Distribution of distances in the solar system

V. Peřinová*, A. Lukš, P. Pintr
Laboratory of Quantum Optics, Faculty of Natural Sciences,
Palacký University, Třída Svobody 26,
77146 Olomouc, Czech Republic

Abstract

The recently published application of a diffusion equation to prediction of distances of planets in the solar system has been identified as a two-dimensional Coulomb problem. A different assignment of quantum numbers in the solar system has been proposed. This method has been applied to the moons of Jupiter on rescaling.

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1 Introduction

The 20th century is held for the golden age of the astronomy and astrophysics, when many persistent questions were solved and the human view of the universe changed radically. In spite of this, at the beginning of the 21st century, one cannot find satisfactory answers to some questions our ancestors posed as early as in the 16th century. For instance, Kepler looked for a universal law, in his Mysterium cosmographicum, to explain the planetary distances in the solar system. Nowadays, when discoveries of other planetary systems occur, such a law could explain the distances of their planets.

[0]*Corresponding author.
E-mail address: perinova@prfnw.upol.cz (V. Peřinová)

In 1766 Titius formulated the law, which described distances of the bodies in the solar system, and it even predicted new bodies at certain distances from the Sun [1]. Actually its being criticized led to the discovery of the remaining planets and new bodies – asteroids – in the solar system. It was the first, controversial, description of the distances of the bodies in this planetary system. But hardly any physical explanation has thus far been given. Is it a mere extravagance, or does this law have some deep physical content? May the planets around stars originate at definite distances?

Quest of the answer developed into invention of new empirical formulae, which describe, with higher or lower accuracy, the distances of the bodies in the solar system. For instance Armelini's empirical formula has the form

$$r_{n\mathrm{A}} = 1.53^n, \tag{1}$$

where n assumes the values: Mercury -2, Venus -1, Earth 0, Mars 1, asteroid Vesta 2, asteroid Camilla 3, Jupiter 4, Saturn 5, asteroid Chiron 6, Uranus 7, Neptune 8, and Pluto 9.

In 1938 Mohorovičić invented an empirical formula [2], which describes the distances of planets and comets with high accuracy, and it also predicts an asteroid belt between Mars and Jupiter. Mohorovičić's law says that the distances of the inner parts of the solar system increase in a sublinear manner and those of the outer parts of this system increase in a superlinear manner. In the paper [3] we have modified this law such that it satisfies also other planetary systems and those of the moons of the giant planets.

Interesting is the empirical formula, which is similar to the laws of quantum mechanics [4]

$$r_{mn} = \frac{1}{2}(m^2 + n^2)r_0, \tag{2}$$

where m are natural numbers, $n = 0, 1, \ldots, m$ and $r_0 = 0.387$ AU. The Bohr–Sommerfeld rule of (allowed) orbits for electrons in the electric fields of the nuclei of various atoms resemble the distribution of planetary distances, but do not let us forget that this rule describes bodies (electrons), which all have the same inertial mass and the same electric charge, which replaces a gravitational mass here. To obtain a distribution of the planetary distances, one either replaces different planetary masses by their mean mass, or makes the quantum of action depend on the actual mass.

Agnese and Festa described the solar system like a gravitational atom [5]. They utilized a quantum law for the hydrogen atom, which they applied to description of major semi-axes of allowed (discretized) elliptical orbits of the bodies of the planetary system

$$r_{n\mathrm{AF}} = r_1 n^2, \tag{3}$$

where n are natural numbers and r_1 is the Bohr radius of the planetary system, which is

$$r_1 = \frac{GM}{\alpha_{\mathrm{g}}^2 c^2}, \tag{4}$$

where G is the gravitational constant, M the mass of the central body, c the vacuum speed of light and α_{g} is a gravitational structure constant, which has the property $\frac{1}{\alpha_{\mathrm{g}}} = 2113 \pm 15$. Agnese and Festa have shown that this description of distances satisfies also the planetary system υ Andromedae [6] and other stellar systems alike on substituting the mass of the appropriate central star for the mass M. A study which elaborates on such ideas has been presented in [7].

Recently, the significance of the Titius–Bode law has been evaluated both by generating random planetary systems [8] and by the help of methods of the modern statistical analysis [9]. In the papers [10, 11] the authors point out quantum features also on large scales, namely discrete values of distances of possible planets and galaxies.

In quantum mechanics one utilizes Schrödinger's equation for the description of a physical system. In the paper [12], the stochastic mechanics is constructed, i. e., the Schrödinger equation is obtained as a classical diffusion equation by the help of the hypothesis that any particle in any interaction also exhibits a universal Brownian motion [13]. The main problem of this kind of derivation is a convincing physical origin for that universal Brownian motion, although a possibility is the quantum nature of space-time [14]. The chaotic behaviour of the solar system during its formation and evolution [15, 16] suggests a diffusion process to be described in terms of a Schrödinger-type equation. The description of the planetary system using a Schrödinger-type diffusion equation has been realized in [17]. There the authors have adapted the Schrödinger equation to the planetary system and shown that there exist very many orbits, on which possible planets may originate. That paper has stimulated us to the following considerations.

2 Discrete distances in the gravitational field of an astronomical body

Let us consider a body of the mass $M_{\rm p}$, which orbits a central body of the mass M and has the potential energy $V(x,y,z)$ in its gravitational field. Because planets and moons of the giant planets revolve approximately in the same plane, we consider $z=0$. Because they revolve in the same direction, we choose directions of the axes x, y and z such that the planets or moons of giant planets revolve counter-clockwise. Then we write the modified Schrödinger equation for the wave function $\psi=\psi(x,y)$ from the part of the Hilbert space $L_2(R^2)\cap C^2(R^2)$ and the eigenvalue $0>E\in R$ in the form

$$-\frac{\hbar_M^2}{2M_{\rm p}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi+V(x,y)\psi=E\psi, \tag{5}$$

where $\hbar_M\approx 1.48\times 10^{15}M_{\rm p}$, $V(x,y)=V(x,y,z)$ and E is the total energy. Negative E classically correspond to the elliptic Kepler orbits and the localization property (bound state) is conserved also in the quantum mechanics for such total energies E. The factor 1.48×10^{15} is not a dimensionless number, but the unit of its measurement is $\rm m^2s^{-1}$. With respect to the unusual unit we do not wonder that Agnese and Festa [5] consider this factor in the form of a product, such that $\hbar_M=\bar\lambda_M cM_{\rm p}$, where $\bar\lambda_M\approx 4.94\times 10^6$ m.

We transform equation (5) into the polar coordinates,

$$-\frac{\hbar_M^2}{2M_{\rm p}}\left(\frac{\partial^2\tilde\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\tilde\psi}{\partial\theta^2}\right)+\tilde V(r)\tilde\psi=E\tilde\psi, \tag{6}$$

where $\tilde\psi\equiv\tilde\psi(r,\theta)=\psi(r\cos\theta,r\sin\theta)$ and $\tilde V(r)=V(r\cos\theta,r\sin\theta)$ does not depend on θ. Particularly we choose

$$\tilde V(r)=-\frac{GM_{\rm p}M}{r}. \tag{7}$$

With respect to the Fourier method we assume a solution of the equation (7) in the form

$$\tilde\psi(r,\theta)=R(r)\Theta(\theta). \tag{8}$$

The original eigenvalue problem is transformed, equivalently, to two eigenvalue problems

$$\Theta''(\theta)=-\Lambda\Theta, \tag{9}$$

$$\Theta(0) = \Theta(2\pi) \tag{10}$$

and

$$R''(r) + \frac{1}{r}R'(r) + \left\{-\frac{\Lambda}{r^2} + \left[E - \tilde{V}(r)\frac{2M_{\rm p}}{\hbar_M^2}\right]\right\} R(r) = 0, \tag{11}$$

$$\lim_{r\to 0+}[\sqrt{r}R(r)] = 0, \quad \sqrt{r}R(r) \in L_2((0,\infty)). \tag{12}$$

The solution of the problem (9)–(10) has the form

$$\Theta_l(\theta) = \frac{1}{\sqrt{2\pi}} \exp{(il\theta)} \tag{13}$$

for $l = \pm\sqrt{\Lambda} \in Z$.

Here $l = 0$ should mean a body, which does not revolve at all. In the classical mechanics such a body moves close to a line segment ending at the central body, and it spends a short time in the vicinity of this body. In this paper we utilize some – not all – of the concepts of quantum mechanics and we will not avoid the case $l = 0$ [17]. In (13) $l = 1, 2, \ldots, \infty$ corresponds to the counter-clockwise revolution.

Respecting (7), the equation (11) becomes

$$R''(r) + \frac{1}{r}R'(r) + \left\{-\frac{l^2}{r^2} - B - \frac{2M_{\rm p}}{\hbar_M^2}\left(-\frac{GM_{\rm p}M}{r}\right)\right\} R(r) = 0, \tag{14}$$

where

$$B = -\frac{2M_{\rm p}E}{\hbar_M^2} = -\frac{2}{(\bar{\lambda}_M c)^2}\frac{E}{M_{\rm p}}. \tag{15}$$

Let us note that

$$\frac{M_{\rm p}GM_{\rm p}M}{\hbar_M^2} = \frac{GM}{(\bar{\lambda}_M c)^2}. \tag{16}$$

On substituting $r = \frac{\rho}{2\sqrt{B}}$ and introducing

$$\tilde{R}(\rho) = R\left(\frac{\rho}{2\sqrt{B}}\right), \tag{17}$$

equation (14) becomes

$$\tilde{R}''(\rho) + \frac{1}{\rho}\tilde{R}'(\rho) + \left(-\frac{1}{4} + \frac{k}{\rho} - \frac{l^2}{\rho^2}\right)\tilde{R}(\rho) = 0, \tag{18}$$

where

$$k = \frac{GM}{(\bar{\lambda}_M c)^2 \sqrt{B}}. \tag{19}$$

For later reference let us note that, inversely,

$$\sqrt{B} = \frac{GM}{(\bar{\lambda}_M c)^2 k}, \tag{20}$$

$$\frac{-E}{M_{\mathrm{p}}} = \frac{(\bar{\lambda}_M c)^2}{2} B \tag{21}$$

$$= \frac{(GM)^2}{2(\bar{\lambda}_M c)^2 k^2}. \tag{22}$$

Expressing $\tilde{R}(\rho)$ in the form

$$\tilde{R}(\rho) = \frac{1}{\sqrt{\rho}} u\left(\rho\right), \tag{23}$$

we obtain an equation for $u(\rho)$

$$u''(\rho) + \left[-\frac{1}{4} + \frac{k}{\rho} - \left(l'^2 - \frac{1}{4} \right) \frac{1}{\rho^2} \right] u\left(\rho\right) = 0, \tag{24}$$

where $l' = l$. It is familiar that this equation has two linear independent solutions $M_{k,l'}(\rho)$, $M_{k,-l'}(\rho)$, if l' is not an integer number. When l' is integer, the solution $M_{k,-l'}(\rho)$ must be replaced with a more complicated solution. It can be proven that the other solution is not regular for $\rho = 0$ (it diverges as $\ln \rho$ for $\rho \to 0$). The remaining solution $M_{k,l}(\rho)$ can be transformed to a wave function from the space $L_2((0, \infty))$ if and only if $k - l - \frac{1}{2} = n_{\mathrm{r}}$ is any nonnegative integer number. We choose this function to be

$$u_{kl}(\rho) = C_{kl} M_{k,l}(\rho), \tag{25}$$

where C_{kl} is an appropriate normalization constant and $M_{k,l}(\rho)$ is a Whittaker function, namely

$$M_{k,l}(\rho) = \rho^{l+\frac{1}{2}} \exp\left(-\frac{\rho}{2}\right) \Phi\left(l - k + \frac{1}{2}, 2l + 1; \rho\right), \tag{26}$$

where Φ is the confluent (or degenerate) hypergeometric function. In (25) the constant C_{kl} has the property

$$\int_0^\infty r[R_{kl}(r)]^2 dr = 1, \tag{27}$$

or it is

$$C_{kl} = 2\sqrt{B}\frac{1}{(2l)!}\sqrt{\frac{(n+l-1)!}{2k(n-l-1)!}}. \tag{28}$$

Then

$$R_{kl}(r) = 2\sqrt{B}\sqrt{\frac{(n-l-1)!}{2k\Gamma(n+l)}}\exp(-r\sqrt{B})(2r\sqrt{B})^l L^{2l}_{n-l-1}(2r\sqrt{B}), \tag{29}$$

where $n = k + \frac{1}{2}$, $L^{2l}_{n-l-1}(x)$ is a Laguerre polynomial, and the relation (20) holds.

3 Interpretation of formulae derived

Having solved the modified Schrödinger equation, we address interpretation of the formulae derived. The probability density $P_{kl}(r)$ of the revolving body occurring at the distance r from the central body is

$$P_{kl}(r) = r[R_{kl}(r)]^2,\ r \in [0, \infty). \tag{30}$$

Mean distances of the planets are given by the relation

$$r_{kl} = \int_0^\infty r P_{kl}(r) dr \tag{31}$$

$$= \frac{(\bar{\lambda}_M c)^2}{4GM}\left[(2k - n_\mathrm{r})(2k - n_\mathrm{r} + 1) + 4n_\mathrm{r}(2k - n_\mathrm{r}) + n_\mathrm{r}(n_\mathrm{r} - 1)\right], \tag{32}$$

where $n_\mathrm{r} = n - l - 1$, $k = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \infty$ and $l = 0, 1, 2, \ldots, n$.

For the solar system $M = M^{\mathrm{Sun}}$ holds and the Bohr radius of the solar system $r_{\frac{1}{2}0} = 0.055$ AU. For survey one finds some expectation values r_{kl} for selected values k, l with the specification of described bodies in table 1 (cf. [17]).

Even though also in this case an empirical formula is tested for distribution of planetary distances, the predicted orbits fit those of the bodies in this solar system.

Using the graphs of the probability densities we have plotted for every predicted orbit of this system, we obtain surprising results. The graphs of the probability densities for each orbit with $k \leq \frac{9}{2}$ and with $\frac{11}{2} \leq k \leq \frac{31}{2}$ are contained, respectively, in figure 1 and in figure 2. The vertical axis denotes the probability density $P_{kl}(r)$ and the longitudinal axis designates the planetary distance r from the Sun. In figure 1 graph no. $p = 1$ is interpreted such that the highest probability density is assigned to the orbit of the radius of 0.055 AU and from the calm shape of the graph we infer that an ideal circular orbit is tested. In figure 1 graph no. $p = 14$ is interpreted such that the highest probability density is assigned to the orbit of the radius of 3.32 AU and, of many peaks, which wave the shape, we infer that no stable circular orbit is tested. After performing the analysis for all the orbits, we obtain only a small number of stable circular orbits. The orbits, on which big bodies – planets – may originate, are listed in table 2.

It emerges that, for every number k, there exists only one stable orbit, on which a big body – a planet – may originate. Then we can interpret the number k as the principal quantum number and l as the orbital quantum number equal to the number of possible orbits, but only for the greatest l there exists a stable orbit of a future body. A planet which does not confirm this theory is the Earth. Since the description based on the modified Schrödinger equation for the planetary system is not fundamental, it could not fit all the stable orbits. Other deviations are likely to be incurred by collisions of the bodies in early stages of the origin of the planets, thus nowadays we already observe elliptical orbits, which are very close to circular orbits.

This procedure has been applied to moons of giant planets by us. It emerges that the moons of giant planets also are fitted by the modified Schrödinger equation and appropriate expectation values. Especially, the predicted stable circular orbits of Jupiter's moons are presented in table 3. For Jupiter it holds that $M = M^{\mathrm{Jup}}$ and the Bohr radius (4) of this system $r_1 = 6287$ km. It emerges that the predicted lunar orbits fit the measured orbits of the moons orbiting Jupiter.

4 Conclusions

In this paper we assume that there exists a law by which big objects – planets and moons of giant planets – do not originate anywhere, but at allowed distances from the central body. Unnegligible number of authors have issued from similar assumptions and derived empirical formulae for parameters of allowed orbits.

The results we have presented in this paper are based on a modified Schrödinger equation, which has been applied to the planetary system by us for the quantum theory contained in the Schrödinger equation to create an interesting view of the birth of such a stellar system, namely the orbits of planets and moons being approximately quantized.

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Table 1. Predicted distances of bodies from the Sun

Body	k	l	r_{kl} [AU]
—	$\frac{1}{2}$	0	0.055
Mercury	$\frac{3}{2}$	1	0.332
Mercury	$\frac{3}{2}$	0	0.387
Venus	$\frac{5}{2}$	2	0.829
Earth	$\frac{5}{2}$	1	0.995
Earth	$\frac{5}{2}$	0	1.050
Mars	$\frac{7}{2}$	3	1.548
Hungaria	$\frac{7}{2}$	2	1.824
Hungaria	$\frac{7}{2}$	1	1.990
Hungaria	$\frac{7}{2}$	0	2.046
Vesta	$\frac{9}{2}$	4	2.488
Ceres	$\frac{9}{2}$	3	2.875
Hygeia	$\frac{9}{2}$	2	3.151
Camilla	$\frac{9}{2}$	1	3.317
Camilla	$\frac{9}{2}$	0	3.372
Jupiter	$\frac{11}{2}$	0	5.031
—	$\frac{13}{2}$	0	7.021
Saturn	$\frac{15}{2}$	0	9.343
Chiron	$\frac{17}{2}$	0	11.997
Chiron	$\frac{19}{2}$	0	14.982
Uranus	$\frac{21}{2}$	0	18.300
—	$\frac{23}{2}$	0	21.948
HA2 (1992), DW2 (1995)	$\frac{25}{2}$	0	25.929
Neptune	$\frac{27}{2}$	0	30.241
—	$\frac{29}{2}$	0	34.885
Pluto	$\frac{31}{2}$	0	39.861

Table 2. Bodies with stable circular orbits.

Body	k	l	r_{kl} [AU]
—	$\frac{1}{2}$	0	0.055
Mercury	$\frac{3}{2}$	1	0.332
Venus	$\frac{5}{2}$	2	0.83
Mars	$\frac{7}{2}$	3	1.54
Vesta	$\frac{9}{2}$	4	2.49
Fayet comet	$\frac{11}{2}$	5	3.64
Jupiter	$\frac{13}{2}$	6	5.03
Neujmin comet	$\frac{15}{2}$	7	6.636
—	$\frac{17}{2}$	8	8.46
Saturn	$\frac{19}{2}$	9	10.5
—	$\frac{21}{2}$	10	12.77
Westphal comet	$\frac{23}{2}$	11	15.26
Pons–Brooks comet	$\frac{25}{2}$	12	17.97
Uranus	$\frac{27}{2}$	13	20.9
—	$\frac{29}{2}$	14	24.055
—	$\frac{31}{2}$	15	27.43
Neptune	$\frac{33}{2}$	16	31.02
—	$\frac{35}{2}$	17	34.84
Pluto	$\frac{37}{2}$	18	38.88
—	$\frac{39}{2}$	19	43.134

Table 3. Moons of Jupiter with stable circular orbits.

Body	k	l	r_{kl} [km]
—	$\frac{1}{2}$	0	6287
—	$\frac{3}{2}$	1	37722
Halo ring	$\frac{5}{2}$	2	94305
Outer ring	$\frac{7}{2}$	3	176036
—	$\frac{9}{2}$	4	282915
Io	$\frac{11}{2}$	5	414942
Europa	$\frac{13}{2}$	6	572117
—	$\frac{15}{2}$	7	754440
—	$\frac{17}{2}$	8	961911
Ganymede	$\frac{19}{2}$	9	1.19×10^6
—	$\frac{21}{2}$	10	1.452×10^6
Callisto	$\frac{23}{2}$	11	1.735×10^6

Figure 1: Probability densities for a particle in states with quantum numbers k, l, which correspond, respectively, (p is an ordinary number) to Mercury ($p = 1$, $l = 1$), Mercury ($p = 2$, $l = 0$, the second possibility), Venus ($p = 3$, $l = 2$), Earth ($p = 4$, $l = 1$), Earth ($p = 5$, $l = 0$, the second possibility), Mars ($p = 6$, $l = 3$), asteroid Hungaria ($p = 7$, $l = 2$), asteroid Hungaria ($p = 8$, $l = 1$, the second possibility), asteroid Hungaria ($p = 9$, $l = 0$, the third possibility), asteroid Vesta ($p = 10$, $l = 4$), asteroid Ceres ($p = 11$, $l = 3$), asteroid Hygeia (p=12, $l = 2$), asteroid Camilla ($p = 13$, $l = 1$), and asteroid Camilla ($p = 14$, $l = 0$, the second possibility). Here $k \in \{\frac{3}{2}, \frac{5}{2}, \ldots, \frac{9}{2}\}$, the quantum number k repeats $n(= k + \frac{1}{2})$ times and r is measured in AU.

Figure 2: Probability densities for a particle in states with quantum numbers k, l, which correspond, respectively, (p is an ordinary number) to Jupiter ($p = 1$), nothing ($p = 2$), Saturn ($p = 3$), Chiron ($p = 4$), Chiron ($p = 5$, the second possibility), Uranus ($p = 6$), nothing ($p = 7$), HA2 (1992), DW2 (1995) ($p = 8$), Neptune ($p = 9$), nothing ($p = 10$) and Pluto ($p = 11$), $k \in \{\frac{11}{2}, \frac{13}{2}, \ldots, \frac{31}{2}\}$, $l = 0$. Here $k = p + \frac{9}{2}$, r is measured in AU.