$\int CUT_1 bdb \simeq b_0^2 \frac{D_1 B}{D} \times \sum_{n=0,1,...} \frac{(-1)^n}{n!(n+2)2^n} \times \sum_{k=0}^{n+1} \left(\begin{array}{c} n+1 \\ k \end{array} \right) c_{n,k} \times (\frac{b_{max}}{b_0})^n \ ,$$

$$c_{n,k} = \frac{i^{n+1-2k}-1}{n+1-2k} for \ \ n \neq 2k-1 \ \ , \ \ c_{n,k} = \frac{i\pi}{2} \ \ for \ \ n = 2k-1 \ \ , \quad (62)$$

Note that for $n = 2k+1 = k$ the coefficient diverges formally and actua

Highly analogous treatment applies to the integral of CUT_2.

1. For the perturbative contribution to $\int CUT_2 bdb$ one obtains

$$I_a = \int_0^{b_{1,max}} CUT_2 bdb = b_1^2 D_2 \sum_{n=0}^{\infty} (n+1) i^n I_n(A,B,C,D) \times (\frac{b_2}{b_1})^{n+2} \ ,$$

$$D_2 = -\frac{sin(\frac{\phi}{2})}{usin(\theta)} \ ,$$

$$b_1 = \frac{\hbar\beta}{m\gamma_1} \ \ , \ \ b_2 = \frac{\hbar}{m\gamma_1} \frac{1}{sin(\theta)cos(\phi)} \ . \quad (63)$$

2. Taylor series converges slowly for

$$\frac{b_1}{b_2} = \frac{sin(\theta)cos(\phi)}{\beta} \rightarrow 0 \ .$$

In this case one can replace $exp(-b/b_2)$ with unity or expand it as Taylor series taking only few terms. This gives the expression for the integral which is of the same general form as in the case of CUT_1

$$I_a = \int_0^{b_{max}} CUT_2 bdb = b_1^2 D_2 \sum_{n=0}^{\infty} \frac{i^n}{n!(n+2)} I_n(A,B,C,D) (\frac{b_{max}}{b_1})^{n+1} \ . \quad (64)$$

3. Also when b/b_1 becomes very large, one must apply stationary phase approximation to calculate the contribution of CUT_2 which gives a result proportional to $\sqrt{b_1/b}$. Assume that $b_m >> b_1$ is the value of impact parameter above which stationary phase approximation is good. This gives for the non-perturbative contribution to the production amplitude the expression

$$I_b = \int_{b_m}^{a} CUT_2 bdb = k\sqrt{\frac{2\pi b_1}{b_2}} b_2^2 \times D_2 \times A_2(\psi = \pi/2) \ ,$$

$$k = \int_{x_1}^{x_2} exp(-x) x^{1/2} dx = 2\int_{\sqrt{x_1}}^{\sqrt{x_2}} exp(-u^2) u^2 du \ ,$$

$$x_1 = \frac{b_m}{b_2} \ \ , \ \ x_2 = \frac{a}{b_2} \ . \quad (65)$$

In good approximation one can take $x_2 = \infty$. $x_1 = 0$ gives the upper bound $k \leq \sqrt{\pi}$ for the integral.

Some remarks relating to the numerics are in order.

1. The contributions of both CUT_1 and CUT_2 are proportional to $1/sin(\theta)$ in the forward direction. The denominators of A_i however behave like $1/sin^2(\theta)$ at this limit so that the amplitude behaves as $sin(\theta)$ at this limit and the amplitude approaches to zero like $sin(\theta)$ Therefore the singularity is only apparent but must be taken into account in the calculation since one has $c_\pm \rightarrow i\infty$ at this limit for CUT_2 and for CUT_1 the roots approach to $c_+ = c_- = i\infty$. One must pose a cutoff θ_{min} below which the contribution of CUT_1 and CUT_2 are calculated directly using approximate he expressions for D_iA_i.

$$\begin{aligned} D_1A_1 &\rightarrow -\frac{i}{K}cos(\psi) \times sin(\theta) \rightarrow 0 \\ D_2A_2 &\rightarrow -\frac{uv_{cm}}{w} \times sin(\theta) \rightarrow 0 \ . \end{aligned} \tag{66}$$

   In good approximation both contributions vanish since also $sin^2(\theta)$ factor from the phase space integration reduces the contribution.

2. A second numerical problem is posed by the possible vanishing of

$$K = \beta\gamma(1 - \frac{v_c m}{\beta}cos(\theta)) \ .$$

   In this case the roots $c_\pm = \pm sin(\phi)$ are real and c_+ gives rise to a pole in the integrand.

   The singularity to the amplitude comes from the logarithmic contributions in the Taylor series expansion of the amplitude. The sum of the singular contributions coming from c_+ and c_- are of form

$$\frac{c_n}{2}(\sqrt{1-sin(\phi)} + \sqrt{1+sin(\phi)}log(\frac{1+u}{1-u}) \ , \ u = \sqrt{\frac{1+sin(\phi)}{1-sin(\phi)}} \ .$$

   Here c_n characterizes the $1/(cos(\psi) - c_\pm)$ term of associated with the $cos(\psi)^n$ term in the Taylor expansion. Logarithm becomes singular for the two terms in the sum at the limit $\phi \rightarrow 0$. The sum however behaves as

$$\frac{c_n}{2}sin(\phi)log(\frac{sin(\phi)}{2}) \ .$$

   so that the net result vanishes at the limit $\phi \rightarrow 0$. It is essential that the logarithmic singularities corresponding to the roots c_+ and c_- cancel each other and this must be taken into account in numerics. There is also apparent singularity at $\phi = \pi/2$ canceled by $cos(\phi)$ factor in D_1. The simplest manner to get rid of the problem is to exclude small intervals $[0, \epsilon]$ and $[\pi/2 - \epsilon, \pi/2]$ from the phase space volume.

4.2.3 Improved approximation to the production cross section

The approximation $J_0(\Delta k_T(b)b) = 1$ and $F(b) = F = constant$ allows to perform the integrations over impact parameter explicitly (for $exp(iS)$ option $F = 1$ holds true identically in the lowest order approximation). An improved approximation is obtained by diving the range of impact parameters to pieces and performing the integrals over the impact parameter ranges exactly using the average values of these functions. This requires only a straightforward generalization of the formulas derived above involving integrals of the functions x^n and $exp(-x)x^n$ over finite range. Obviously this is still numerically well-controlled procedure.

References

[1] M. Pitkänen (2006), *Physics in Many-Sheeted Space-Time.* http://www.helsinki.fi/~matpitka/tgdclass/tgdclass.html.

[2] M. Pitkänen (2006), *p-Adic length Scale Hypothesis and Dark Matter Hierarchy.* http://www.helsinki.fi/~matpitka/paddark/paddark.html.

[3] The chapter *Cosmic Strings* of [1]. http://www.helsinki.fi/~matpitka/tgdclass/tgdclass.html#cstrings.

[4] The chapter *TGD and Nuclear Physics* of [2]. http://www.helsinki.fi/~matpitka/paddark/paddark.html#padnucl.

[5] The chapter *Nuclear String Model* of [2]. http://www.helsinki.fi/~matpitka/paddark/paddark.html#nuclstring.

[6] The chapter *The Recent Status of Lepto-hadron Hypothesis* of [2]. http://www.helsinki.fi/~matpitka/paddark/paddark.html#leptc.

[7] The chapter *The Status of Lepto-pion Hypothesis* of "TGD and p-Adic Numbers". http://www.helsinki.fi/~matpitka/padtgd/padtgd.html#lepto.

[8] M. Pitknen (1990). Int. J. of Theor. Phys, 29, 275. Lepto-hadron hypothesis is represented in preliminary form.

[9] A. Chodos (1987) , Comments Nucl. Part. Phys., Vol 17, No 4, pp. 211, 223.

[10] L. Kraus and M. Zeller (1986), Phys. Rev. D 34, 3385.

[11] W. Koenig *et al*(1987), Zeitschrift fur Physik A, 3288, 1297.

[12] S. Judge *et al* (1990) , Phys.Rev. Lett., 65(8), 972.

[13] K. Dantzman *et al* (1989), Phys. Rev. Lett., 62, 2353.

[14] T. Cowan et al.(1985), Phys. Rev. Lett. 54, 1761 and T. Cowan et al.(1986), Phys. Rev. Lett. 56, 444.

[15] H.Tsertos et al. (1985) , Phys. Lett. 162B, 273, H.Tsertos et al.(1987) , Z. Phys. A 326, 235.

[16] T. Akesson *et al*(1987), Phys. Lett. B192, 463, T. Akesson *et al*(1987), Phys. Rev. D36, 2615.

[17] A.T. Goshaw *et al*(1979), Phys. Rev. Lett. 43, 1065.

[18] P.V. Chliapnikov *et al*(1984), Phys. Lett. B 141, 276.

[19] S. Barshay (1992) , Mod. Phys. Lett. A, Vol 7, No 20, p. 1843.

[20] J.Schweppe et al.(1983), Phys. Rev. Lett. 51, 2261.

[21] M. Clemente et al. (1984), Phys. Rev. Lett. 137B, 41.

[22] P. Salabura *et al* (1990), Phys. Lett. B 245, 2, 153.

[23] KARMEN Collaboration, B. Armbruster *et al* (1995) , Phys. Lett. B 348, 19.

[24] V. Barger, R. J. N. Phillips, S. Sarkar (1995), Phys. Lett. B 352,365-371.

[25] C. I. Westbrook ,D. W Kidley, R. S. Gidley, R. S Conti and A. Rich (1987), Phys. Rev. Lett. 58 , 1328.

[26] R. Escribabno,E. Masso, R. Toldra (1995), Phys. Lett. B. 356, 313-318.

[27] PAMELA Collaboration (2008), *Observation of an anomalous positron abundance in the cosmic radiation.* http://arxiv.org/PS_cache/arxiv/pdf/0810/0810.4995v1.pdf.
M. Boexio (2008), talk represented at IDM 2008, Stockholm, Sweden.

[28] J. Chang et al. (ATIC) (2005), prepared for 29th International Cosmic Ray Conferences (ICRC 2005), Pune, India, 31 Aug 03 - 10 2005.

[29] N. Arkani-Hamed and N. Weiner (2008), *LHC Signals for a SuperUnified Theory of Dark Matter*, arXiv:0810.0714arXiv.org.

[30] E. Churazov, R. Sunyaev, S. Sazonov, M. Revnivtsev, and D. Varshalovich, *Positron annihilation spectrum from the Galactic Center region observed by SPI/INTEGRAL*, Mon. Not. Roy. 17. Astron. Soc. 357, 1377 (2005), astro-ph/0411351.

[31] G. Weidenspointner *et al* (2006), *The sky distribution of positronium annihilation continuum emission measured with SPI/INTEGRAL*, Astron. Astrophys. 450, 1013, astro-ph/0601673.

[32] X.-G. He, J. Tandean, G. Valencia (2007), *Has HyperCP Observed a Light Higgs Boson?*,Phys. Rev. D74.
http://arxiv.org/abs/hep-ph/0610274.

[33] X.-G. He, J. Tandean, G. Valencia (2007), *Light Higgs Production in Hyperon Decay*, Phys. Rev. Lett. 98.
http://arxiv.org/abs/hep-ph/0610362.

[34] G. W. Bennett *et al* (2004), *Measurement of the Negative Muon Anomalous Magnetic Moment to 0.7 ppm.* Phys. Rev. Lett. 92; 1618102.
The E821 Muon (g-2) Home Page, http://www.g-2.bnl.gov/.

[35] Peter Woit (2008), *A new long-lived particle by CDF experiment*, http://www.math.columbia.edu/∼woit/wordpress/?p=1045.

[36] *Pamela is coming out* (2008), http://resonaances.blogspot.com/2008/10/pamelas-coming-out.html.

[37] T. Dorigo (2008), *Saving a good text from a few mistakes.*
http://dorigo.wordpress.com/2008/11/21/saving-a-good-text-from-a-few-mistakes/.

[38] T. Dorigo (2008), *Some notes on the multi-muon analysis - part I.*
http://dorigo.wordpress.com/2008/11/08/some-notes-on-the-multi-muon-analysis-part-i/.

[39] CDF Collaboration (2008), *Study of multi-muon events produced in p-pbar collisions at sqrt(s)=1.96 TeV.* http://arxiv.org/PS_cache/arxiv/pdf/0810/0810.0714v1.pdf.

[40] P. Giromini, F. Happacher, M. J. Kim, M. Kruse, K. Pitts, F. Ptohos, S. Torre (2008), *Phenomenological interpretation of the multi-muon events reported by the CDF collaboration.* http://arxiv.org/abs/0810.5730.

[41] K. V. Shajesh (2008), *Eikonal approximation*, http://www.nhn.ou.edu/∼shajesh/eikonal/sp.pdf.

[42] *Astronomers claim dark matter breakthrough*, New Scientist, 6 October, 2003. http://www.newscientist.com/news/news.jsp?id=ns99994214.