A new solution of the Schrödinger equation

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Abstract

We obtained a new solution of Schrödinger equation by the method of Euclidean approach (Wick rotation). This is a wave motion which is fluctuating.

1. Introduction

In the Feynman's path integral, the solution of the Schrödinger equation is expressed as

$$\psi(x,t) = \int Dx \exp\left[\frac{i}{\hbar}\int_{t_0}^{t}\frac{1}{2}mv^2 dt'\right]\left[\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}V(x,t')\,dt'\right)\psi(x_0,t_0)\right].$$

And

$$\int Dx \exp\left[\frac{i}{\hbar}\int\frac{1}{2}mv^2 dt\right] = \int Dx \exp\left[-\frac{1}{\hbar}\int\frac{1}{2}m\frac{(dx)^2}{d(it)}\right]$$

corresponds to the normal distribution after replacing $it \to t$, so wave motion function can be written as

$$\psi(x,t) = E\left[\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}V(x,t')\,dt'\right)\psi(x_0,t_0)\right].$$

$E[\cdots]$ means the expected value is taken. This is an expression of so-called Feynman-Kac formula. The purpose of our study is to discover the correct wave motion function that had been concealed by the operation of taking the expected value.

2. Discussion

We try to obtain a new solution of Schrödinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x,t)\,\psi(x,t) \tag{1}$$

by the technique of the Euclidean approach (Wick rotation). The Euclidean approach is one of the techniques of the quantum electrodynamics to come and go in the quantum mechanics and the statistical mechanics by doing a $it \to t$ replacement.

(1) becomes

$$-\frac{\partial\psi(x,t)}{\partial t} + \frac{\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{V(x,t)}{\hbar}\psi(x,t) \tag{2}$$

after using the Euclidean approach ($it \to t$) and dividing both sides by $\hbar$.

The normal distribution

$$\int Dx \exp\left[-\frac{1}{\hbar}\int\frac{1}{2}m\frac{(dx)^2}{dt}\right]$$

means that x follows the stochastic process of

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$dx = \sqrt{\frac{\hbar}{m}} dW^* (t)$ (Refer to Appendix A) (3).

Here,

$dW^* (t) = \sqrt{-dt}\xi \; (dt < 0) \cdots$Standard Brownian motion (conjugate Wiener process)

$\xi \cdots$Standard regular random variable.

(By the replacement of $it \rightarrow \tau$, path of integration t is converted into imaginary number time like $[t_1, t_2] \rightarrow [\tau_1/i, \tau_2/i] \rightarrow [-i\tau_1, -i\tau_2]$. On the other hand, because real time τ is $\tau_1 - \tau_2 > 0$, the direction of time increasing becomes contrary to the direction of integration ($d\tau = \tau_2 - \tau_1 < 0$). In a word, it becomes $dt \rightarrow -id\tau$ for $it \rightarrow \tau$.)

If Ito's lemma (Refer to Appendix B) is used, a function $\psi (x, t)$ that has variable x and t follows the stochastic process of

$$d\psi (x, t) = \left(\frac{\partial \psi (x, t)}{\partial t} - \frac{1}{2} \frac{\partial^2 \psi (x, t)}{\partial x^2} \left\{ \sqrt{\frac{\hbar}{m}} \right\}^2 \right) dt + \frac{\partial \psi (x, t)}{\partial x} \sqrt{\frac{\hbar}{m}} dW^* (t) \quad (4).$$

If (2) is substituted for (4), it becomes

$$d\psi (x, t) = -\frac{V (x, t)}{\hbar} \psi (x, t)\, dt + \frac{\partial \psi (x, t)}{\partial x} \sqrt{\frac{\hbar}{m}} dW^* (t) \quad (5).$$

This is Ornstein-Uhlenbeck process. So the solution is obtained as follows[1].

$$\psi (x, t) = \exp \left(-\frac{1}{\hbar} \int_{t_0}^{t} V (x, t')\, dt' \right) [\; \psi (x_0, t_0)$$
$$+ \int_{t_0}^{t} \exp \left(\frac{1}{\hbar} \int_{t_0}^{u} V (x, t')\, dt' \right) \frac{\partial \psi (x, u)}{\partial x} \sqrt{\frac{\hbar}{m}} dW^* (u) \;]$$

or

$$\psi (x, t) = \exp \left(-\frac{1}{\hbar} \int_{t_0}^{t} V (x, t')\, dt' \right) \psi (x_0, t_0)$$
$$+ \int_{t_0}^{t} \exp \left(-\frac{1}{\hbar} \int_{u}^{t} V (x, t')\, dt' \right) \frac{\partial \psi (x, u)}{\partial x} \sqrt{\frac{\hbar}{m}} dW^* (u) \quad (6).$$

Clause 2 of the right side shows fluctuation. Then we can obtain a new solution of Schrödinger equation (fluctuating wave function) by reverse Euclidean approach ($t \rightarrow it$).

$$\psi (x, t) = \exp \left(-\frac{i}{\hbar} \int_{t_0}^{t} V (x, t')\, dt' \right) [\; \psi (x_0, t_0)$$
$$+ \int_{t_0}^{t} \exp \left(\frac{i}{\hbar} \int_{t_0}^{u} V (x, t')\, dt' \right) \frac{\partial \psi (x, u)}{\partial x} \sqrt{\frac{\hbar}{m}} \sqrt{-idu}\xi \;]$$
$$= \exp \left(-\frac{i}{\hbar} \int_{t_0}^{t} V (x, t')\, dt' \right) [\; \psi (x_0, t_0)$$
$$+ \exp \left(-\frac{\pi}{4} i \right) \int_{t_0}^{t} \exp \left(\frac{i}{\hbar} \int_{t_0}^{u} V (x, t')\, dt' \right) \frac{\partial \psi (x, u)}{\partial x} \sqrt{\frac{\hbar}{m}} \sqrt{du}\xi \;]$$

or

$$\psi (x, t) = \exp \left(-\frac{i}{\hbar} \int_{t_0}^{t} V (x, t')\, dt' \right) \psi (x_0, t_0)$$

$$+\exp\left(-\frac{\pi}{4}i\right)\int_{t_0}^{t}\exp\left(-\frac{i}{\hbar}\int_{u}^{t}V\left(x,t'\right)dt'\right)\frac{\partial\psi\left(x,u\right)}{\partial x}\sqrt{\frac{\hbar}{m}}dW\left(u\right) \qquad (7).$$

This is the correct expression of a fluctuating wave motion. Clause 1 shows the appearance that the principal ingredient of the wave develops at time by exponential with potential in the shoulder. Only this clause 1 is considered in a present quantum theory. The effect of fluctuation is added by clause 2.

We will confirm the fluctuation disappears from the expression when the expected value of this new solution (fluctuating solution) of the Schrödinger equation is taken.

After taking expected value of the right side of (6) by

$$\int Dx\exp\left[-\frac{1}{\hbar}\int\frac{1}{2}m\frac{\left(dx\right)^{2}}{dt}\right],$$

(6) becomes

$$\psi\left(x,t\right)=\int Dx\exp\left[-\frac{1}{\hbar}\int\frac{1}{2}mv^{2}dt\right]\left[\exp\left(-\frac{1}{\hbar}\int_{t_0}^{t}V\left(x,t'\right)dt'\right)\psi\left(x_0,t_0\right)\right] \quad (8)$$

because clause 2 of the right side of (6) is 0 by Ito's integral.

It is understood that clause 2 of (6) disappears and the fluctuation has disappeared on the expression. And replace $t\rightarrow it$ in (8), then (8) becomes

$$\psi\left(x,t\right)=\int Dx\exp\left[\frac{i}{\hbar}\int\frac{1}{2}mv^{2}dt\right]\left[\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}V\left(x,t'\right)dt'\right)\psi\left(x_0,t_0\right)\right] \quad (9).$$

It is corresponding to Feynman's path integral formula (Lagrangian Path Integrals).

In Hamiltonian Path Integrals

$$\psi\left(x,t\right)=\int DxDp_x\exp\left(\frac{i}{\hbar}\int\left[p_x\dot{x}-\frac{p_x^2}{2m}-V\left(x,t\right)\right]dt\right)\psi\left(x_0,t_0\right)$$

$$=\int DxDp_x\exp\left(\frac{i}{\hbar}\int\left[-\frac{1}{2m}\left(p_x-m\frac{dx}{dt}\right)^{2}+\frac{1}{2}m\left(\frac{dx}{dt}\right)^{2}-V\left(x,t\right)\right]dt\right)\psi\left(x_0,t_0\right)$$

, if the Gauss integration concerning the momentum is executed, it becomes Lagrangian Path Integrals. When a kinetic energy paragraph of Hamiltonian Path Integrals

$$\int Dp_x\exp\left(\frac{i}{\hbar}\int\left[-\frac{1}{2m}\left(p_x-m\frac{dx}{dt}\right)^{2}\right]dt\right)$$

is compared with the normal distribution function, it is meant that the momentum fluctuates like

$$dp_x\cong\sqrt{\frac{\hbar m}{dt}}.$$

(3) means coordinates fluctuates like

$$dx\cong\sqrt{\frac{\hbar}{m}dt}.$$

So, It becomes

$$\Delta x\Delta p_x\cong\hbar$$

from this two. It is a so-called uncertainty principle.

3. The image of fluctuating wave function

If (8) is differentiated by x , it becomes

$$\frac{\partial\psi(x,t)}{\partial x}=\left\{-\frac{1}{\hbar}\frac{V(x,t)\,dt}{dx}\right\}\psi(x,t) \qquad (10).$$

So after (10) is substituted for (5), it becomes

$$d\psi(x,t)=-\frac{V(x,t)}{\hbar}\psi(x,t)\,dt+\left\{-\frac{1}{\hbar}\frac{V(x,t)\,dt}{dx}\right\}\psi(x,t)\sqrt{\frac{\hbar}{m}}dW^*(t) \qquad (11).$$

If both sides is divided by $\psi(x,t)$, it becomes

$$\frac{d\psi(x,t)}{\psi(x,t)}=-\frac{V(x,t)}{\hbar}dt+\left\{-\frac{1}{\hbar}\frac{V(x,t)\,dt}{dx}\right\}\sqrt{\frac{\hbar}{m}}dW^*(t) \qquad (12).$$

$$\frac{d\psi(x,t)}{\psi(x,t)}=d\log[\psi(x,t)]=\log[\psi(x,t)]-\log[\psi(x(t-dt),t-dt)]=\log\frac{\psi(x,t)}{\psi(x(t-dt),t-dt)}$$

, So (12) becomes

$$\log\frac{\psi(x,t)}{\psi(x(t-dt),t-dt)}=-\frac{V(x,t)}{\hbar}dt+\left\{-\frac{1}{\hbar}\frac{V(x,t)\,dt}{dx}\right\}\sqrt{\frac{\hbar}{m}}dW^*(t) \qquad (13).$$

Therefore, $\psi(x,t)$ becomes

$$\psi(x,t)=\exp\left(-\frac{1}{\hbar}\left[V(x,t)\,dt+\left\{\frac{V(x,t)\,dt}{dx}\right\}\sqrt{\frac{\hbar}{m}}dW^*(t)\right]\right)\psi(x(t-dt),t-dt)$$

$$=\exp\left(-\frac{1}{\hbar}V(x,t)\left[dt+\frac{dt}{dx}\sqrt{\frac{\hbar}{m}}dW^*(t)\right]\right)\psi(x(t-dt),t-dt)$$

$$=\exp\left(-\frac{1}{\hbar}V(x,t)\left[dt+\frac{1}{v_x}\sqrt{\frac{\hbar}{m}}dW^*(t)\right]\right)\psi(x(t-dt),t-dt) \qquad (14).$$

A Brownian motion paragraph appears to the shoulder of exponential, and, as a result, the wave function fluctuates.

Considering that the potential is a function of also y like $V(x)=V(x,y)$, the discussion when the electron is turning to the vertical direction (y) against the incidence direction (x) receiving power is as follows.

$$\psi(x,t)=\exp\left(\frac{1}{\hbar}p_y\cdot dy\right)\psi(x(t-dt),t-dt) \qquad (15).$$

$$p_y=-\frac{\partial V(x,y)}{\partial y}\left[dt+\frac{1}{v_x}\sqrt{\frac{\hbar}{m}}dW^*(t)\right] \qquad (16).$$

It is understood that the momentum in the direction of y fluctuates by the Brownian motion.

When we replace t with it in (14), it becomes

$$\psi(x,t)=\exp\left(-\frac{i}{\hbar}V(x,t)\left[dt+\frac{1}{v_x}\sqrt{\frac{\hbar}{m}}\sqrt{-idt}\xi\right]\right)\psi(x(t-dt),t-dt)$$

$$=\exp\left(-\frac{i}{\hbar}V(x,t)\left[dt+(1-i)\frac{1}{v_x}\sqrt{\frac{\hbar}{2m}}\sqrt{dt}\xi\right]\right)\psi(x(t-dt),t-dt)$$

$$=\exp\left(-\frac{1}{\hbar}V(x,t)\left[\frac{1}{v_x}\sqrt{\frac{\hbar}{2m}}dW(t)\right]\right)\exp\left(-\frac{i}{\hbar}V(x,t)\left[dt+\frac{1}{v_x}\sqrt{\frac{\hbar}{2m}}dW(t)\right]\right)\psi(x(t-dt),t-dt)$$

(17).

When we replace t with it in (15) and (16), it becomes

$$\psi\left(x,t\right)=\exp\left(-\frac{1}{\hbar}\frac{\partial V\left(x,t\right)}{\partial y}\left[\frac{1}{v_x}\sqrt{\frac{\hbar}{2m}}dW\left(t\right)\right]\cdot dy\right)\exp\left(\frac{i}{\hbar}p_y\cdot dy\right)\psi\left(x\left(t-dt\right),t-dt\right)$$

(18).

$$p_y=-\frac{\partial V\left(x,y\right)}{\partial y}\left[dt+\frac{1}{v_x}\sqrt{\frac{\hbar}{2m}}dW\left(t\right)\right] \quad (19).$$

To describe the fluctuating wave directly, (7),(17) and (18) are needed.

The wave motion itself fluctuates because of the existence of this solution. We expect that the result of the two-slit experiment can be explained using this new solution of Schrödinger equation: fluctuating wave motion.

For instance, there is an two-slit experiment with electron carried out by Tonomura et al[2]. According to Tonomura et al this experiment is explained by the following theories.

4. The theory of two-slit experiment in advanced research laboratory, Hitachi Limited; Tonomura et al[2].

The biprism consists of two parallel grounded plates with a fine filament between them, the latter having a positive potential relative to the former. The electrostatic potential is given by $V\left(x,z\right)$ and the incoming electron wave by $\exp i\left(k_z z\right)$, the deflected wave is given by

$$\psi\left(x,z\right)=\exp i\left(k_z z-\frac{me}{\hbar^2 k_z}\int_{-\infty}^{z}V\left(x,z'\right)dz'\right), \quad \text{T-(1)}$$

The two waves having passed on each side of the filament can be approximated by $\exp i\left(k_z z\pm k_x x\right)$ up to a constant factor, where

$$k_x=-\frac{me}{\hbar^2 k_z}\int_{-\infty}^{\infty}\left(\frac{\partial V\left(x,z'\right)}{\partial x}\right)_{x=a}dz', \quad \text{T-(2)}$$

and the symmetry $V\left(x,z\right)=V\left(-x,z\right)$ has been taken into account. This can be interpreted classically also:

$-e\left[\partial V\left(x,z'\right)/\partial x\right]_{x=a}$ is the x component of force exerted on the electron. Its integral with respect to $dz/v_z=dt$ ($v_z=\hbar k_z/m$) gives the impulse imparted to it, which is the same in absolute value but reversed in sign, depending on which side of the filament the electron passes. If the two waves overlap in the observation plane to give

$$\psi\left(x,z\right)=\exp\left(ik_z z\right)\left[\exp\left(-ik_x x\right)+\exp\left(ik_x x\right)\right], \quad \text{T-(3)}$$

then this leads to the interference fringes

$$\left|\psi\left(x,z\right)\right|^2=4\cos^2 k_x x. \quad \text{T-(4)}$$

The Copenhagen interpretation has concluded that $\psi\left(x,z\right)$ is the "probability amplitude".

5. Alternative explanation to Copenhagen interpretation: Waviness

If v_z is deleted from

$dz/v_z=dt$, $v_z=\hbar k_z/m$

that exists between the expression T-(2) and the expression T-(3) of the Tonomura thesis, it becomes $\frac{m}{\hbar k_z}dz=dt$.

If this expression is substituted for the expression T-(1) of the Tonomura thesis, it becomes

$$\psi(x,z) = \exp i\left(k_z z - \frac{1}{\hbar}\int_{t_0}^{t} eV(x,z')\,dt'\right).$$

And the expression T-(2) of the Tonomura thesis becomes

$$k_x = \frac{p_x}{\hbar} = -\frac{1}{\hbar}\int_{t_0}^{t} e\left(\frac{\partial V(x,z')}{\partial x}\right)_{x=a} dt'.$$

However, because it is necessary to use the fluctuating wave motion (18),(19) accurately, the impulse that electron receives is not

$$-e\left[\partial V(x,z')/\partial x\right]_{x=a} dt$$

but

$$-e\left[\partial V(x,z')/\partial x\right]_{x=a}\left(dt + \frac{1}{v_z}\sqrt{\frac{\hbar}{2m}}dW(t)\right).$$

(Note : While the direction of incidence is x and the direction where electron that receives power bends is y in (18),(19), the direction of incidence is z and the direction where electron that receives power bends is x in Tonomura thesis.) This is the cause of fluctuation.

The momentum of electron fluctuates by this fluctuation of impulse, and the wave number vector of electron also fluctuates.

As understood when seeing (18),(19), the wave fluctuates only when it is in potential, and the wave doesn't fluctuate in the area that can be considered potential to be 0.

So, k_x is a different value in each wave that has occurred from biprism. Expression T-(2) means that k_x is determined by the accumulation of the impulse that the electron wave received from potential energy when it went in the biprism. In general, because the impulse fluctuates, the impulse that the first wave received in the biprism and the impulse that the second wave received are different. As a result, the value of k_x is different according to each wave.

As shown by the expression T-(3)

$$\psi(x,z) = \exp(ik_z z)\left[\exp(-ik_x x) + \exp(ik_x x)\right]$$

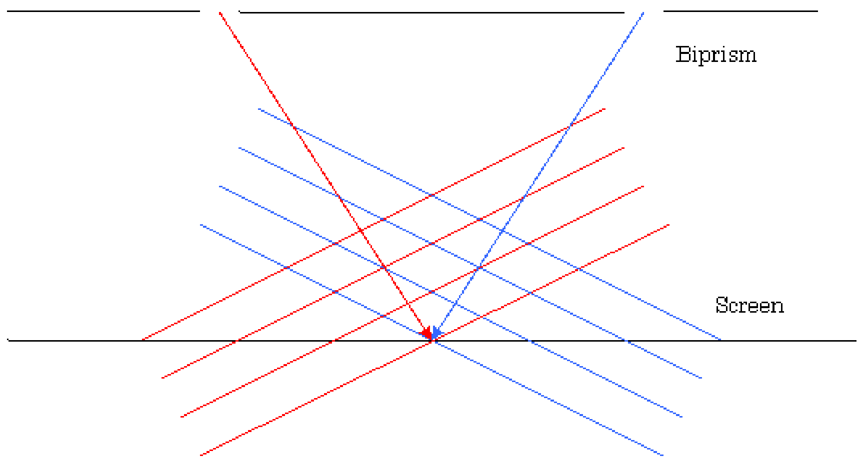
of the Tonomura thesis, there is k_x in the wave function that is reaching the screen from right and left biprism. So the phase of wave fluctuates because it receives the influence of fluctuation of the impulse.

The appearance of the interference when wave number k_x fluctuates is seen as follows.

By the expression T-(3) of the Tonomura thesis, the wave number vector of the wave that comes from the left can be written as $(k_x, 0, k_z)$ and the wave number vector that comes from the right, $(-k_x, 0, k_z)$.

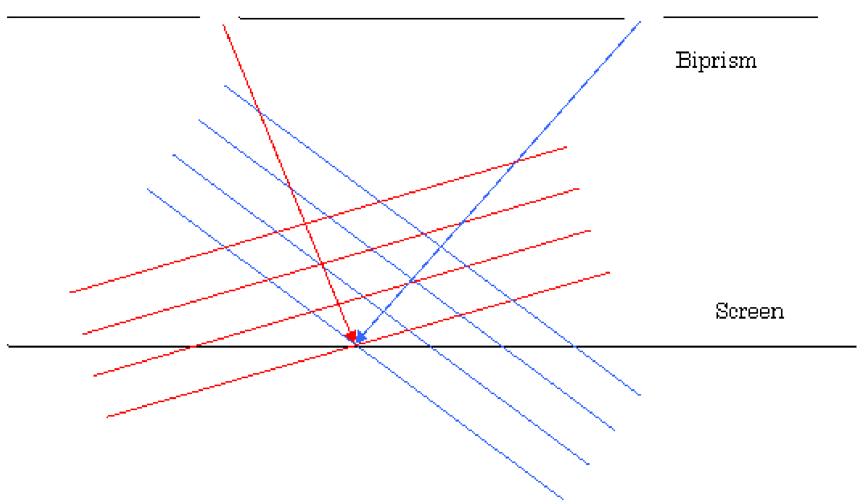
In general, because the impulse doesn't fluctuate symmetrically, neither k_x from the left nor k_x from the right are equal. Then, the wave number vector of the wave that comes from the left is written as $(k_x(L), 0, k_z)$, and the wave number vector of the wave that comes from the right is written as $(-k_x(R), 0, k_z)$. It becomes a different value because of the first wave, the second wave, and the third wave even if paying attention only to $k_x(L)$ (paying attention only to $k_x(R)$).

Figure 1. When the wave number $k_x\,(L)$ that comes from the left and the wave number $k_x\,(R)$ that comes from the right are equal:



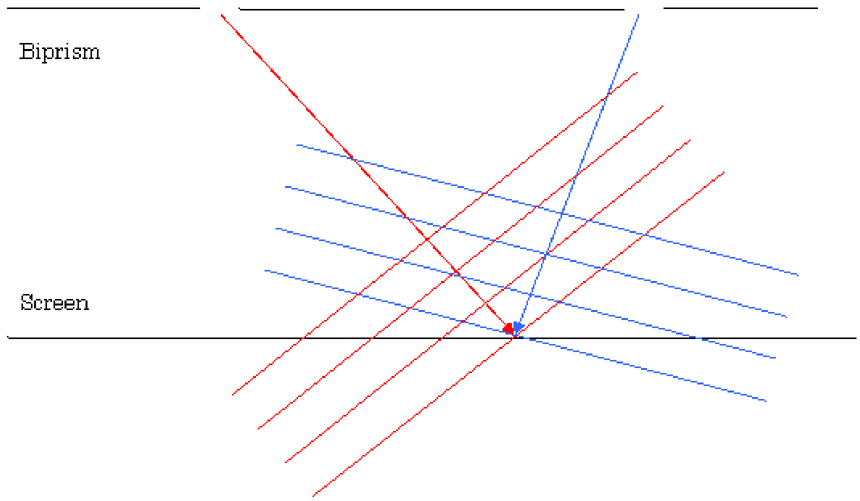
Bright spot appears at the center of the screen.

Figure 2: When the wave number $k_x\,(R)$ that comes from the right is larger than the Wave number $k_x\,(L)$ that comes from the left:



Bright spot shifts left.

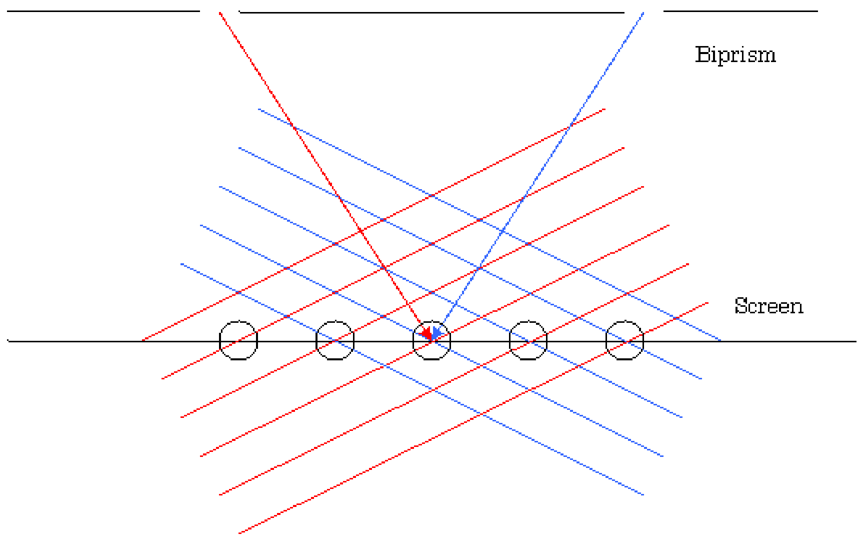
Figure 3: When the wave number $k_x\,(R)$ that comes from the right is smaller than the Wave number $k_x\,(L)$ that comes from the left:



Bright spot shifts right.

We suggest that this is the mechanism by which the bright spot is observed at a random position. Because the impulse that the electron receives in the potential energy that the filament makes fluctuates, the wave number $k_x\left(L\right)$ of the wave that passes the left prism sometimes becomes large and at other times it becomes small. The wave number $k_x\left(R\right)$ of the wave that passes the right prism is also similar. As a result, the position that two waves strengthen each other is different on each occasion as shown in the above figure. In current quantum theory, only the case of Figure 1 is considered, and it is said that the place enclosed in the following figure is a position of the interference fringes.

Figure 4: The position of the interference fringes:

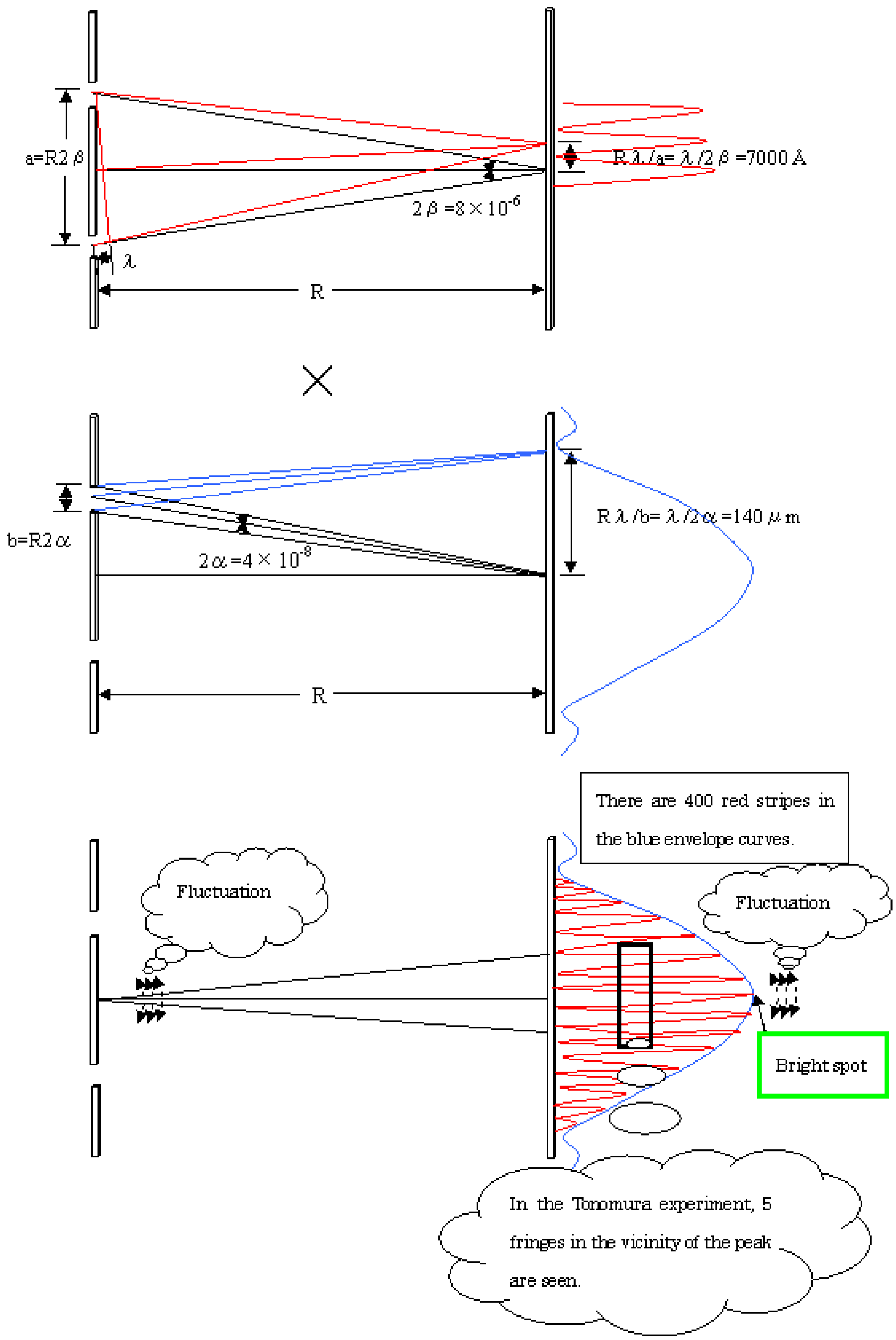


It follows from this that our interpretation is different from the current interpretation of the quantum theory in that it becomes a very dynamic image like two moving searchlights independently scattering waves of light into the night sky. On the other hand, the image of current quantum theory is very static.

The second point that requires clarification is why it is observed as “a spot” in the experiment when the electron wave is weakened. The reason why it is observed as “a spot” is that the effect of the diffraction (so-called Fraunhofer diffraction) exacerbates the above-mentioned interference because the opening of biprism is not the ideal one like the delta function but has some size in an actual experiment. Therefore, strength of the electron wave on the screen becomes narrowed shape like the interference fringes shown by cos function narrowed by *sinc* function ($\sin x/x$).

(Refer to the figure below. A part of the numerical value is excerpted from the Tonomura thesis.)

Figure 5: The Fraunhofer diffraction in the two-slit experiment carried out by Tonomura et al:



Strength of the electron wave on the screen becomes shaped like a sliced mountain. There were an estimated 400 slices in the Tonomura experiment. In addition, only a very narrow area (center part of Airy disk so-called) in the top of a mountain will reflect because the pictures in this experiment were taken with very limited sensitivity. It is concluded that this is the bright spot observed on the screen.

Furthermore, the top of the mountain shakes at random due to the above-mentioned fluctuation. The peak of the distribution of the Fraunhofer diffraction appears at random because the potential energy (electric field) fluctuates and the electron wave fluctuates. Also, because the electron wave discharged from the electron gun is weak, only the part of the peak is taken of a picture.

Up to this point we have explained the two-slit experiment by only waviness.

6. Proposal to experiment

According to Tonomura theses, the width of the interference fringes is $7000\mathring{A}$ and the transverse coherence length is $140\mu m$. If the Fraunhofar diffraction pattern is a probability wave said by a present quantum theory, the bright spot is sure to scatter over $280\mu m$, and to appear by many hundreds of (400 theoretically) interference fringes. It is because the electron reaches in the edge of the Fraunhofar diffraction even though the probability is low.

On the other hand, the number of interference fringes is sure to be only ten or more in our proposal. It is because only peak of that figure is taken of picture, and it fluctuates. The rough estimate is as follows. The standard deviation (volatility) of fluctuating of the position of the peak of the Fraunhofar diffraction is

$$\sigma = \sqrt{\frac{\hbar}{2m}}\sqrt{\Delta t}$$

by (19).

According to Tonomura theses,

Distance from slit to screen : $1.5m$

Velocity of electron : $1.3 \times 10^8 m/s$ (Accelerating voltage : $50kV$).

So σ becomes about $0.8\mu m$.

Therefore, ranges where the peak of the Fraunhofar diffraction is distributed are $2 \times 0.8 = 1.6\mu m$ in 1σ (cover rate of 68%). ($\times 2$ is a meaning of both sides of normal distribution.) Therefore, the number of interference fringes $= 1.6\mu m \div 900\mathring{A} = 18$. (Note : According to Tonomura theses, the width of the interference fringes is $900\mathring{A}$ theoretically while it is $7000\mathring{A}$ experimentally. It is because a spherical wave instead of a plane wave is incident on the biprism in the actual experiment.) So it is sure to be taken a picture of only ten or more interference fringes. Moreover, because fluctuating of the peak of the Fraunhofar diffraction is normal distribution, it is expected that the interference fringe in the center part is bright, and it darkens to the surrounding.

7. Conclusion

The wave motion itself fluctuates because of the existence of this solution

$$\psi(x,t) = \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} V(x,t')\,dt'\right)\psi(x_0,t_0)$$

$$+\exp\left(-\frac{\pi}{4}i\right)\int_{t_0}^{t}\exp\left(-\frac{i}{\hbar}\int_{u}^{t}V\left(x,t'\right)dt'\right)\frac{\partial\psi\left(x,u\right)}{\partial x}\sqrt{\frac{\hbar}{m}}dW\left(u\right).$$

We expect that the result of the two-slit experiment carried out by Tonomura et al can be explained as follows using this new solution of Schrödinger equation: fluctuating wave motion.

In the two-slit experiment, the wave number vector of each wave that occurs from biprism fluctuates by normal distribution. The wave that occurred from biprism is launched in various directions for this fluctuation. This fluctuation is expressed by the probability distribution of normal distribution.

In conclusion, we should note that while in present quantum mechanics it is the wave function that determines the probability distribution of the electron, our observations show that it is not the wave function but the kinetic energy exponential part that determines the probability distribution. As a result, wave motion itself fluctuates. Furthermore, the bright spot observed on the screen is not "an electron" but the peak of the distribution of the Fraunhofer diffraction.

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Appendix A "Normal distribution and kinetic energy"

In general, when x follows the Stochastic process

$$x-x_0=dx=\mu dt+\sigma dW=\mu dt+\sigma\sqrt{dt}\xi$$

(dW : Standard Brownian motion, ξ : Standard regular random variable), x becomes normal distribution

$$\Phi\left(x,t\right)=\frac{1}{\sqrt{2\pi\sigma^2 dt}}\exp\left[-\frac{\left(x-x_0-\mu dt\right)^2}{2\sigma^2 dt}\right] \quad (20).$$

On the other hand, a kinetic energy paragraph of path integral becomes

$$\sqrt{\frac{m}{2\pi\hbar dt}}\exp\left[-\frac{1}{\hbar}\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 dt\right]$$

after Euclidean approach.

$$\sqrt{\frac{m}{2\pi\hbar dt}}\exp\left[-\frac{1}{\hbar}\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 dt\right]=\sqrt{\frac{m}{2\pi\hbar dt}}\exp\left[-\frac{1}{\hbar}\frac{1}{2}m\frac{\left(dx\right)^2}{dt}\right]=\sqrt{\frac{m}{2\pi\hbar dt}}\exp\left[-\frac{1}{\hbar}\frac{1}{2}m\frac{\left(x-x_0\right)^2}{dt}\right]$$

(21),

So, (20) corresponds to (21) after replacing

$\sigma=\sqrt{\frac{\hbar}{m}}$, $\mu=0$.

In a word, the process x follows is not

$dx=\mu dt+\sigma dW$

but

$dx = \sigma dW$
in path integral.

Appendix B "Ito's lemma"

When X follows the Ito process
$dX = a(X,t)\,dt + b(X,t)\,dW(t)$,
the movement of function $f(X,t)$ of X and t follows
$$df = \left(a(X,t)\frac{\partial f}{\partial X} + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\{b(X,t)\}^2 \right) dt + \frac{\partial f}{\partial X} b(X,t)\,dW(t).$$
Here, $dW(t)$ means Standard Brownian motion.

References

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