Born's Reciprocal General Relativity Theory and Complex Nonabelian Gravity as Gauge Theory of the Quaplectic Group : A novel path to Quantum Gravity

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Abstract

Born's Reciprocal Relativity in flat spacetimes is based on the principle of a *maximal* speed limit (speed of light) and a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) and where coordinates and momenta are unified on a single footing. We extend Born's theory to the case of curved spacetimes and construct a Reciprocal General Relativity theory (in curved spacetimes) as a local Gauge Theory of the Quaplectic Group and given by the semi-direct product $\mathcal{Q}(1,3) \equiv U(1,3) \otimes_s H(1,3)$, where the Nonabelian Weyl-Heisenberg group is $H(1,3)$. The gauge theory has the same structure as that of Complex Nonabelian Gravity. Actions are presented and it is argued why such actions based on Born's Reciprocal Relativity principle, involving a maximal speed limit and a maximal proper force, is a very promising avenue to Quantize Gravity that does *not* rely in *breaking* the Lorentz symmetry at the Planck scale, in contrast to other approaches based on deformations of the Poincare algebra, Quantum Groups. It is discussed how one could embed the Quaplectic gauge theory into one based on the $U(1,4), U(2,3)$ groups where the observed cosmological constant emerges in a natural way. We conclude with a brief discussion of Complex coordinates and Finsler spaces with symmetric and nonsymmetric metrics studied by Eisenhart as relevant closed-string target space backgrounds where Born's principle may be operating.

1 Introduction : On Born's Reciprocal Theory of Relativity

In this introductory section we will review in detail Born's Reciprocal ("Dual") Relativity [1] and the principle of Maximal-acceleration Relativity from the perspective of $8D$ Phase Spaces and the role of the invariance $U(1,3)$ Group. We will focus for simplicity on a $flat$ $8D$ Phase Space. A *curved* case scenario has been analyzed by Brandt [8] within the context of the Finsler geometry of the $8D$ tangent bundle of spacetime and written the generalized $8D$ gravitational equations that reduce to the ordinary Einstein-Riemannian gravitational equations in the $infinite$ acceleration limit. Vacaru [35] has constructed the Riemann-Finsler geometries endowed with non-holonomic structures induced by *nonlinear* connections and developed the formalism to build a Noncommutative Riemann-Finsler Geometry by introducing suitable Clifford structures. A curved *momentum* space geometry was studied by [27] . Toller [18] has explored the different possible geometries associated with the maximal acceleration principle and the physical implications of the meaning of an "observer", "measuring device" in the cotangent space.

The $U(1,3) = SU(1,3) \otimes U(1)$ Group transformations, which leave invariant the phase-space intervals under rotations, velocity and acceleration boosts, were found by Low [20] and can be simplified drastically when the velocity/acceleration boosts are taken to lie in the z-direction, leaving the transverse directions x, y, p_x, p_y intact ; i.e., the $U(1,1) = SU(1,1) \otimes U(1)$ subgroup transformations that leave invariant the phase-space interval are given by (in units of $\hbar = c = 1$)

$$(d\omega)^2 = (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} =$$

$$(d\tau)^2 [1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2}] = (d\tau)^2 [1 - \frac{m^2 g^2(\tau)}{m_P^2 A_{max}^2}]. \tag{1.1}$$

where we have factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in eq-(1-1) and the maximal proper-force is set to be $b \equiv m_P A_{max}$. m_P is the Planck mass $1/L_P$ so that $b = (1/L_P)^2$, may also be interpreted as the maximal string tension when L_P is the Planck scale.

The quantity $g(\tau)$ is the proper four-acceleration of a particle of mass m in the z-direction which we take to be defined by the X coordinate. The interval $(d\omega)^2$ described by Low [20] is $U(1,3)$-invariant for the most general transformations in the $8D$ phase-space. These transformations are rather elaborate, so we refer to the references [20] for details. The appearance of the $U(1,3)$ group in $8D$ Phase Space is not too surprising since it could be seen as the " complex doubling " version of the Lorentz group $SO(1,3)$. Low discussed the irreducible unitary representations of such $U(1,3)$ group and the relevance for the strong interactions of quarks and hadrons since $U(1,3)$, with 16 generators, contains the $SU(3)$ group.

The analog of the Lorentz relativistic factor in eq-(1-1) involves the ratios of two proper *forces*. One variable force is given by $mg(\tau)$ and the maximal proper force sustained by an *elementary* particle of mass m_P (a *Planckton*) is assumed to be $F_{max} = m_{Planck} c^2 / L_P$. When $m = m_P$, the ratio-squared of the forces appearing in the relativistic factor of eq-(1-1) becomes then g^2 / A_{max}^2, and the phase space interval coincides with the geometric interval discussed by [9], [5], [11] , [12] .

The transformations laws of the coordinates in that leave invariant the interval (1-1) were given by [20]:

$$T' = T cosh\xi + (\frac{\xi_v X}{c^2} + \frac{\xi_a P}{b^2}) \frac{sinh\xi}{\xi}. \tag{1.2a}$$

$$E' = E cosh\xi + (-\xi_a X + \xi_v P) \frac{sinh\xi}{\xi}. \tag{1.2b}$$

$$X' = X cosh\xi + (\xi_v T - \frac{\xi_a E}{b^2}) \frac{sinh\xi}{\xi}. \tag{1.2c}$$

$$P' = P cosh\xi + (\frac{\xi_v E}{c^2} + \xi_a T) \frac{sinh\xi}{\xi}. \tag{1.2d}$$

The ξ_v is velocity-boost rapidity parameter and the ξ_a is the force/acceleration-boost rapidity parameter of the primed-reference frame. They are defined respectively :

$$tanh(\frac{\xi_v}{c}) = \frac{v}{c}. \quad tanh(\frac{\xi_a}{b}) = \frac{ma}{m_P A_{max}}. \tag{1.3}$$

The *effective* boost parameter ξ of the $U(1,1)$ subgroup transformations appearing in eqs-(1-2a, 1-2d) is defined in terms of the velocity and acceleration boosts parameters ξ_v, ξ_a respectively as:

$$\xi \equiv \sqrt{\frac{\xi_v^2}{c^2} + \frac{\xi_a^2}{b^2}}. \tag{1.4}$$

Our definition of the rapidity parameters are *different* than those in [20].

Straightforward algebra allows us to verify that these transformations leave the interval of eq- (1-1) in classical phase space invariant. They are are fully consistent with Born's duality Relativity symmetry principle [1] $(Q, P) \to (P, -Q)$. By inspection we can see that under Born duality, the transformations in eqs-(1-2a, 1-2d) are *rotated* into each other, up to numerical b factors in order to match units. When on sets $\xi_a = 0$ in (1-2a, 1-2d) one recovers automatically the standard Lorentz transformations for the X, T and E, P variables *separately*, leaving invariant the intervals $dT^2 - dX^2 = (d\tau)^2$ and $(dE^2 - dP^2)/b^2$ separately.

When one sets $\xi_v = 0$ we obtain the transformations rules of the events in Phase space, from one reference-frame into another $uniformly$-accelerated frame of reference, $a = constant$, whose acceleration-rapidity parameter is in this particular case:

$$\xi \equiv \frac{\xi_a}{b}. \quad tanh(\xi) = \frac{ma}{m_P A_{max}}. \tag{1.5}$$

The transformations for pure acceleration-boosts in Phase Space are:

$$T' = T cosh\xi + \frac{P}{b} sinh\xi. \tag{1.6a}$$

$$E' = E cosh\xi - b X sinh\xi. \tag{1.6b}$$

$$X' = X cosh\xi - \frac{E}{b} sinh\xi. \tag{1.6c}$$

$$P' = P cosh\xi + b T sinh\xi. \tag{1.6d}$$

It is straightforwad to verify that the transformations (1-6a, 1-6c) leave invariant the fully phase space interval (1-1) but *does not* leave invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$. Only the *combination*:

$$(d\omega)^2 = (d\tau)^2 (1 - \frac{m^2 g^2}{m_P^2 A_{max}^2}) \tag{1.7}$$

is truly left invariant under pure acceleration-boosts in Phase Space. Once again, can verify as well that these transformations satisfy Born's duality symmetry principle:

$$(T, X) \rightarrow (E, P). \ (E, P) \rightarrow (-T, -X). \tag{1.8}$$

and $b \rightarrow \frac{1}{b}$. The latter Born duality transformation is nothing but a manifestation of the large/small tension duality principle reminiscent of the T-duality symmetry in string theory; i.e. namely, a small/large radius duality, a winding modes/ Kaluza-Klein modes duality symmetry in string compactifications and the Ultraviolet/Infrared entanglement in Noncommutative Field Theories. Hence, Born's duality principle in exchanging coordinates for momenta could be the underlying physical reason behind T-duality in string theory.

The composition of two succesive pure acceleration-boosts is another pure acceleration-boost with acceleration rapidity given by $\xi'' = \xi + \xi'$. The addition of *proper* forces (accelerations) follows the usual relativistic composition rule:

$$tanh\xi'' = tanh(\xi + \xi') = \frac{tanh\xi + \tanh\xi'}{1 + tanh\xi tanh\xi'} \Rightarrow \frac{ma''}{m_P A} = \frac{\frac{ma}{m_P A} + \frac{ma'}{m_P A}}{1 + \frac{m^2 a a'}{m_P^2 A^2}}. \tag{1.9}$$

and in this fashion the upper limiting *proper* acceleration is never *surpassed* like it happens with the ordinary Special Relativistic addition of velocities.

The group properties of the full combination of velocity and acceleration boosts eqs-(1-2a, 1-2d) in Phase Space requires much more algebra [4]. A careful study reveals that the composition *rule* of two succesive full transformations is given by $\xi'' = \xi + \xi'$ and the transformation laws are *preserved* if, and only if, the $\xi; \xi'; \xi''$...... parameters obeyed the suitable relations:

$$\frac{\xi_a}{\xi} = \frac{\xi'_a}{\xi'} = \frac{\xi''_a}{\xi''} = \frac{\xi''_a}{\xi + \xi'}. \tag{1.10a}$$

$$\frac{\xi_v}{\xi} = \frac{\xi'_v}{\xi'} = \frac{\xi''_v}{\xi''} = \frac{\xi''_v}{\xi + \xi'}. \tag{1.10b}$$

Finally we arrive at the compostion law for the effective, velocity and acceleration boosts parameters $\xi''; \xi''_v; \xi''_a$ respectively:

$$\xi''_v = \xi_v + \xi'_v. \tag{1.11a}$$

$$\xi''_a = \xi_a + \xi'_a. \tag{1.11b}$$

$$\xi'' = \xi + \xi'. \tag{1.11c}$$

The above relations among the parameters are required in order to prove the $U(1,1)$ *group* composition law of the transformations in order to have a truly Maximal-Acceleration Phase Space Relativity theory resulting from a Phase-Space change of coordinates in the cotangent bundle of spacetime.

Planck-Scale Areas are Invariant under Acceleration Boosts

Having displayed explicity the Group transformations rules of the coordinates in Phase space we will show why $infinite$ acceleration-boosts (which is *not* the same as infinite proper acceleration) preserve Planck-Scale $Areas$ [4] as a result of the fact that $b = (1/L_P^2)$ equals the $maximal$ invariant force, or string tension, if the units of $\hbar = c = 1$ are used.

At Planck-scale L_P intervals/increments in one reference frame we have by definition (in units of $\hbar = c = 1$): $\Delta X = \Delta T = L_P$ and $\Delta E = \Delta P = \frac{1}{L_P}$ where $b \equiv \frac{1}{L_P^2}$ is the maximal tension. From eqs-(1-6a, 1-6d) we get for the transformation rules of the finite intervals $\Delta X, \Delta T, \Delta E, \Delta P$, from one reference frame into another frame, in the $infinite$ acceleration-boost limit $\xi \to \infty$,

$$\Delta T' = L_P(cosh\xi + sinh\xi) \to \infty \tag{1.12a}$$

$$\Delta E' = \frac{1}{L_P}(cosh\xi - sinh\xi) \to 0 \tag{1.12b}$$

by a simple use of L'Hopital's rule or by noticing that both $cosh\xi; sinh\xi$ functions approach infinity at the same rate.

$$\Delta X' = L_P(cosh\xi - sinh\xi) \to 0. \tag{1.12c}$$

$$\Delta P' = \frac{1}{L_P}(cosh\xi + sinh\xi) \to \infty \tag{1.12d}$$

where the discrete displacements of two events in Phase Space are defined: $\Delta X = X_2 - X_1 = L_P$, $\Delta E = E_2 - E_1 = \frac{1}{L_P}$, $\Delta T = T_2 - T_1 = L_P$ and $\Delta P = P_2 - P_1 = \frac{1}{L_P}$.

Due to the identity:

$$(cosh\xi + sinh\xi)(cosh\xi - sinh\xi) = cosh^2\xi - sinh^2\xi = 1 \tag{1.13}$$

one can see from eqs-(1.12a, 1.12d) that the Planck-scale *Areas* are truly *invariant* under *infinite* acceleration-boosts $\xi = \infty$:

$$\Delta X'\Delta P' = 0 \times \infty = \Delta X \Delta P(cosh^2\xi - sinh^2\xi) = \Delta X \Delta P = \frac{L_P}{L_P} = 1. \tag{1.14a}$$

$$\Delta T'\Delta E' = \infty \times 0 = \Delta T \Delta E(cosh^2\xi - sinh^2\xi) = \Delta T \Delta E = \frac{L_P}{L_P} = 1. \tag{1.14b}$$

$$\Delta X'\Delta T' = 0 \times \infty = \Delta X \Delta T(cosh^2\xi - sinh^2\xi) = \Delta X \Delta T = (L_P)^2. \quad (1.14c)$$

$$\Delta P'\Delta E' = \infty \times 0 = \Delta P \Delta E(cosh^2\xi - sinh^2\xi) = \Delta P \Delta E = \frac{1}{L_P^2}. \quad (1.14d)$$

It is important to emphasize that the invariance property of the minimal Planck-scale *Areas* (maximal Tension) is *not* an exclusive property of *infinite* acceleration boosts $\xi = \infty$, but, as a result of the identity $cosh^2\xi - sinh^2\xi = 1$, for all values of ξ, the minimal Planck-scale *Areas* are *always* invariant under *any* acceleration-boosts transformations. Meaning physically, in units of $\hbar = c = 1$, that the Maximal Tension (or maximal Force) $b = \frac{1}{L_P^2}$ is a true physical *invariant* universal quantity. Also we notice that the Phase-space areas, or cells, in units of $\hbar$, are also invariant ! The pure-acceleration boosts transformations preserve the symplectic form $dX \wedge dP + dE \wedge dT$.

The infinite acceleration-boosts are closely related to the infinite red-shift effects when light signals barely escape Black hole Horizons reaching an asymptotic observer with an infinite redshift factor. The important fact is that the Planck-scale *Areas* are truly maintained invariant under acceleration-boosts. This could reveal very important information about Black-holes Entropy and Holography.

A natural action associated with the invariant interval in Phase-Space given by eq-(1.1) is :

$$S = m \int d\tau \sqrt{1 + \frac{m^2}{m_P^2 a^2}(d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2)}. \quad (1.15)$$

The proper-acceleration is orthogonal to the proper-velocity and this can be easily verified by differentiating the timelike proper-velocity squared:

$$V^2 = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = V^\mu V_\mu = 1 > 0 \Rightarrow \frac{dV^\mu}{d\tau}V_\mu = \frac{d^2x^\mu}{d\tau^2}V_\mu = 0. \quad (1.16)$$

which implies that the proper-acceleration is spacelike:

$$-g^2(\tau) = \frac{d^2x^\mu}{d\tau^2}\frac{d^2x_\mu}{d\tau^2} < 0 \Rightarrow S = m\int d\tau\sqrt{1 - \frac{m^2g^2}{m_P^2a^2}} = m\int d\omega. \quad (1.17)$$

where the analog of the Lorentz time-dilation factor in Phase-space is now given by

$$d\omega = d\tau\sqrt{1 - \frac{m^2g^2(\tau)}{m_P^2a^2}}. \quad (1.18a)$$

namely,

$$(d\omega)^2 = \Omega^2 d\tau^2 = [1 - \frac{m^2g^2(\tau)}{m_P^2a^2}]g_{\mu\nu}dx^\mu dx^\nu. \quad (1.18b)$$

The invariant proper interval is no longer the standard proper-time τ but is given by the quantity $\omega(\tau)$. The deep connection between the physics of maximal acceleration and Finsler geometry has been analyzed by [8]. The action is real-valued if, and only if, $m^2g^2 < m_P^2a^2$ in the same fashion that the action in Minkowski spacetime is real-valued if, and only if, $v^2 < c^2$. This is the physical reason why there is an upper bound in the proper-four force acting on a fundamental particle given by $(mg)_{bound} = m_P(c^2/L_P) = m_P^2$ in natural units of $\hbar = c = 1$.

In the next section we construct a local gauge theory of the Quaplectic group $\mathcal{Q}(1,3) \equiv U(1,3) \otimes_s H(1,3)$, where the Nonabelian Heisenberg group is $H(1,3)$ and which has been studied extensively over the years by Low [20]. The $U(1,3)$ arises as the group that leaves invariant the interval in $8D$ Phase $dx_\mu dx^\mu + dp_\mu dp^\mu$ space as well as invariant the symplectic two-form $\omega = \omega_{\mu\nu} dx^\mu \wedge dp^\nu$, simultaneously. For further details on the Quaplectic group $\mathcal{Q}(1,3) \equiv U(1,3) \otimes_s H(1,3)$, the Nonabelian Phase Space defined as the coset $\mathcal{Q}^{1,3} \equiv \mathcal{Q}(1,3)/SU(1,3)$; Casimir invariant field equations; unitary irreducible representations based on Mackey's theory of induced representations; relativistic harmonic oscillator, coherent states, the granularity of spacetime, the Schrodinger-Robertson inequality, multi-mode squeezed states, "non-commutative" relativistic phase space geometry, ... in which position and momentum are *interchangeable* and frame-dependent, ... see [20], [21].

2 Complex Nonabelian Gravity as a Gauge Theory of the Quaplectic Group and Born's Reciprocal General Relativity

Einstein's General Theory of Relativity admits a reformulation as a gauge theory of the Poincare group that is the semi-direct product of the Lorentz group and the (Abelian) Translation group $SO(1,3) \otimes_s T_4$. The spin connection $\omega_\mu^{[ab]}$ gauges the (local) Lorentz symmetry of the tangent space while the tetrad e_μ^a gauges the (local) Abelian translations. The latter fields can be incorporated into a connection $A_\mu = \omega_\mu^{[ab]} L_{[ab]} + e_\mu^a P_a$. The Lorentz and translation generators are $L_{[ab]}, P_a$ respectively with $a, b = 1, 2, 3, 4$ and where $[ab]$ denotes the anti-symmetry property of the indices of the Lorentz generators. Supergravity also admits a reformulation as a gauge theory of the super-Poincare group where the spin $\frac{3}{2}$ gravitino field Ψ_μ^α gauges the (local) supersymmetry algebra and whose generators are given by the spinorial Q_α "charges". The Metric Affine Theories of Gravity are based on the semi-direct product of $GL(4,R) \otimes_s T_4$ [19]. The group $GL(4,R)$ admits *infinite*-dimensional spinorial representations and for this reason it has not been widely used, compared to the Lorentz group that admits finite-dimensional spinorial representations.

The main purpose of this section is to show how one can construct an Extended General Relativity Theory based on Born's Reciprocal Relativity principe with a *maximal* speed limit c and a *maximal* proper force $\mathcal{F}_{max} = b$ by gauging the Quaplectic-group [20] given by the semi-direct product of the $U(1,3)$ group and the Nonabelian Heisenberg $H(1,3)$ group : $\mathcal{Q}(1,3) \equiv U(1,3) \otimes_s H(1,3)$. We should *stress* that our results (as far as we know) are new and *differ* from the prior constructions of a Complex Gravity theory initiated by Einstein-Strauss [25] and continued later on by Moffat and Boal [27] by recurring to a symmetric and non-symmetric metrics. A $U(1,3)$ gauge formulation of Complex Gravity has been studied before by [22], however these authors did not include the Nonabelian Heisenberg algebra $H(1,3)$ into account, nor they and the prior authors studying Complex Gravity realized the crucial existence of an underlying Born's Reciprocal Relativity principle operating in the Complex Nonabelian Gravity Theory to be constructed next. For these reasons we must emphasize that the results of this section are *new* (to our knowledge) and are *not* a repetition of previous work on Complex Gravity.

Furthermore, we must also emphasize that Low [20] has already explained that it is possible to *avoid* the $no - go$ theorem of Schuller [9], stating that one cannot lift the identification of the Quaplectic coset space $\mathcal{Q}^{1,n} = \mathcal{Q}(1,n)/SU(1,n)$ to the tangent space of a more general *Nonabelian* manifold with *curvature*, due precisely to the presence of the central charge generator I of the Nonabelian Weyl-Heisenberg algebra and which furnishes a crucial $\eta^{ab}\ I\ Z_{ab}$ term in the quadratic Casimir invariant associated with the Quaplectic group. Z_{ab} are the $U(1,n)$ generators. Interestingly enough is the fact that Armand Wyler [30] long ago studied the geometry of the coset spaces $\mathcal{Q}^{1,n}$ within the context of Symplectic Groups, Spinors and the Complex Lightcone that permitted him to derive the value of the fine structure constant $1/137$ from purely geometric means.

Also worth mentioning is that the real dimensions of the Quaplectic coset space $\mathcal{Q}^{1,3} = \mathcal{Q}(1,3)/SU(1,3)$ is $25 - 15 = 10$ which coincides with the ten dimensions of the anomaly free superstring theory. This may be just a numerical coincidence, nevertheless it deserves further investigation. Quaternionic and Octonionic Gravity have been studied in [26], [29], [32]. It is warranted to study the physical implications of these theories within the context of Grand Unfication. A Chern-Simons E_8 Gauge theory of Gravity and Grand Unification was proposed recently by [34].

The Nonabelian Heisenberg algebra $H(1,3)$ involves the generators

$$Z_a \ = \ \frac{1}{\sqrt{2}} \ (\ \frac{X_a}{\lambda_l} \ - \ i\frac{P_a}{\lambda_p} \ ); \quad \bar{Z}_a \ = \ \frac{1}{\sqrt{2}} \ (\ \frac{X_a}{\lambda_l} \ + \ i\frac{P_a}{\lambda_p} \ ); \quad a = 1,2,3,4. \tag{2.1a}$$

Notice that we must *not* confuse the generators X_a, P_a (associated with the fiber coordinates of the internal space of the fiber bundle) with the ordinary spacetime coordinates and momenta x_μ, p_u. If one writes

$$x_\mu \ = \ e^a_\mu \ X_a; \quad p_\nu \ = \ e^b_\mu \ p_b \ \Rightarrow \ [x_\mu, \ p_\nu] \ = \ i\hbar \ g_{\mu\nu} \ I. \tag{2.1b}$$

where the curved spacetime metric in terms of the tetrads is $g_{\mu\nu} = e^a_\mu \ e^b_\nu \ \eta_{ab}$. The Gauge theory is constructed in the fiber bundle over the base manifold which is a $4D$ curved spacetime with *commuting* coordinates $x^\mu = x^0, x^1, x^2, x^3$. The Quaplectic group acts as the Automorphism group along the internal Fiber coordinates. The $8D$ curved Phase space is defined as the cotangent bundle of the $4D$ curved spacetime. Since X_a does not commute with P_a one has a Noncommutative Relativistic Phase space. However, we must stress that the coordinates x^μ are *commuting* among themselves. Also the momenta p^μ variables are *commuting* among themselves. Therefore we must *not* confuse the Complex Nanbelian Gravity constructed here with the Noncommutative Gravity work in the literature [15] where the spacetime coordinates x^μ are *not* commuting.

The four fundamental length, momentum, temporal and energy scales are respectively

$$\lambda_l = \sqrt{\frac{\hbar c}{b}}; \quad \lambda_p = \sqrt{\frac{\hbar b}{c}}; \quad \lambda_t = \sqrt{\frac{\hbar}{bc}}; \quad \lambda_e = \sqrt{\hbar bc}. \tag{2.2}$$

where b is the *maximal* proper force associated with the Born's reciprocal Relativity theory. In the natural units $\hbar = c = b = 1$ all four scales become *unity*. The gravitational coupling is given by

$$G = \alpha_G \ \frac{c^4}{\mathcal{F}_{max}} = \alpha_G \ \frac{c^4}{b}. \tag{2.3}$$

if, and only if, $\alpha_G = 1$ the four scales coincide then with the Planck length, momentum, time and energy, respectively and we can verify that

$$\mathcal{F}_{\max} \ = \ m_{Planck} \ \frac{c^2}{L_{Planck}} \ \sim \ M_{Universe} \ \frac{c^2}{R_{Hubble}}. \tag{2.4}$$

it was proposed in [33] that a certain large (Hubble) /small (Planck) scale *duality* was operating in this Born Reciprocal Relativity theory reminiscent of the T-duality in string theory compactifications. The numerical value α_G needs to be determined experimentally.

The generators $Z_{ab}, Z_a, \bar{Z}_a, I$ of the $U(1,3)$ algebra and the non-abelian Heisenberg algebra obey the relations

$$(Z_{ab})^\dagger \ = \ Z_{ab}; \quad (Z_a)^\dagger \ = \ \bar{Z}_a; \quad I^\dagger = I; \quad a,b = 1,2,3,4. \tag{2.5}$$

The Quaplectic group [20] is given by the semi-direct product of the $U(1,3)$ group and the Heisenberg $H(1,3)$ group : $\mathcal{Q}(1,3) \equiv U(1,3) \otimes_s H(1,3)$ and is defined in terms of the generators $Z_{ab}, Z_a, \bar{Z}_a, I$ with $a,b = 1,2,3,4$. The commutation relations of the Non-abelian Heisenberg algebra generators Z_a

$$Z_a \ = \ \frac{1}{\sqrt{2}} \ (\ \frac{X_a}{\lambda_l} \ - \ i\frac{P_a}{\lambda_p} \ ); \quad \bar{Z}_a \ = \ \frac{1}{\sqrt{2}} \ (\ \frac{X_a}{\lambda_l} \ + \ i\frac{P_a}{\lambda_p} \ ). \tag{2.6}$$

are

$$[Z_a, \ \bar{Z}_b] = -\alpha_\hbar \ \eta_{ab} \ I; \quad [\bar{Z}_a, \ Z_b] \ = \ \alpha_\hbar \ \eta_{ab} \ I; \quad [Z_a, \ Z_b] \ = \ [\bar{Z}_a, \ \bar{Z}_b] \ = \ 0. \tag{2.7}$$

with the Planck constant given by $\hbar = \alpha_\hbar \ \lambda_l \ \lambda_p$ from which we can infer that $\alpha_\hbar = 1$ as a result of the relations (2.2).

The generators Z_{ab} of the $U(1,3)$ algebra can be decomposed into the Lorentz-subalgebra generators L_{ab} and the "shear"-like generators M_{ab} as

$$Z_{ab} \equiv M_{ab} - iL_{ab}; \quad L_{ab} = L_{[ab]} \ = \ \frac{i}{2}(Z_{ab} - Z_{ba}); \quad M_{ab} = M_{(ab)} = \ \frac{1}{2}(Z_{ab} + Z_{ba}), \tag{2.8}$$

one can see that the "shear"-like generators M_{ab} are *Hermitian* and the Lorentz generators L_{ab} are $anti-Hermitian$ with respect to the fiber internal space indices. The explicit commutation relations of the Quaplectic algebra are then

$$[L_{ab}, \ L_{cd}] \ = \ (\eta_{bc}L_{ad} - \eta_{ac}L_{bd} - \eta_{bd}L_{ac} + \eta_{ad}L_{bc}). \tag{2.9a}$$

$$[M_{ab}, \ M_{cd}] \ = \ i \ (\eta_{bc}M_{ad} + \eta_{ac}M_{bd} + \eta_{bd}M_{ac} + \eta_{ad}M_{bc}). \tag{2.9b}$$

$$[L_{ab}, \ M_{cd}] \ = \ (\eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc}). \tag{2.9c}$$

$$[L_{ab},\ X_c] \ = \ (\eta_{bc} X_a - \eta_{ac} X_b); \qquad [L_{ab},\ P_c] \ = \ (\eta_{bc} P_a - \eta_{ac} P_b). \tag{2.10a}$$

$$[M_{ab},\ X_c] \ = \ -\, i\ (\eta_{bc} P_a - \eta_{ac} P_b); \quad [M_{ab},\ P_c] \ = \ i\ (\eta_{bc} X_a - \eta_{ac} X_b) \tag{2.10b}$$

supplemented by the commutators of the Weyl-Heisenberg non-Abelian algebra

$$[\frac{X_a}{\lambda_l},\ \frac{P_b}{\lambda_p}] \ = \ i\ \alpha_\hbar\ \eta_{ab}\ I; \quad [X_a, X_b] = [P_a, P_b] = 0. \tag{2.11}$$

where the metric $\eta_{ab} = (+1, -1, -1, -1)$ is used to raise and lower indices and $\hbar = \alpha_\hbar\ \lambda_l\ \lambda_p$. From the definitions (2.2) one can infer that $\alpha_\hbar = 1$.

The *complex* tetrad E^a_μ transforming under the fundamental representation of $U(1,3)$ is defined as

$$E^a_\mu \ = \ \frac{1}{\sqrt{2}}\ (\ e^a_\mu \ + \ i f^a_\mu\ ); \quad \bar{E}^a_\mu \ = \ \frac{1}{\sqrt{2}}\ (\ e^a_\mu \ - \ i f^a_\mu\ ). \tag{2.12}$$

The complex metric is given by

$$G_{\mu\nu} \ = \ \bar{E}^a_\mu\ E^b_\nu\ \eta_{ab} \ = \ g_{(\mu\nu)} \ + i g_{[\mu\nu]} \ = \ g_{(\mu\nu)} \ + i B_{\mu\nu}. \tag{2.13}$$

such that

$$(G_{\mu\nu})^\dagger \ = \ \bar{G}_{\nu\mu} \ = \ G_{\mu\nu}; \quad \bar{G}_{\mu\nu} \ = \ G_{\nu\mu}. \tag{2.14}$$

where the bar denotes complex conjugation. Despite that the metric is complex the infinitesimal line element is *real*

$$ds^2 \ = \ G_{\mu\nu}\ dx^\mu\ dx^\nu \ = \ g_{(\mu\nu)}\ dx^\mu\ dx^\nu, \ \ because\ \ i\ g_{[\mu\nu]}\ dx^\mu\ dx^\nu \ = \ 0. \tag{2.15a}$$

We have identified the anti-symmetric components of the metric $g_{[\mu\nu]} = B_{\mu\nu}$ with the anti-symmetric tensor Kalb-Ramond-like field present in the massless spectrum of closed-strings and the symmetric components $g_{(\mu\nu)}$ with the ordinary spacetime Riemannian metric. Under infinitesimal $U(1,3)$ gauge transformations the complex tetrad transforms as

$$\delta E^a_\mu = (\xi^a_{b(1)} + i \xi^a_{b(2)}) E^b_\mu. \tag{2.15b}$$

where the real $\xi^{(1)}_{[ab]}$ and imaginary $\xi^{(2)}_{(ab)}$ components of the complex parameter are anti-symmetric and symmetric respectively with respect to the indices a, b for anti-Hermitian infinitesimal $U(1,3)$ gauge transformations.

The Quaplectic-algebra-valued anti-Hermitian gauge field $(\mathbf{A}_\mu)^\dagger \ = \ -\,\mathbf{A}_\mu$ is given by

$$\mathbf{A}_\mu \ = \ \Omega^{ab}_\mu\ Z_{ab} \ + \frac{i}{L_P}\ (\ E^a_\mu\ Z_a \ + \ \bar{E}^a\ \bar{Z}_a\ ) \ + i\ \Omega_\mu\ I\ . \tag{2.16}$$

where the Planck *length* scale L_P needs to be introduced in the second terms in the r.h.s since the connection $\mathbf{A}_\mu$ must have units of $(length)^{-1}$. In natural units of $\hbar = c = 1$ the gravitational coupling in $4D$ is $G = L_P^2$. A length scale squared l^2 given as the product of the Hubble and Planck scale $l^2 = R_{Hubble}\ L_{Planck}$ and such that the observed value of the vacuum energy is $l^{-4} \sim 10^{-122} M^4_{Planck}$ was derived in [16]. Since Born's Reciprocal Relativity is consistent with a maximal (Hubble) and minimal (Planck) scale *duality* [33] it is not surprising to see why the observed vacuum energy might involve the product of an upper and lower scale.

From eq-(2.1) one obtains the *dimensionless* quantity

$$E^a_\mu \; Z_a \; + \; \bar{E}^a \; \bar{Z}_a \; = \; e^a_\mu \; \frac{X_a}{\lambda_l} \; + \; f^a_\mu \; \frac{P_a}{\lambda_p}. \tag{2.17}$$

In Born's Reciprocal Relativity X and P are interchangeable so there is no discrepancy in assigning f^a_μ to the P_a generator and e^a_μ to the X_a generator.

Decomposing the anti-Hermitian components of the connection Ω^{ab}_μ into anti-symmetric $[ab]$ and symmetric (ab) pieces with respect to the internal indices

$$\Omega^{ab}_\mu \; = \; \Omega^{[ab]}_\mu \; + i \;\; \Omega^{(ab)}_\mu. \tag{2.18}$$

gives the anti-Hermitian $U(1,3)$-valued connection

$$\Omega^{ab}_\mu \; Z_{ab} \; = \; (\Omega^{[ab]}_\mu \; + i \;\; \Omega^{(ab)}_\mu) \; (M_{ab} - i \; L_{ab}) \; =$$

$$- \, i \; \Omega^{[ab]}_\mu \; L_{ab} \; + \; i \; \Omega^{(ab)}_\mu \; M_{ab} \; \Rightarrow \; (\Omega^{ab}_\mu \; Z_{ab})^\dagger \; = \; - \; \Omega^{ab}_\mu \; Z_{ab}. \tag{2.19}$$

since $(Z_{ab})^\dagger \; = \; Z_{ab}$

In the standard complex gravity theory introduced by Einstein-Strauss the complex connection (affinity) Λ^{ab}_μ which defines the covariant derivatives of the complex tetrads is also anti-Hermitian with respect to the indices of the tangent space

$$\Lambda^{ab}_\mu \; = \; \Lambda^{[ab]}_\mu \; + \; i \; \Lambda^{(ab)}_\mu \; \Rightarrow \; (\Lambda^{ab}_\mu)^\dagger \; = \; - \; \Lambda^{ba}_\mu. \tag{2.20}$$

The Quaplectic-algebra-valued field strength is :

$$\mathbf{F}_{\mu\nu} \; = \; \partial_\mu \mathbf{A}_\nu \; - \; \partial_\nu \mathbf{A}_\mu \; + \; [\mathbf{A}_\mu, \; \mathbf{A}_\nu] \; =$$

$$F^{ab}_{\mu\nu} \; Z_{ab} \; + \; F^a_{\mu\nu} \; Z_a \; + \; \bar{F}^a_{\mu\nu} \; \bar{Z}_a \; + \; F_{\mu\nu} \; I \tag{2.21}$$

The components of the curvature two-form associated with the anti-Hermitian connection $\Omega^{ab}_\mu \; = \; \Omega^{[ab]}_\mu + i\Omega^{(ab)}_\mu$ are

$$F^{ab}_{\mu\nu} \; = \; \partial_\mu \Omega^{ab}_\nu \; - \; \partial_\nu \Omega^{ab}_\mu \; + \; \Omega^a_{\mu c} \; \Omega^{cb}_\nu \; - \; \Omega^a_{\nu c} \; \Omega^{cb}_\mu. \tag{2.22}$$

where $\Omega^a_{\mu c} \; = \; \Omega^{ad}_\mu \; \eta_{dc}$.

The components of the Torsion two-form are :

$$F^a_{\mu\nu} \; = \; \partial_\mu E^a_\nu \; - \; \partial_\nu E^a_\mu \; + \; \Omega^a_{\mu c} \; E^c_\nu \; - \; \Omega^a_{\nu c} \; E^c_\mu. \tag{2.23}$$

the $\bar{F}^a_{\mu\nu}$ components are obtained by replacing $E^a_\mu \to \bar{E}^a_\mu$ in (2.23). The remaining field strength has roughly the same form as a $U(1)$ field strength in Noncommutative spaces due to the additional contribution of $B_{\mu\nu}$ resulting from the Nonabelian nature of the Weyl-Heisenberg algebra in the internal space (fibers) and which is reminiscent of the Noncommutativity of the coordinates with the momentum :

$$F_{\mu\nu} \; = \; i \; \partial_\mu \Omega_\nu \; - \; i \; \partial_\nu \Omega_\mu \; + \; [\; iE^a_\mu \; Z_a, \; i\bar{E}^b_\nu \; \bar{Z}_b \;] \; + [\; i\bar{E}^a_\mu \; \bar{Z}_a, \; iE^b_\nu \; Z_b \;] \;\; =$$

$$i(\partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu) - \alpha_\hbar (G_{\mu\nu} - G_{\nu\mu}) =$$

$$i \Omega_{[\mu\nu]} - (G_{\mu\nu} - G_{\nu\mu}) = i \Omega_{[\mu\nu]} - 2i B_{\mu\nu} \quad (2.24)$$

upon setting $\alpha_\hbar = 1$ and recurring to the commutation relations

$$[Z_a, \bar{Z}_b] = - \alpha_\hbar \eta_{ab} I; \quad [\bar{Z}_a, Z_b] = \alpha_\hbar \eta_{ab} I. \quad (2.25)$$

and the Hermitian property of the metric

$$G_{\mu\nu} = \bar{E}^a_\mu E^b_\nu \eta_{ab} = [E^a_\mu \bar{E}^b_\nu \eta_{ab}]^* = (G_{\nu\mu})^* \Rightarrow (G_{\mu\nu})^* = G_{\nu\mu}. \quad (2.26)$$

where $*$ stands for (bar) complex conjugation.

In the Complex Gravity formulation of Einstein-Strauss the components $F_{\mu\nu}$ are *absent*. One may constrain the Torsion $F^a_{\mu\nu}$ to zero (which is *not* required to do so in the most general case and especially if we wish to incorporate fermions). The curvature tensor is defined in terms of the anti-Hermitian connection $\Omega^{[ab]}_\mu + i \Omega^{(ab)}_\mu$ as

$$\mathcal{R}^\rho_{\mu\nu\lambda} \equiv (F^{[ab]}_{\mu\nu} + i F^{(ab)}_{\mu\nu}) E^\rho_a E_{b\lambda}. \quad (2.27)$$

where the explicit components $F^{[ab]}_{\mu\nu}$ and $F^{(ab)}_{\mu\nu}$ can be read from the defining relations :

$$F^{[ab]}_{\mu\nu} L_{[ab]} = (\partial_\mu \Omega^{[ab]}_\nu - \partial_\nu \Omega^{[ab]}_\mu) L_{[ab]} + [\Omega^{[ac]}_\mu L_{[ac]}, \Omega^{[eb]}_\nu L_{[eb]}]. \quad (2.28a)$$

$$F^{(ab)}_{\mu\nu} M_{(ab)} = (\partial_\mu \Omega^{(ab)}_\nu - \partial_\nu \Omega^{(ab)}_\mu) M_{(ab)} + [\Omega^{(ac)}_\mu M_{(ac)}, \Omega^{(eb)}_\nu M_{(eb)}] +$$

$$[\Omega^{[ac]}_\mu L_{[ac]}, \Omega^{(eb)}_\nu M_{(eb)}] + [\Omega^{(ac)}_\mu M_{(ac)}, \Omega^{[eb]}_\nu L_{[eb]}]. \quad (2.28b)$$

after recurring to the commutation relations (2.9). The above relations (2.28) are consistent with the definitions in eqs-(2.22, 2.23).

Contracting indices yields a complex-valued Ricci tensor

$$\mathcal{R}_{\mu\lambda} = \delta^\nu_\rho R^\rho_{\mu\nu\lambda} = R_{(\mu\lambda)} + i R_{[\mu\lambda]}. \quad (2.29)$$

A further contraction yields the generalized (real-valued) Ricci scalar

$$\mathcal{R} = (g^{(\mu\lambda)} + i g^{[\mu\lambda]}) (R_{(\mu\lambda)} + i R_{[\mu\lambda]}) =$$

$$\mathcal{R} = g^{(\mu\lambda)} R_{(\mu\lambda)} - B^{\mu\lambda} R_{[\mu\lambda]}; \quad g^{[\mu\lambda]} \equiv B^{\mu\lambda}. \quad (2.30)$$

where the first term $g^{(\mu\lambda)} R_{(\mu\lambda)}$ corresponds to the usual scalar curvature of the ordinary Riemannian geometry. The presence of the extra terms $B^{\mu\lambda} R_{[\mu\lambda]}$ due to the anti-symmetric components of the metric and the Ricci tensors are one of the hallmarks of Complex Gravity.

The real-valued action linear in the generalized Ricci scalar curvature is then given by the Einstein-Srauss form

$$\frac{1}{2\kappa^2} \int_{M_4} d^4x \sqrt{| \ det \ (g_{(\mu\nu)} + iB_{\mu\nu}) \ |} \ \mathcal{R} =$$

$$\frac{1}{2\kappa^2}\int_{M_4} d^4x\ \sqrt{|\ det\ (g_{(\mu\nu)} + iB_{\mu\nu})\ |}\ (\ g^{(\mu\lambda)}\ R_{(\mu\lambda)}\ -\ B^{\mu\lambda}\ R_{[\mu\lambda]}\ ). \tag{2.31}$$

where $\kappa^2 = 8\pi G$ is the gravitational coupling and in natural units $\hbar = c = 1$ one has $G = L^2_{Planck}$. We should not forget the presence of the components $F_{\mu\nu}$ (and in general not to constrain the Torsion to zero) . Therefore, one could add an *extra* contribution to the action stemming from the terms $iB^{\mu\nu}F_{\mu\nu}$ which is very reminiscent of the BF terms in Topological Gravity and Plebanksi's formulation of gravity. In the most general case, one must include both the contributions from the Torsion and the $i\ B^{\mu\nu}F_{\mu\nu}$ terms.

The contractions involving $G^{\mu\nu} = g^{(\mu\nu)} + iB^{\mu\nu}$ with the components $F_{\mu\nu}$ (due to the antisymmetry property of $F_{\mu\nu} = -F_{\nu\mu}$) lead to

$$i\ B^{\mu\nu}\ F_{\mu\nu}\ =\ -\ B^{\mu\nu}\ (\ \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu\ ) + 2\ B^{\mu\nu}\ B_{\mu\nu}\ =\ -\ B^{\mu\nu}\ \Omega_{\mu\nu} + 2\ B^{\mu\nu}\ B_{\mu\nu}\ . \tag{2.32}$$

Therefore, the above terms (2.32) are *real*-valued as they should be if one wishes to build a real-valued action involving these terms.

When the Torsion is not constrained to vanish one must include those contributions as well. Hence, to sum up, the net contribution of the $\mathcal{R}$ scalar curvature, including the $i\ B^{\mu\nu}F_{\mu\nu}$ and Torsion terms, to the candidate action is of the form

$$\frac{1}{2\kappa^2}\int_{M_4} d^4x\ \sqrt{|\ det\ (g_{(\mu\nu)} + iB_{\mu\nu})\ |}\ [\ a_1\mathcal{R}\ + i\ a_2B^{\mu\nu}F_{\mu\nu}\ +\ Torsion\ ] =$$

$$\frac{1}{2\kappa^2}\int_{M_4} d^4x\ \sqrt{|\ det\ (g_{(\mu\nu)} + iB_{\mu\nu})\ |}\ [\ a_1g^{(\mu\lambda)}\ R_{(\mu\lambda)}\ - a_1\ B^{\mu\lambda}\ R_{[\mu\lambda]}\ + \ldots\ldots\ldots\ ]. \tag{2.33}$$

where the ellipsis .... stand for

$$-a_2\ B^{\mu\nu}\ \Omega_{\mu\nu}\ +\ 2a_2\ B^{\mu\nu}\ B_{\mu\nu}\ +\ Torsion. \tag{2.34}$$

The Torsion terms are

$$a_3\ T_{\mu\nu\rho}\ T^{\mu\nu\rho}\ +\ a_4\ T_\mu\ T^\mu. \tag{2.35}$$

where the Torsion tensor and Torsion vector are defined in terms of the components of the Torsion two-form

$$(\ F^a_{\mu\nu}\ Z_a\ +\ \bar{F}^a_{\mu\nu}\ \bar{Z}_a\ )\ dx^\mu \wedge dx^\nu. \tag{2.36a}$$

as

$$T^\rho_{\mu\nu}\ \equiv\ (F^a_{\mu\nu}\ E^\rho_a\ +\ \bar{F}^a_{\mu\nu}\ \bar{E}^\rho_a);\quad T_\mu\ =\ \delta^\nu_\rho\ T^\rho_{\mu\nu}. \tag{2.36b}$$

a_1, a_2, a_3, a_4 are suitable numerical coefficients that will be constrained to have certain values if one wishes to avoid the presence of propagating ghosts degrees of freedom.

In the very special case when $\Omega_{\mu\nu}\ =\ B_{\mu\nu}$ one has

$$-\ B^{\mu\nu}\ \Omega_{\mu\nu}\ +\ 2\ B^{\mu\nu}\ B_{\mu\nu}\ =\ B^{\mu\nu}\ B_{\mu\nu}. \tag{2.37}$$

such that the BF terms reduce to a mass-like term for the $B_{\mu\nu}$ field. Mass terms for the $B_{\mu\nu}$ and a *massive* graviton formulation of bi-gravity (in addition to a massles graviton) based on a $SL(2, C)$ gauge formulation have been studied by [22], [28], [27] .

For closed-strings propagating in curved backgrounds the Kalb-Ramond field $B_{\mu\nu}$ appears in the closed-string effective action only through the presence of its field strength $H = H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + ...$ which has the same role as a propagating Torsion and it exhibits the gauge symmetry $B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ leaving H invariant. It is interesting that if $\Omega_{\mu\nu} = B_{\mu\nu}$ this means that the Kalb-Ramond field is pure gauge with trivial kinetic degrees of freedom since $H = dB = dd\Omega = 0$. Therefore, one could gauge away the field $B_{\mu\nu}$ from the action and which is consistent with the fact that there are *no* mass terms $B_{\mu\nu}B^{\mu\nu}$ in the closed-string effective action action to lowest order, because the $B_{\mu\nu}$ field in closed string theory is *massless*. We need to explore further the connection of Born's Reciprocal Relativity with closed-string theory, in particular to find the place for the massless dilaton which is not present here. However, one knows that the conformal group is $SU(2,2)$ which would be associated with a gauge theory based on the group $U(2,2) = U(1) \otimes SU(2,2)$ rather than $U(1,3)$ group studied here; i.e. the signature of spacetime would be $(+,+,-,-)$ instead of $(+,-,-,-)$.

Notice that in general the extra contribution from the $iB^{\mu\nu}F_{\mu\nu}$ terms *is not present* in the Einstein-Strauss formulation of Complex Gravity. These extra iBF terms in the action are the *signature* of the Nonabelian Weyl-Heisenberg algebraic sector of the Quaplectic-group and should be essential in the construction of a Quantum Gravity based on Born's Reciprocal General Relativity theory in a curved spacetime. It is our belief that the action (2.33) based on Born's Reciprocal Relativity principle involving a *maximal* speed limit and a *maximal* proper force is a very promising avenue to Quantize Gravity that does *not* rely in *breaking* the Lorentz symmetry at the Planck scale in contrast to other approaches based on deformations of the Poincare algebra, Quantum Groups [10], [17], [14] .

A Quaplectic-group invariant Killing metric κ_{AB} is given in terms of the structure constants of the algebra as $\kappa_{AB} = f_{ACD}\ f^{CD}_{B}$ where the indices $A, B...$ span over all the indices of the generators $Z_{ab}, Z_a, \bar{Z}_a$ and the central element I of the Quaplectic-algebra with $a, b = 1,2,3,4$. A Quaplectic-group-invariant Yang-Mills-like action is then

$$\int_{M_4} d^4x\ \sqrt{\mid det\ (g_{(\mu\nu)} + iB_{\mu\nu})\mid}\ \kappa_{AB}\ \mathbf{F}^A_{\mu\nu}\ \mathbf{F}^B_{\rho\sigma}\ G^{\mu\rho}\ G^{\nu\sigma}\ +\ complex\ conjugate. \tag{2.38}$$

A Quaplectic-group-invariant topological action like the theta term in QCD is

$$\int_{M_4} d^4x\ \sqrt{\mid det\ (g_{(\mu\nu)} + iB_{\mu\nu})\mid}\ \kappa_{AB}\ \mathbf{F}^A_{\mu\nu}\ \mathbf{F}^B_{\rho\sigma}\ \epsilon^{\mu\nu\rho\sigma}\ +\ complex\ conjugate. \tag{2.39}$$

where $\mathbf{F}^A_{\mu\nu}$ are the components of the Quaplectic-algebra-valued two-form

$$\mathbf{F}_{\mu\nu}\ dx^\mu \wedge dx^\nu\ =\ \mathbf{F}^A_{\mu\nu}\ T_A\ dx^\mu \wedge dx^\nu\ =\ (F^{ab}_{\mu\nu}\ Z_{ab}\ +\ F^a_{\mu\nu}\ Z_a\ +\ \bar{F}^a_{\mu\nu}\ \bar{Z}_a\ +\ F_{\mu\nu}\ I)\ dx^\mu \wedge dx^\nu. \tag{2.40}$$

These actions (2.38, 2.39) are very appealing because we know from the work of McDowell-Mansouri and Chamseddine-West that a topological action like (2.39) based on the Anti de Sitter group $SO(3,2)$ in $4D$, with signature $(+,+,+,1)$ and after setting the Torsion to zero, leads to the Einstein-Hilbert action with a cosmological constant and the

Gauss-Bonnet curvature squared terms (a topological invariant in four dimensions). The Einstein-Hilbert action stems form the terms $R \wedge e \wedge e$ present in the $\mathbf{F}^{ab} \wedge \mathbf{F}^{cd}$ expansion of the action resulting from the decomposition of the $SO(3,2)$ field strength $F^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} + \frac{1}{l^2} e^a_\mu \wedge e^b_\mu$. The tetrad $e^a_\mu \equiv l \ A^{a5}_\mu$ and the spin connection is $\omega^{ab}_\mu \equiv A^{ab}_\mu$ since the internal $SO(3,2)$ group indices range from $1,2,3,4,5$. The length scale l required to match units is related to the throat size of Anti de Sitter space (the cosmological constant).

One may ask whether or not something similar might occur with the action (2.39). The reason is that the Quaplectic group $\mathcal{Q}(1,3)$ has 25 real generators. 16 generators from the $U(1,3)$ group. 8 generators from the Nonabelian translation generators $Z_a, \bar{Z}_a$ (or X_a and P_a) and 1 central element generator I. Thus the 25 real generators of $\mathcal{Q}(1,3)$ match *precisely* the number of generators of the group $U(1,4) = U(1) \otimes SU(1,4)$. Thus a Topological action based on the group $U(1,4)$ would be the unitary group analogy of the McDowell-Mansouri and Chamseddine-West procedure to generate an Einstein-Hilbert action with a cosmological constant and the Gauss-Bonnet curvature-squared topological invariant.

The indices A, B associated with a $U(1,4)$ gauge theory range now from $1,2,3,4,5$ so the gravitational term should emerge from products of the form $\mathcal{F}^{ab}_{\mu\nu} \wedge \mathcal{F}^{cd}_{\mu\nu}$ where now

$$\mathcal{F}^{ab}_{\mu\nu} = (\partial_\mu \Omega^{ab}_\nu - \partial_\nu \Omega^{ab}_\mu + \Omega^a_{\mu c} \Omega^{cb}_\nu - \Omega^a_{\nu c} \Omega^{cb}_\mu) + (\Omega^a_{\mu 5} \Omega^{5b}_\nu - \Omega^a_{\nu 5} \Omega^{5b}_\mu) +$$
$$(\Omega^{ab}_\mu \Omega^{55}_\nu - \Omega^{ab}_\nu \Omega^{55}_\mu) =$$
$$F^{ab}_{\mu\nu} + \frac{i}{l^2} (E^a_\mu \wedge E^b_\nu + \bar{E}^a_\mu \wedge \bar{E}^b_\nu) + i \ \Omega^{ab}_\mu \wedge \Omega_\nu. \tag{2.41}$$

where one makes the following identifications of the components Ω^{AB}_μ when $A, B = 1,2,3,4,5$

$$\Omega^{55}_\mu = i \ \Omega_\mu; \quad \frac{1}{2}(E^a_\mu + \bar{E}^a_\mu) = \Omega^{(a5)}_\mu; \quad \frac{1}{2i}(E^a_\mu - \bar{E}^a_\mu) = \Omega^{[a5]}_\mu. \tag{2.42a}$$

$$\Omega^{AB}_\mu = \Omega^{ab}_\mu, \ when \ A, B = a, b = 1,2,3,4. \tag{2.42b}$$

Therefore the products of the form $\mathcal{F}^{ab}_{\mu\nu} \wedge \mathcal{F}^{cd}_{\mu\nu}$ will contain the gravitational term :

$$\frac{1}{l^2} \ \mathcal{R}^{ab}_{\mu\nu} \wedge E^c_\rho \wedge E^d_\sigma \ \epsilon_{abcd} \ \epsilon^{\mu\nu\rho\sigma}. \tag{2.43a}$$

since $\mathcal{R}^{ab}_{\mu\nu} = F^{ab}_{\mu\nu}$ in last terms of (2.41), in addition to curvature squared terms and the cosmological constant term

$$\frac{1}{l^4} \ E^a_\mu \wedge E^b_\nu \wedge E^c_\rho \wedge E^d_\sigma \ \epsilon_{abcd} \ \epsilon^{\mu\nu\rho\sigma} + \ complex \ conjugate \tag{2.43b}$$

There is one problem when $l^2 \sim G$ because it yields a huge value of the cosmological constant. The way to solve this problem in the case of the Anti de Sitter group $SO(3,2)$ in $4D$ and generate the correct value of the cosmological constant was found in [16] by working with a BF-Chern-Simons-Higgs model

$$S = \int_{M_4} \Phi\ F \wedge F + \Phi\ d\Phi \wedge d\Phi \wedge d\Phi \wedge d\Phi - V_H(\Phi). \qquad (2.44)$$

after adjoining a family of 5 Higgs scalars ϕ^A belonging to a $SO(3,2)$-valued scalar-multiplet Φ. It was shown that the solutions to the equations of motion when the Higgs potential $V_H(\Phi)$ was minimized at tree level upon choosing the vacuum expectation values

$$< \phi^5 > = v; \quad < \phi^a > = 0;\ a = 1,2,3,4. \qquad (2.45)$$

yielded the zero Torsion condition, without having to *impose* it by hand, and the Einstein-Hilbert action with a cosmological constant plus the curvature squared terms (topological Gauss-Bonnet terms) such that now one has the following numerical relations

$$\frac{1}{l^2}\ v \sim \frac{1}{G} = \frac{1}{L^2_{Planck}}; \quad \frac{1}{l^4}\ v \sim \rho_{vacuum}. \qquad (2.46)$$

by eliminating the value of the vacuum expectation value $< \phi^5 > = v$ from (2.46) we can infer that the vacuum energy density ρ_{vacuum} is given then by the geometric mean

$$\rho_{vacuum} \sim \frac{1}{l^2}\ \frac{1}{L^2_{Planck}}. \qquad (2.47)$$

if one identifies the length scale l with the throat size of Anti de Sitter space given by the Hubble scale one arrives at the correct value for the observed vacuum energy density ρ_{vacuum}

$$\rho_{vacuum} = \frac{1}{R^2_{Hubble}}\ \frac{1}{L^2_{Planck}} = \frac{L^2_{Planck}}{R^2_{Hubble}}\ \frac{1}{L^4_{Planck}} \sim 10^{-122}\ M^4_{Planck} \qquad (2.48)$$

Of course, the BF-Chern-Simons-Higgs theory based on the $U(1,4), U(2,3)$ groups needs to be investigated thoroughly, in particular, the topological nature of the curvature squared terms. These ideas were the basis of our original motivation to construct the novel Chern-Simons E_8 gauge theory of Gravity in $D = 15$ within the context of a E_8 Grand Unification scheme [34].

To conclude, it is warranted to explore the relationship of Finsler Geometry and Maximal proper acceleration found by Brandt [8] and the Complex Nonabelian Gravity formulation of Born's Reciprocal Relativity with a maximal speed and maximal proper force in spacetime. A covariant acceleration in curved space-times in units of $c = 1$ is given by

$$\frac{Dv^\mu}{d\tau} = \frac{dv^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} v^\nu v^\rho. \qquad (2.49)$$

A particle in free fall follows a geodesic with *zero* covariant acceleration. The norm of the spacelike proper acceleration is :

$$a^2 = -g_{\mu\nu} \frac{Dv^\mu}{d\tau} \frac{Dv^\nu}{d\tau}. \qquad (2.50)$$

The condition $a^2 \leq a_o^2 = (\frac{c^2}{L_P})^2 = \frac{1}{L_P^2}$ after the substitution $v^\mu = dx^\mu/d\tau$ leads to

$$d\sigma^2 = g_{\mu\nu} dx^\mu dx^\nu \ + \ L_P^2 \ g_{\alpha\beta} \ (dv^\alpha + \Gamma^\alpha_{\lambda\mu} v^\lambda dx^\mu) \ (dv^\beta + \Gamma^\beta_{\delta\nu} v^\delta dx^\nu) \ \geq 0. \qquad (2.51)$$

as was shown by Brandt. This is the $8D$ curved-phase space interval associated with the cotangent space of the curved spacetime. The Lagrange-Finsler geometric approach to the interval (2.43) requires that the metric components $g_{\mu\nu}, g_{\alpha\beta}$ and $\Gamma^\alpha_{\mu\nu}$ (the nonlinear connection) depend both on x^μ and v^μ !

We treated the spacetime coordinates of the Quaplectic valued gauge theory as real-valued. One should be able to extend our construction to the case of Complex Coordinates as well. For example, to build a Quaplectic Gauge theory as a Fiber bundle over the curved $8D$ Phase Space rather than having a Quaplectic Gauge theory as a Fiber bundle over the $4D$ curved spacetime. Curved Phase spaces, Conformal groups, wavelets and the geometry of the complex domains studied by Wyler [30] in his derivation of the fine structure constant have been reviewed by [31].

Chamseddine has studied Gravity in Complex Hermitian Spacetimes based on the results by Witten [24] on the high-energy behaviour of string scattering amplitudes where it was observed that the *imaginary* parts of the string coordinates of the target manifold appear. In this picture the metric tensor and the Kalb-Ramond anti-symmetric tensor $B_{\mu\nu}$ are unified in one field, the complex metric tensor of the Hermitian manifold first proposed by Einstein [25]. Chamseddine arrived at an action in an $8D$ Real space (four-complex dimensional spacetime) which was compatible with the equations of motion associated with the consistent propagation of closed-strings on Complex target backgrounds. The action was expressed entirely in terms of the integrable complex structure J (a two-form) of the complex Hermitian spacetime given by

$$\int_{M_8} d^4z \ d^4\bar{z} \ J \wedge \partial J \wedge \bar{\partial} J \qquad (2.52)$$

A Complex Hermitian but non-Kahlerian space has nontrivial Torsion. A Kahler space has vanishing Torsion. Finsler metrics with symmetric and nonsymmetric components have been studied by Eisenhart [35]. It is these spaces studied by Eisenhart which deserve further investigation as those particular closed-string target space backgrounds where Born's Reciprocal Relativity principle of a *maximal* proper force and *maximal* speed may be operating simultaneously. The propagation of D branes in curved backgrounds with anti-symmetric tensor fields $B_{\mu\nu}$ is another interesting project, after all the effective action governing the behaviour of D branes in curved backgrounds is precisely given by a Born-Infeld action. Thus the principle of maximal force which is compatible with a minimal length [36] may be operating in string theory with the main advantage that one does *not* need to break the Lorentz invariance. Finally, the connections to Noncommutative Geometry [23], Noncommutative Gravity [22], [15] and the Extended Relativity theory in Clifford-spaces [13], [33] warrants further investigation.

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