Foundations of Physics, Vol. 35, No. 7, July 2005 (© 2005)
DOI: 10.1007/s10701-005-6407-y

On the Unification of Geometric and Random Structures through Torsion Fields: Brownian Motions, Viscous and Magneto-fluid-dynamics

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Received April 14, 2005

We present the unification of Riemann–Cartan–Weyl (RCW) space-time geometries and random generalized Brownian motions. These are metric compatible connections (albeit the metric can be trivially euclidean) which have a propagating trace-torsion 1-form, whose metric conjugate describes the average motion interaction term. Thus, the universality of torsion fields is proved through the universality of Brownian motions. We extend this approach to give a random symplectic theory on phase-space. We present as a case study of this approach, the invariant Navier–Stokes equations for viscous fluids, and the kinematic dynamo equation of magnetohydrodynamics. We give analytical random representations for these equations. We discuss briefly the relation between them and the Reynolds approach to turbulence. We discuss the role of the Cartan classical development method and the random extension of it as the method to generate these generalized Brownian motions, as well as the key to construct finite-dimensional almost everywhere smooth approximations of the random representations of these equations, the random symplectic theory, and the random Poincaré–Cartan invariants associated to it. We discuss the role of autoparallels of the RCW connections as providing polygonal smooth almost everywhere realizations of the random representations.

KEY WORDS: Brownian motions; Riemann–Cartan–Weyl connections: trace-torsion; electromagnetism; autoparallels; Navier–Stokes equations; kinematic dynamo; turbulence; Reynolds decomposition; stochastic differential equations.

1. INTRODUCTION

In contemporary theoretical physics, there are two major approaches which are presented as irreconciliable: the differential geometrical structures

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of gauge theories and general relativity, and the approach to statistical mechanics and quantum field theories in terms of the Feynman integral and stochastic processes. Remarkably, the genesis of both can be traced back to the seminal works of Einstein in general relativity, and his foundational works on Brownian motion. Yet, mainstream physics has left unconsidered the possibility of an integration, which in fact, has been developed initially by the present author in a number of fields: equilibrium and non-equilibrium stastistical mechanics,[24,25] quantum mechanics [36] and the problem of equivalence between the Maxwell equation (note the singular on equation) in a Clifford Geometric Calculus[44] setting and the non-linear Dirac–Hestenes equation which integrates Maxwell's electromagnetism and relativistic quantum mechanics in the framework of a non-linear Heisenberg field theory,[21,27] and specially, the non-linear Navier–Stokes equations (NS, in the following) of fluid-dynamics and further, the equations of passive transport of fields on fluids, particularly, the kinematic dynamo equation (KDE, in the following) of magnetohydrodynamics.[3–7,35] The key to this unification between generalized Brownian motion as the paradigma of non-equilibrium phenomenae whose dynamics are ruled by invariant non-linear stochastic differential equations, and the geometries of space-time (and still, the random symplectic canonical structures of phase-space), stems from the simple fact that the generators of these motions are placed in one-to-one correspondance with linear connections of Riemann–Cartan–Weyl. These connections (or covariant derivative operators) have the particularity of being metric compatible (no historicity problem, which lead to the rejection by Einstein of Weyl's first ever gauge theory in 1918[46]) and have a Cartan–Weyl differential 1-form which is described by the trace-torsion. The laplacian operators of these covariant derivative operators are the differential generators (or still, the infinitesimal generators of the diffusion semigroups) of the random continuous dynamics, and viceversa, given this dynamics, one can retrieve the connections. The role of the metric, even if trivial Euclidean (in any case, it has to be properly Riemannian, i.e. positive-definite) is that its square root describes the noise tensor, and together with the metric conjugate of the Cartan–Weyl form which describes the average motion (drift) in their description by stochastic differential equations, they appear in a unified setting in which, as we said, it is the generalized laplacian associated to the Riemann–Cartan–Weyl (RCW, in the following) connections. These connections are of course related still to an extension of scale fields, and thus they will have appropiate physical constants linked to the noise term. Thus, this setting can accomodate fluctuations, whatever their origin, quantum, thermodynamical, or still, due to viscosity (kinematical for NS, magnetic for KDE).[1] The latter equations of fluid-dynamics, are somewhat a remarkable example of this approach, since it leads to the analytical representations for these equations in terms of stochastic differential equations, both for the case of compact manifolds with or without boundaries, and still, in half-space and Euclidean domains; indeed, no general analytical representations for NS on smooth manifolds were known previous to this approach (see vol. III, Ref. 15). Although this article will present some aspects of this example, the author feels obliged to remark that in no case this example should be seen as final in the formulation of the relation of this approach with regards to fluid-dynamics, since one can envision that a new perspective can be forged of theoretical physics and of one of its standing biggest unsolved problem, that of the characterization of fluid turbulence, in terms of the derivation of the equations of classical electrodynamics (which as we shall see in the following accompanying article in this volume, can be unified with the Dirac–Hestenes equations of relativistic quantum mechanics in a Clifford Geometric Calculus setting)

[1] The possibility of space-time having torsion, has been discussed mainly in the context of extensions of general relativity to accomodate for angular momentum.[28] A notable exception is condensed matter physics, with its applications to metallurgy, defects, etc., where the role of torsion has for long been known to be essential, precisely associated to a dislocation tensor.[22,23] Most of these theories, have proven to be beyond the reach of present experimental measurements, and thus torsion, whether non-propagating or propagating, is been regarded as a problem of academic exercise. The present article, asseses the contrary view: Torsion fields are extremely common as much as Brownian motion is – and one should stress here the that the quest for verifying experimentally Einstein's Brownian motion theory lead Perrin and his student Chaudesaiges to determine Avogrado's number and basically to establish the reality of the existance of atoms[20], encompassing as much Chemistry, and in the description of chemical reactions through the Fokker–Planck operator and stochastic differential equations (see the works of Schuss and Gardiner[29],) and even polymer physics[29] and as we shall see in this article, the velocity field of viscous fluids as described by the Navier–Stokes equations. There is another role of torsion-Brownian motions that should be stressed, which is the determination of the asymptotic description of chaotic systems, through the determination of their low dimensional attractors in terms of ordinary differential equations, for example, the Lorenz attractor. Yet, it has been proved by T. Taylor[33] that the solutions of these systems converge (in the sense of weak convergence of processes) to the solutions of stochastic differential equations, and thus are equivalenty described by generalized Brownian motions, which as we shall prove in this article, define a connection with trace-torsion. Taylor reveals in his article his debt to discussions with K. D. Elworthy, one of the founders of Stochastic Differential Geometry, i.e. the theory of diffusion processes on differentiable manifolds, which is the framework in which we shall develop this article, in connection with the Riemann–Cartan–Weyl connections and their generalized laplacian operators. Finally, we should like to remark that Taylor's work was completely ignored by the chaos industry, which about that time was concerned precisely with the appearance of random behavior of deterministic dynamics.

from the equations of fluid-dynamics. Such a striking derivation was provided by H. Marmanis in the framework of 19th century Gibbs vector calculus, and a gauge-theory of turbulence was constructed[19] (where as in our formulation, it is the potential one-form of velocity that stands at the basis of the theory, and not the "curvature" 2-form describing the field intensities, the vorticity as it turns out to be in NS, and we should recall, in agreement with the Aharonov–Bohm phenomena), and the extension to the Clifford Geometric Calculus[44] remains as a problem of singular importance. Having made these clarifications, we state the objective of this article to be the formulation of the geometry of Brownian motions and their application to fluid and magnetohydrodynamics, while we keep for a second accompanying article, the presentation of the constitutive equations of Brownian motions, particulary electrodynamics and the relation with the Dirac–Hestenes equation of relativistic quantum mechanics, and still, the connection with gravitation.

Thus, our first subject will be the general presentation of the equivalence between the RCW connections and random continuous generalized Brownian motions, to further provide the description of the random transport of differential forms, as the basic setting for the analytical representations of NS and KDE, and still, the construction of a natural random symplectic structure, in terms of the Cartan classical development method[26] (from which the gauge theories with torsion appear[34]) and its implementation to give an approximation of the random motions by ordinary differential equations (in almost all times), and furthermore, the appearance of random invariants which were obtained recently by the present author.[7,35]

Some historical considerations are in order. Geometrical and topological invariants in hydrodynamics and magnetohydrodynamics have been extensively considered by several authors.[10,16,18,31] Thus, for the Euler equation for perfect fluids, an infinite-dimensional symplectic geometry theory was constructed by V. I. Arnold[31] (see refs. in 10), followed by work by Ebin and Marsden,[17] which is widely perceived as a beautiful example of the differential–geometrical methods in fluid-dynamics, while a theory for the case of viscous fluids and for magnetohydrodynamics has only been recently constructed by the present author.[3–7,35] This theory stems from stochastic differential geometry,[1,2,11,12] i.e. a geometricaly invariant theory of diffusion processes or still, a stochastic theory of gauge-theoretical structures (linear connnections of RCW in the current author's approach), so that geometrical and probabilistic structures become unified in a single theory which has been applied to several areas of mathematical and theoretical physics (see Ref. 5 and references therein). In particular, this theory has yielded a new class of random symplectic invariants for NS, from which in the case of vanishing kinematical viscosity, we retrieve the Arnold–Ebin–Marsden theory. Furthermore, this approach has yielded analytical representations for NS on smooth compact manifolds with or without smooth boundaries, and still on Euclidean spaces and semispaces.[3,7] The subject of this article is in giving for a start a rather sketchy (unfortunately due to page limitations) review of the fundamental elements of stochastic differential geometry, yet stating the central role of the trace-torsion, which is absent in the classics,[1,2,11,12] to further extend this methodology previously used to construct the theory for NS and KDE. Of course, such a presentation of Brownian motion bears little resemblance with the pioneering work of Einstein, Langevin and Smoluchowski, undoubtedly the founding fathers of the subject, at least from the physics point of view.[20] This pioneering approach required the work of several generations of mathematicians working in probability, starting with Kolmogorov and Wiener, the work of P. Levy as the founder of the school of probability in France and main contributor to Brownian motion theory and its extensions that carry his name, to follow with the founding of stochastic analysis by Ito and Stratonovich and the development of potential theory by Doob. A third conceptual generation carried its development to yield a geometricaly invariant theory of Brownian motion, and the main activity was carried out by S. Bochner in the USA, J. Eells, K. D. Elworthy and P. Baxendale in the United Kingdom, the contributors of the French school P. Malliavin,[2,49] J. M. Bismut,[11] P. Meyer[49] and L. Schwartz (the founder of the theory of generalized functions), the Japanese school of Ito (Ikeda, Watanabe, Kunita and Takayashi[1] and references therein), the Russian school with Dynkin, Dalecki, Belopolskaya and Molchanov[51] which lead to the integration by Gliklikh of NS on the flat torus as a random perturbation of the Arnold–Ebin–Marsden approach[32]; this was followed by the work in the USA by D. Stroock and S. R. S. Varadhan[50] who developed the martingale problem approach to the solutions of elliptic and parabolic partial differential equations for scalar fields through stochastic differential equations, thus paving the way to its extension to the solution of such equations for differential forms, which in the work of this author became the method to solve NS and KDE.

In this article we shall follow a chain of constructions of stochastic differential geometry following Ref. 5 whose naturality we would like to stress in remarking that a similar line of development of our ideas, can be found in the last work by Elworthy (jointly with X. Li and Y. Le Jan)[47] which was developed independently by the author at about the same time of appearance of their work: We introduce linear connections[2]and define their laplacians on scalars to further extend them to define generalized laplacians on differential forms; then we give the stochastic differential equations determined by them through the rules of stochastic analysis, i.e. the Ito–Elworthy formula of the transformation rules of differential forms along the random continuous curves related to the generalized scalar laplacian; thus we shall see that the transformation rules for scalars will determine completely the random evolution of differential forms. Thus, keeping in mind that odd (even) degree differential forms are bosons (fermions) in a theory – that we shall present in an accompanying article (see footnote 5 below) – of supersymmetric systems defined by the Dirac and Laplacian operators for differential forms introduced in this theory in complementing it with the Clifford geometric calculus, then the quantization of the motions of bosons and fermions through stochastic differential equations will appear to be completely determined by the random quantization of the paths of scalar fields[3] As a final comment, the

[2] In Ref. 47 it is the universal connection which will lead to the driftless non-interaction representations of random motion which also show up in NS as pure noise motions whose noise tensor incorporates the velocity(5).

[3] Starting with the pioneering work by Schroedinger in the thirties of relating his equation to stochastic processes and in particular diffusion processes, much work has been done to relate quantum mechanics to diffusion processes, the most known is E. Nelson's stochastic mechanics.(52) Nelson's work stirred much interest and developments that follow presently,(53) as well as alternative approaches in terms of Bernstein's stochastic processes.(58) Yet we must stress that although Nelson's approach was centered on the study of diffusion processes and Schroedinger non-relativistic equations on smooth manifolds, in his conception stochastic processes and in particular quantum fluctuations bear no relation with geometrical gauge-theoretical structures, i.e. connections. Thus, it missed the mark that was being set at the same time by stochastic differential geometry: diffusion processes are determined by a geometry, or still, determine a geometry. (Our difference with the foundational treatment of stochastic differential geometry, is the unified setting that provides the RCW connections, their Dirac and Laplacian operators.); at the same time it was elaborated inadvertedly of the fact that the definition of a diffusion process on a manifold, requires the introduction of a connection (see Ref. 11 and P. Meyer(49)). Indeed, the problem is to define intrinsically the noise and drift term to yield a diffeomorphism invariant construction, and this problem is precisely solved by the introduction of a RCW connection(5) which places them in a unified setting. Thus, stochastic mechanics could not treat quantum fluctuations in the setting of a unified theory which can be established by a gauge theoretical formulation for quantum (or arbitrary) fluctuations that admit a random continuous dynamics. Nelson's approach requires a forward and backward stochastic derivative to yield a time-t-reversible theory through a stochastic Newton equation, in contrast with the present one, which is a τ-irreversible theory (we shall identify τ further below). Yet, it has been argued(57) that the backward derivative also depends on the single-event distribution, entailing that the drift will depend as well on it; consequently, the backward process thus introduced is not Markovian and it yields results in contradiction with quantum mechanics; indeed, the position correlation

random quantization of scalar fields, such as the Schroedinger equation for the scalar wave function, such not be understood *a priori* as the quantization of a spinless particle; indeed, from Hestenes follows that it corresponds to a fixed isotropic spin-eigenvalue (equal to $\sqrt{-1}\hbar$) scalar field.(54) This quantization in our approach is determined by the generalized laplacian on scalar fields defined from the connection with torsion, which we shall see reduces to the RCW connections, since only the trace of the torsion tensor will appear in the expression for the laplacian. This, as we shall see in the accompanying article,[4]does not mean that angular momentum or still spinor fields are excluded from this theory in regards of the usual association between them and skew-symmetric torsion, but on the contrary, they are built in the trace-torsion as we shall see in the accompanying article to the present one. Finally, having completed this basic introduction, we shall apply these constructions to NS and KDE and give the analytical representations for them, in the boundaryless case.

Furthermore, stochastic differential geometry introduced in the configuration manifold M, allows to construct a random symplectic theory on the cotangent manifold, T^*M, and in particular, a random symplectic structure for fluid-dynamics. This construction is established through the realization by sequences of ordinary differential equations of the random generalized Brownian motions, and in particular, those that yield the flow of NS and KDE on smooth compact connected manifolds without boundary, M, which are further isometrically embedded in Euclidean space. In fact, this construction will follow the formulation of stochastic differential geometry that stems from the developing (or transfer) method due to E. Cartan(26), for which a smooth curve lying on an Euclidean n-space is roled (keeping first-order contact) on a smooth n-manifold,[5] extending it to the random development of Wiener processes on the same Euclidean

function for the quantum harmonic oscillator was computed in the framework of both quantum and stochastic mechanics, and was found that for the latter yields an exponentially damped expression, which cannot be associated thus with a reversible process.(57)

[4] D. Rapoport, "Cartan–Weyl Dirac and Laplacian Operators, Brownian Motions: The Quantum Potential and Scalar Curvature, Maxwell´s and Dirac–Hestenes Equations, and Supersymmetric Systems", to appear in *Foundations of Physics*, special issue of the Fourth Biennial Conference on Relativistic Dynamics, Saas Fee, Switzerland, June 12–19, 2004; L. Horwitz *et al.* (eds.).

[5] Remarkably in the beautiful recent account by Sharpe(26) of the Cartan transfer method and the theory of the Cartan connections and their formalization by Ehressmann, not even a comment appears about its role in the foundations of stochastic differential geometry. While E. Cartan and P. Lévy were contemporaries in Paris, and there is no proof of communication between them (P. Malliavin, personal communication); it is suprising (or probably we should say that is the signature of the fragmentation of knowledge and interests of scholars of the present times) that till today followers of E. Cartan continue

space, as the geometrical construction of the most general diffusion processes on general manifolds.[1,2,11] In this extension, we can approximate smoothly with exception of a zero measure set, the Wiener process and more generally the noise term and the whole random continuous process, by a sequence of almost everywhere differentiable smooth paths, that converge in probability to the latter. From this development method, albeit in configuration space, stemmed our representations for NS and KDE for smooth boundary compact manifolds.[6] These constructions can be extended in a seemingly natural presentation (yet a tour de force in stochastic analysis was necessary for this) to T^*M, which provided with the canonical symplectic structure, will lead us to construct a random symplectic theory for NS and KDE, and a new class of random invariants of generalized Brownian motions and a fortiori of quantum mechanics, yet in this article we shall restrict ourselves to NS and KDE. In the case of NS, the random lagrangian paths that integrate the fluid flow, can be associated with a decomposition of the velocity, like the Reynolds' classical approach to turbulence in fluids, in a classical velocity which obeys NS and a random term, which is defined by the noise tensor from the Riemannian metric which is given; thus, in distinction with the Reynolds approach, we have no closure problem since the random term is not an unknown and the only problem which motivates the definition of the random Hamiltonian system, is the definition of the derivative of the Wiener process which is multiplied by the noise tensor. Yet, as we remarked already, these random flows can be realized by a sequence of almost everywhere classical paths that approximate in probability the random paths both in configuration and phase spaces generated by the generalized laplacians of our geometries, or still, in terms of a generalized stochastic process: the white noise process. This process is defined on a Hilbert rigged space, or still, a Gel'fand triplet, which is the natural analytical setting for quantum mechanics as an operator theory, for quantum field theory in terms of the Feynman path integral and its implementation for gauge theories (see Rapoport and Tilli[48]), and Prigogine's spectral approach to dynamical systems and the problem of the time-arrow.[45] We shall also briefly discuss the non-uniqueness of these a.e. smooth approximations, as they can also be realized in terms of almost everywhere (a.e) smooth polygonals RCW autoparallel[6] paths (i.e. the 'straightest' paths defined by

to be unaware of the role of his work in the foundations of a geometrically invariant theory of stochastic processes.

[6] The Poincaré group theory of gravitation is based on the extension of General Relativity (GR) by considering Cartan connections with torsion, say ∇, while GR is framed in terms of metrics g and the associated Levi-Civita connections ∇^g. In this theory,[28,56]

the RCW connection), and thus autoparallels which hitherto could not be identified with the motion of any physical system (spinless test-particles submitted to the curvature and torsion of a Cartan connection follow the geodesic flow, not the autoparallels[(34)]) adquire a meaning in terms of the random structure determined by the RCW connection. As an application of our general constructions, the Euclidean cases R^2 and R^3 for NS and KDE will be fully presented.

2. RIEMANN–CARTAN–WEYL GEOMETRY OF DIFFUSIONS

In this section we follow Refs. (3, 5). In this article M denotes a smooth connected compact orientable n-dimensional manifold (without boundary). We shall further provide M with a linear (or, still, affine[(36)]) connection described by a covariant derivative operator ∇ which we assume to be compatible with a given metric g on M, i.e. $\nabla g = 0$. Given a coordinate chart (x^α) $(\alpha = 1, \ldots, n)$ of M, a system of functions on M (the Christoffel symbols of ∇) are defined by $\nabla_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\gamma} = \Gamma(x)^\alpha_{\beta\gamma} \frac{\partial}{\partial x^\alpha}$. The Christoffel coefficients of ∇ can be decomposed as:

$$\Gamma^\alpha_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{1}{2} K^\alpha_{\beta\gamma}. \tag{1}$$

The first term in (1) stands for the metric Christoffel coefficients of the Levi–Civita connection ∇^g associated to g, i.e. $\left\{ {\alpha \atop \beta\gamma} \right\} = \frac{1}{2}(\frac{\partial}{\partial x^\beta} g_{\nu\gamma} + \frac{\partial}{\partial x^\gamma} g_{\beta\nu} - \frac{\partial}{\partial x^\nu} g_{\beta\gamma}) g^{\alpha\nu}$, and

$$K^\alpha_{\beta\gamma} = T^\alpha_{\beta\gamma} + S^\alpha_{\beta\gamma} + S^\alpha_{\gamma\beta}, \tag{2}$$

is the contortion tensor, with $S^\alpha_{\beta\gamma} = g^{\alpha\nu} g_{\beta\kappa} T^\kappa_{\nu\gamma}$, and $T^\alpha_{\beta\gamma} = (\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta})$ the skew-symmetric torsion tensor. We are interested in (one-half) the Laplacian operator associated to ∇, i.e. the operator acting on smooth functions on M defined as

$$H(\nabla) := 1/2\nabla^2 = 1/2 g^{\alpha\beta} \nabla_\alpha \nabla_\beta. \tag{3}$$

the term 'autoparallels' has been coined to indicate the equations of inertial motion derived from ∇. A more proper name would be ∇-geodesics (which is seldom used): Indeed, if we consider the restricted case of null torsion so that ∇ becomes ∇^g, the ∇^g-geodesic flow is the usual geodesic flow of a metric g. Nevertheless, since the term 'autoparallels' is the one identifiable by the workers in the field, we shall keep it to facilitate its interpretation.

A straightforward computation shows that $H(\nabla)$ only depends in the trace of the torsion tensor and g, since it is

$$H(\nabla) = 1/2\triangle_g + \hat{Q}, \tag{4}$$

with $Q := Q_\beta dx^\beta = T^\nu_{\nu\beta} dx^\beta$ the trace-torsion one-form and where $\hat{Q}$ is the vector field associated to Q via g: $\hat{Q}(f) = g(Q, df)$, for any smooth function f defined on M. Finally, $\triangle_g$ is the Laplace–Beltrami operator of g: $\triangle_g f = \mathrm{div_g}\ \mathrm{grad} f,\ f \in C^\infty(M)$, with $\mathrm{div_g}$ and grad are the Riemannian divergence and gradient, respectively. Thus for any smooth function, we have $\triangle_g f = 1/[det(g)]^{\frac{1}{2}} g^{\alpha\beta} \frac{\partial}{\partial x^\beta}([det(g)]^{\frac{1}{2}} \frac{\partial}{\partial x^\alpha} f)$. Consider the family of 0-th order differential operators acting on smooth k-forms, i.e. differential forms of degree k ($k = 0, \ldots, n$) defined on M:

$$H_k(g, Q) := 1/2\triangle_k + L_{\hat{Q}}, \tag{5}$$

In the first summand of the r.h.s. of (5), we have the Hodge operator acting on k-forms:

$$\triangle_k = (d - \delta)^2 = -(d\delta + \delta d), \tag{6}$$

with d and δ the exterior differential and codifferential operators, respectively, i.e. δ is the adjoint operator of d defined through the pairing of k-forms on M: $(\omega_1, \omega_2) := \int \otimes^k g^{-1}(\omega_1, \omega_2)\mathrm{vol}_g$, for arbitrary k-forms ω_1, ω_2, where $\mathrm{vol}_g(x) = \det(g(x))^{\frac{1}{2}} dx$ is the volume density, g^{-1} denotes the induced metric on 1-forms and $\otimes^k g^{-1}$ the induced metric on k-forms. The last identity in (6) follows from the fact that $d^2 = 0$ so that $\delta^2 = 0$. Furthermore, the second term in eq. (5) denotes the Lie-derivative with respect to the vectorfield $\hat{Q}$: $L_{\hat{Q}} = i_{\hat{Q}} d + d i_{\hat{Q}}$, where $i_{\hat{Q}}$ is the interior product with respect to $\hat{Q}$: for arbitrary vectorfields $X_1, \ldots, X_{k-1}$ and ϕ a k-form defined on M, we have $(i_{\hat{Q}}\phi)(X_1, \ldots, X_{k-1}) = \phi(\hat{Q}, X_1, \ldots, X_{k-1})$. Then, for f a scalar field, $i_{\hat{Q}} f = 0$ and

$$L_{\hat{Q}} f = (i_{\hat{Q}} d + d i_{\hat{Q}}) f = i_{\hat{Q}} df = g(Q, df) = \hat{Q}(f). \tag{7}$$

Since $\triangle_0 = (\nabla^g)^2 = \triangle_g$, we see that from the family defined in (5) we retrieve for scalar fields ($k = 0$) the operator $H(\nabla)$ defined in (3 and 4). The Hodge laplacian can be further written expliciting the Weitzenbock metric curvature term, so that when dealing with $M = R^n$ provided with the Euclidean metric, $\triangle_k$ is the standard Euclidean laplacian acting on the components of a k-form defined on R^n ($0 \leqslant k \leqslant n$).

Proposition 1. Assume that g is non-degenerate. There is a one-to-one mapping

$$\nabla \rightsquigarrow H_k(g, Q) = 1/2\triangle_k + L_{\hat{Q}}$$

between the space of g-compatible linear connections ∇ with Christoffel coefficients of the form

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{2}{(n-1)} \left\{ \delta^{\alpha}_{\beta}\ Q_{\gamma}\ -\ g_{\beta\gamma}\ Q^{\alpha} \right\}, \quad n \neq 1 \tag{8}$$

and the space of elliptic second order differential operators on k-forms $(k = 0, \ldots, n)$.

3. RIEMANN–CARTAN–WEYL DIFFUSIONS ON THE TANGENT MANIFOLD

In this section we shall present the setting for the extension of the correspondence of Proposition 1 to a correspondence between RCW connections defined by (8) and diffusion processes of k-forms $(k = 0, \ldots, n)$ having $H_k(g, Q)$ as infinitesimal generators (i.g. for short, in the following).[7] For this, we shall see this correspondence in the case of scalars, and then prepare the extension by defining diffusion processes on the tangent manifold. We have already seen that introduction of more general covariant derivative operators (or still, of linear connections) than the Levi–Civita connection, is naturally associated with the appearance of an interaction term in the generalized laplacians, which is the vectorfield given by the g-conjugate of a trace-torsion 1-form and thus with a RCW connection. We shall further see that in introducing the Wiener processes (white noise) and the rules of stochastic analysis,[(1)] the present approach will lead us to associate the noise tensor of a generalized diffusion process with the Riemannian metric and the trace-torsion interaction term with the drift of a diffusion process.

For the sake of generality, in the following we shall further assume that $Q = Q(\tau, x)$ is a time-dependent 1-form, and we assume $\tau \geqslant 0$.[8] The stochastic flow associated to the diffusion generated by $H_0(g, Q)$ has

[7] Thus, naturally we shall call these processes as *RCW diffusion processes*.

[8] We have written the dependance of Q in terms of the evolution parameter τ which should not be confused with the time variable that may exist in the Riemannian manifold M. Thus, we are in a situation similar to relativistic dynamics based in the evolution

for sample paths the continuous curves $\tau \mapsto x(\tau) \in M$ satisfying the Ito invariant non-degenerate s.d.e. (stochastic differential equation)

$$dx(\tau) = X(x(\tau))dW(\tau) + \hat{Q}(\tau, x(\tau))d\tau. \tag{9}$$

In this expression, $X : M \times R^m \to TM$ is such that $X(x) : R^m \to TM$ is linear for any $x \in M$, so that we write $X(x) = (X_i^\alpha(x))$ $(1 \leqslant \alpha \leqslant n,\ 1 \leqslant i \leqslant m)$ which satisfies $X_i^\alpha X_i^\beta = g^{\alpha\beta}$, where $g = (g^{\alpha\beta})$, and $\{W(\tau), \tau \geqslant 0\}$ is a standard Wiener process on R^m. Taking in account the rules of stochastic analysis[29] for which $dW^\alpha(\tau)dW^\beta(\tau) = \delta^\alpha_\beta d\tau$ (the Kronecker tensor), $d\tau dW(\tau) = 0$ and $(d\tau)^2 = 0$, we find that if $f : R \times M \to R$ is a C^2 function on the M-variables and C^1 in the τ-variable, then a Taylor expansion yields

$$\begin{aligned} f(\tau, x(\tau)) = f(0, x(0)) &+ \left[\frac{\partial f}{\partial \tau} + H_0(g, Q)f\right](\tau, x(\tau))d\tau \\ &+ \frac{\partial f}{\partial x^\alpha}(\tau, x(\tau))X_i^\alpha(x(\tau))dW^i(\tau) \end{aligned} \tag{10}$$

and thus $\frac{\partial}{\partial \tau} + H_0(g, Q)$ is the infinitesimal generator of the diffusion represented by integrating the s.d.e. (9). Furthermore, this identity sets up the so-called martingale problem approach to the random integration of linear evolution equations for scalar fields,[1] and further, for differential forms as we shall see next. Note, that if we start with Eq. (9), we can reconstruct the associated RCW connection.

Our next step, is to extend the above results to differential forms, for which we have to construct diffusions on the tangent manifold. Consider

parameter introduced by Stuckelberg,[37] later elaborated in several pioneering works by Piron and Horwitz,[38] Fanchi,[39] and which bears a relation with Prigogine's Liouvillian time.[45] The τ-invariant case of this theory, makes Q independant of τ, but we stress that it still may depend on t (whenever M is a space-time). In keeping the Riemannian metric (so we are dealing with the local group of orthogonal transformations, in relation with the random Wiener process that has a positive covariance, and is invariant by these transformations, and further invariant by a full conformal group in which the rotations are orthogonal[41]) in contrast with a Lorentzian metric, say Minkowski space, we are formulating a theory which while being invariant by diffeomorphisms, is not relativistic in the sense of having a Lorentz group gauge invariance. This problem has been elaborated recently by Horwitz and Oron,[42] and it will be discussed in the context of Clifford algebras when discussing the random quantization of the Dirac–Hestenes equations, in our accompanying article; a fortiori, the present approach will lead to the non-relativistic equations of viscous fluid-dynamics since M will be a space manifold, in contrast with the theory for relativistic fluids as approached by Horwitz and Sklarz[43] in which M is Minkowski space-time.

the canonical Wiener space Ω of continuous maps $\omega: R \to R^m, \omega(0) = 0$, with the canonical realization of the Wiener process $W(\tau)(\omega) = \omega(\tau)$. The (stochastic) flow of the s.d.e. (9) is a mapping $F_\tau : M \times \Omega \to M, \quad \tau \geqslant 0$, such that for each $\omega \in \Omega$, the mapping $F_.(.\ , \omega)$: $[0, \infty) \times M \to M$, is continuous and such that $\{F_\tau(x)\tau \geqslant 0\}$ is a solution of Eq. (9) with $F_0(x) = x$, for any $x \in M$. Under very general analytical conditions on the components of the noise tensor and drift vectorfield, for each fixed $\omega \in \Omega$, the flow of Eq. (9) defines a diffeomorphism of M.[9] It is most remarkable that the random flow generated by a RCW connection, and a fortiori, by the Levi–Civita connection which only accounts for the noise tensor, define for each fixed Wiener sample path an active diffeomorphism of M, so that the random flows reproduce diffeomorphormically the differentiable structure of M. Let us describe the (first) derivative (or *jacobian*) flow of eq. (9), i.e. the stochastic process $\{v(\tau) := T_{x_0} F_\tau(v(0)) \in T_{F_\tau(x_0)} M, v(0) \in T_{x_0} M\}$; here $T_z M$ denotes the tangent space to M at z and $T_{x_0} F_\tau$ is the linear derivative of F_τ at x_0. The process $\{v_\tau, \tau \geqslant 0\}$ can be described[(12)] as the solution of the invariant Ito s.d.e. on TM:

$$dv(\tau) = \nabla^g \hat{Q}(\tau, v(\tau))d\tau + \nabla^g X(v(\tau))dW(\tau) \tag{11}$$

If we take U to be an open neighbourhood in M so that the tangent space on U is $TU = U \times R^n$, then $v(\tau) = (x(\tau), \tilde{v}(\tau))$ is described by the system given by integrating Eq. (9) and the invariant Ito s.d.e.

$$d\tilde{v}(\tau)(x(\tau)) = \nabla^g X(x(\tau))(\tilde{v}(\tau))dW(\tau) + \nabla^g \hat{Q}(\tau, x(\tau))(\tilde{v}(\tau))d\tau, \tag{12}$$

[9] These analytical conditions are commonplace in p.d.e theory, yet the elaboration of the diffeomorphism theory for random flows, has been an exceptional piece of art in analysis and probability, which is not possible to present otherwise than in a telegram style. Say, we assume the components X_i^α, $\hat{Q}^\alpha$, $\alpha, \beta = 1, \ldots, n$ of the vectorfields X and $\hat{Q}_\tau$ on M in Eq. (9) are predictable (i.e. measurable with respect to the Borel σ algebra of sets defined up to time τ) functions which further belong to $C_b^{m,\epsilon}$ $(0 < \epsilon < 1$, m a non-negative integer), the space of Hölder bounded continuous functions of degree $m \geqslant 1$ and exponent ϵ, and also that $\hat{Q}^\alpha(\tau) \in L^1(R)$, for any $\alpha = 1, \ldots, n$. With these regularity conditions, if we further assume that $x(\tau)$ is a semimartingale on a probability space $(\Omega, \mathcal{F}, P)$,[(8)] then it follows that the flow of Eq. (9) has a modification (which with abuse of notation we denote as) $F_\tau(\omega) : M \to M, \qquad F_\tau(\omega)(x) = F_\tau(x, \omega)$, which is a diffeomorphism of class C^m, almost surely for $\tau \geqslant 0$ and $\omega \in \Omega$.[(8)] We would like to point out that a similar result follows from working with Sobolev space regularity conditions instead of Hölder continuity. Indeed, assume that the components of X and $\hat{Q}$, $X_i^\alpha \in H^{s+2}(M)$ and $\hat{Q}^\beta \in H^{s+1}(M)$, $1 \leqslant i \leqslant m$, $1 \leqslant \beta \leqslant n$, where the Sobolev space $H^s(M) = W^{2,s}(M)$ with $s > \frac{n}{2} + m$, $m \geqslant 1$.[(9)] Then, the flow of Eq. (9) for fixed ω defines a diffeomorphism in $H^s(M, M)$, and hence by the Sobolev embedding theorem, a diffeomorphism in $C^m(M, M)$.[(9)]

with initial condition $\tilde{v}(0) = v_0$. Thus, $\{v(\tau) = (x(\tau), \tilde{v}(\tau)), \tau \geqslant 0\}$ defines a random flow on TM.

4. REALIZATION OF THE RCW DIFFUSIONS BY ODE'S

To realize the s.d.e's by o.d.e's it is mandatory to pass to the Stratonovich pre-prescription, which are well known to have the same transformation rules in stochastic analysis that those of classical flows.(1,2,29) The need for such approximations is obvious whenever the noise tensor is not trivial, and thus the random integration may be extremely difficult; in the trivial noise case it becomes superfluous, as we shall see when dealing with the Euclidean space case further below of this article. Thus, instead of Eq. (9) we consider the Stratonovich s.d.e. (here denoted, as usual, by the symbol $\circ$) for it given by:

$$\begin{aligned} dx(\tau) &= X(x(\tau)) \circ dW(\tau) + b^{Q,X}(\tau, x(\tau))d\tau, \\ &\text{where } b^{Q,X}(\tau, x(\tau)) = \hat{Q}(\tau, x(\tau)) + S(\nabla^g, X)(x(\tau)), \end{aligned} \tag{13}$$

where the drift now contains an additional term, the Stratonovich correction term, given by $S(\nabla^g, X) = \frac{1}{2}\mathrm{tr}(\nabla^g_X X)$, where $\nabla^g_X X$, the Levi–Civita covariant derivative of X in the same direction and thus it is an element of TM, so that in local coordinates we have $S(\nabla^g, X)^\beta = \frac{1}{2} X^\beta_i \nabla^g_{\frac{\partial}{\partial x^\alpha}} X^\alpha_i$. Now we also represent the jacobian flow using the Stratonovich prescription

$$d\tilde{v}(\tau) = \nabla^g X(x(\tau))(\tilde{v}(\tau)) \circ dW(\tau) + \nabla^g b^{Q,X}(\tau, x(\tau))(\tilde{v}(\tau))d\tau. \tag{14}$$

Now we shall construct classical flows to approximate the random flow $\{x(\tau) : \tau \geqslant 0\}$. We start by constructing a piecewise linear approximation of the Wiener process. Thus, we set for each $k = 1, 2, \ldots,$

$$\begin{aligned} W_k(\tau) &= k\left[\left(\frac{j+1}{k} - \tau\right) W\left(\frac{j}{k}\right) + \left(\tau - \frac{j}{k}\right) W\left(\frac{j+1}{k}\right)\right], \\ &\text{if } \frac{j}{k} \leqslant \tau \leqslant \frac{j+1}{k}, \quad j = 0, 1, \ldots \end{aligned} \tag{15}$$

and we further consider the sequence $\{x_k(\tau)\}_{k \in N}$ satisfying

$$\frac{dx_k(\tau)}{d\tau} = X(x_k(\tau))\frac{dW_k}{d\tau}(\tau) + b^{Q,X}(\tau, x_k(\tau)), \tag{16}$$

$$\frac{d\tilde{v}_k(\tau)}{d\tau} = \nabla^g X(x_k(\tau))(\tilde{v}_k(\tau))\frac{dW_k}{d\tau}(\tau) + \nabla^g b^{Q,X}(\tau, x_k(\tau))(\tilde{v}_k(\tau)), \quad (17)$$

$$\frac{dW_k}{d\tau}(\tau) = k\left[W\left(\frac{j+1}{k}\right) - W\left(\frac{j}{k}\right)\right] \quad \text{for } \frac{j}{k} < \tau < \frac{j+1}{k}, \quad (18)$$

(otherwise, it is undefined), so that $\frac{dW_k}{d\tau}(\tau)$ exists for almost all values of τ (a.e., in short in the following). Since $\{W_k(\tau)\}_{k\in N}$ is differentiable a.e., thus $\{x_k(\tau) : x_k(0) = x(0)\}_{k\in N}$ is a sequence of flows obtained by integration of well defined o.d.e's on M a.e., for all $W \in \Omega$. We remark that $\{x_k(\tau)\}_{k\in n}$ depends on the (here chosen canonical) realization of $W \in \Omega$ so that in rigour, we should write $\{x_k(\tau, W, x_0)\}_{k\in N}$ to describe the flow; the same observation is valid for the approximation of the derivative flow below. With the additional assumption that X and Q are smooth, then the previous sequence defines for almost all τ and for all $W \in \Omega$, a flow of smooth diffeomorphisms of M, and thus, the flow $\{v_k(\tau) = (x_k(\tau), \tilde{v}_k(\tau)) : v_k(0) = (x(0), v(0))\}$ defines a flow of smooth diffeomorphisms of TM. In this case, this flow converges uniformly in probability, in the group of smooth diffeomorphisms of TM, to the the flow of random diffeomorphisms on TM defined by Eqs. (13) and (14).[1,2,11]

Remarks 1. There is not an unique construction for the approximation of these random diffeomorphisms by o.d.e's; indeed, the noise term can be alternatively presented in terms of the extension of the Cartan development method, as a sequence of polygonal geodesic paths.[11] Furthermore, in the case of manifolds being immersed in Euclidean space (which will be the case further below) and complete (autoparallels exist for any τ), the latter construction can be extended to a unified setting in which the random diffeomorphisms of a RCW diffusion can be realized (with convergence in probability) by sequences of polygonal autoparallel paths, i.e. smooth a.e. curves of the form $\frac{\nabla^2 x(\tau)}{\partial\tau^2} = 0$, where ∇ is a RCW connection. These approximations are irreversible *per se* in distinction with the above ones, since autoparallels just like geodesics can focus in a point; they can be constructed through the image of the exponential map of ∇ as the image of the parallel random transport by ∇ of a family of linear frames in TM; the presentation of these constructions would increase greatly the length of this article, and can be found in a somewhat long and intricate presentation in Chapter 8, of the masterpiece due to Bismut.[11] This is of great importance, as it allows to establish an original understanding of the role of the autoparallels of ∇ as we shall argue next. Firstly, autoparallels are not the paths followed by spinless particles submitted to an exterior gravitational field described by a linear connection with torsion (the latter a common mistake as in Ref. 56), or more restricted, a RCW connection, which is the geodesic flow as proved independantly of any lagrangian nor Hamiltonian dynamics.[34] This resulted from applying the ideas of E. Cartan's classical developing method and symplectic geometry, to derive the dynamics of relativistic spinning test-particles on exterior gravitational fields turned out to be an outstanding success of this approach, yielding extensions of the well known Papapetrou–Dixon–Souriau equations.[34] So RCW autoparallel polygonal a.e. smooth paths provide approximations of the random continuous of RCW diffusions (or still, of the Feynman path integral representation of their transition density), which as we already remarked, not necessarily should be thought as spinless particles, furthermore, *vis a vis* the construction of a theory of supersymmetric systems which have these motions as their support for the motions of arbitrary degree differential forms; we shall address the latter problem in the next Section.[10]

[10] Most remarkably, in the path integral representation due to Kleinert of the classical action for a scalar path on a time-sliced Euclidean space which through anholonomic coordinate transformation adquires both torsion and curvature, the classical motions appear to be autoparallels and by applying discretization on them, a short-time-t Feynman propagator has been built for *arbitrary* Q which yields the non-relativistic Schroedinger equation where the Schroedinger operator is the non-relativistic version of our present $H_0(g, Q)$. Yet, in this work, the rule for discretization is the Hanggi–Klimontovich (post-point) rule and thus it is not Ito's (middle point) nor the Stratonovich (pre-point) rules; see chapters 10 and 11.[56] Now, the appearance in the present article of $H_0(g, Q)$ as the differential generator of a diffusion process in terms of which the whole theory is constructed, has to do with the need of a diffeomorphism invariant description of a diffusion process and its generator, which requires the introduction of a linear connection,[11] here a RCW connection whose laplacian is $H_0(g, Q)$. Such an approach fixes the discretization rule to be Ito's, and thus the Brownian integral of the theory is given by the random integral flow of Ito's Eq. (9), and thus the Feynman integral which corresponds by analytical continuation on τ of the flow of Eq. (9) still corresponds to a medium-point rule. In the remarkable computational work due to Kleinert (which has a number of intriguing postulates for the definition of the Feynman measure such as a so-called principle of democracy between differentials and increments; see p. 335 in Ref. 56), no connection is made between diffusion processes, the Schroedinger wave function and the exact term of Q, as it shall appear in the accompanying article to the present one. Another result of this approach is that it will yield a modification of the (controversial) coefficient affecting the metric scalar curvature term (see Ref. 56 and references therein), which in the accompanying article to the present one due to this author it will be associated with a generalization of Bohm's quantum potential in a relativistic setting. We would like to remark that in a recent formulation of a $1+1$-dimensional relativistic theory of Brownian motion in phase space, it is claimed that when studying the equilibrium distribution of a free Brownian particle submitted to a heat bath, the post-point rule is the one that leads to the relativistic Maxwell distribution for the velocities; see J. Dunkel and P. Hanggi, arXiv:cond-mat/0411011.

5. RCW GRADIENT DIFFUSIONS OF DIFFERENTIAL FORMS

Assume that there is an isometric immersion of an n-dimensional manifold M into a Euclidean space R^m given by the mapping $f\colon M \to R^m$, $f(x) = (f^1(x), \dots, f^m(x))$. For example, $M = S^n, T^n$, the n-dimensional sphere or torus, respectively, and f is an isometric embedding into R^{n+1}, or still $M = R^m$ with f given by the identity map. The existance of such a smooth immersion is proved by the Nash theorem in the compact manifold case, yet the result is known to be valid as well for non compact manifolds (see vol. I[15]). Assume further that $X(x) : R^m \to T_x M$, is the orthogonal projection of R^m onto $T_x M$ the tangent space at x to M, considered as a subset of R^m. Then, if $e_1, \dots, e_m$ denotes the standard basis of R^m, we have

$$X(x) = X^i(x) e_i, \quad \text{with} \quad X^i(x) = \text{grad } f^i(x), \quad i = 1, \dots, m. \tag{19}$$

We should remark for the benefit of the reader, that although the noise term is provided by the isometric immersion and thus associated as in the general case with the Levi–Civita covariant derivative operator, we still have a more general covariant derivative, in fact a RCW connection, since the drift of the diffusion process will continue to be associated with the g-conjugate of the trace-torsion of this connection, which together with the metric, yields the RCW connection.

So we are interested in the RCW *gradient* diffusion processes on compact manifolds isometrically immersed in Euclidean space, given by Eq. (9) with the diffusion tensor X given by eq. (19). We shall now give the Ito–Elworthy formula for k-forms ($0 \leqslant k \leqslant n$) on compact manifolds which are isometrically immersed in Euclidean space. Recall that the k-th exterior product of k time-dependant vector fields $v_1, \dots, v_k$ is written as $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ and $\Lambda^k(R \times TM)$ is the vector space generated by them. We further denote by $C_c^{1,2}(\Lambda^k(R \times M))$ the space of time-dependant k-forms on M continuously differentiable with respect to the time variable and of class C^2 with respect to the M variable and of compact support with its derivatives.

Theorem 1 (Ito–Elworthy Formula for k-forms[12]). Let M be isometrically immersed in R^m as above. Let $V_0 \in \Lambda^k T_{x_0} M$, $0 \leqslant k \leqslant n$. Set $V_\tau = \Lambda^k(TF_\tau)(V_0)$, the k-th Grassmann product of the jacobian flow of the RCW gradient diffusion with noise tensor $X = \nabla f$. Then $\partial_\tau + H_k(g, \hat{Q})$ is the i.g. (with domain of definition the differential forms of degree k in $C_c^{1,2}(\Lambda^k(R \times M))$) of $\{V_\tau\colon \tau \geqslant 0\}$.

Remarks 2 Therefore, starting from the flow $\{F_\tau : \tau \geqslant 0\}$ of the s.d.e. (9) (or its Stratonovich version given by Eq. (12)) with i.g. given by $\partial_\tau + H_0(g, Q)$, we construct (fibered on it) the derived velocity process $\{v(\tau) : \tau \geqslant 0\}$ given by Eq. (11) (or Eqs. (9) and (12), with the diffusion tensor given by Eq. (19), or still, its Stratonovich version given by Eqs. (13) and (14)) which has $\partial_\tau + H_1(g, Q)$ for i.g. Finally, if we consider the diffusion processes of differential forms of degree $k \geqslant 1$, we further get that $\partial_\tau + H_k(g, Q)$ is the i.g. of the process $\{\Lambda^k v(\tau) : \tau \geqslant 0\}$, on the Grassmannian bundle $\Lambda^k(R \times TM)$, $(k = 0, \ldots, n)$. Note that consistent with our notation, and since $\Lambda^0(TM) = M$ we have that $\Lambda^0 v(\tau) \equiv x(\tau), \forall \tau \geqslant 0$. In particular, $\partial_\tau + H_2(g, Q)$ is the i.g. of the stochastic process $\{v(\tau) \wedge v(\tau) : \tau \geqslant 0\}$ on $(R \times TM) \wedge (R \times TM)$. Thus, as we previously commented, for manifolds isometrically immersed in Euclidean space, the diffusion of differential forms is determined by those of scalar fields, by a tower of Laplacian operators that extends the scalar Laplacian, and by taking simply exterior products of the Jacobian process. We want to remark that this condition of isometric immersion is not essential, and can be dropped completely, building instead inhomogeneous Levi–Civita geodesic equations fibered ontop of the diffusions of scalar fields.(5)

Consider on a smooth manifold M isometrically immersed in Euclidean space, the following initial value problem: We want to solve

$$\frac{\partial}{\partial \tau}\beta = H_k(g, Q)\beta_\tau, \quad \text{with} \quad \beta(0, x) = \beta_0(x), \quad 0 \leqslant k \leqslant n, \tag{20}$$

for an arbitrary time-dependant k-form $\beta = \beta_\tau(x) = \beta(\tau, x)$ defined on M which belongs to $C_c^{1,2}(\Lambda^k(R \times M))$. Then, the formal solution of this problem is as follows:(13) Consider the stochastic differential equation given by running backwards in time Eq. (15):[11]

$$dx^{\tau,s,x} = X(x^{\tau,s,x}) \circ dW(s) + b^{Q,X}(\tau - s, x^{\tau,s,x})ds, x^{\tau,0,x} = x \in M. \tag{21}$$

and the derived velocity process $\{v^{\tau,s,v(x)}, v^{\tau,0,v(x)} = v(x) \in T_x M, 0 \leqslant s \leqslant \tau\}$ which in a coordinate system we write as $v^{\tau,s,v(x)} = (x^{\tau,s,x}, \tilde{v}^{\tau,s,v(x)})$ verifying Eq. (21) and the s.d.e.

$$d\tilde{v}^{\tau,s,v(x)} = \nabla^g X(x^{\tau,s,x})(\tilde{v}^{\tau,s,v(x)}) \circ dW(s) + \nabla^g b^{Q,X}(\tau - s, x^{\tau,s,x})(\tilde{v}^{\tau,s,v(x)})\, ds,$$
$$\tilde{v}^{\tau,0,v(x)} = v(x). \tag{22}$$

[11] We can, of course, solve this problem by running the Ito form(12)

Notice that this system is nothing else that the jacobian process running backwards in time until the beginning $\tau = 0$.

Theorem 2. [12] The formal solution of the initial value problem given by Eqs. (20) is

$$\beta(\tau, x)(\Lambda^k v(x)) = E_x[\beta_0(x^{\tau,\tau,x})(\Lambda^k \tilde{v}^{\tau,\tau,v(x)})]. \tag{23}$$

where the l.h.s. $\Lambda^k v(x)$ denotes the exterior product of k linearly independant tangent vectors at x, and in the r.h.s. $\Lambda^k v^{\tau,\tau,v(x)}$ denotes the exterior product of the flows having initial condition given by $\Lambda^k v(x)$.

Proof. It follows from the Ito–Elworthy formula. □

Remarks 3. Thus, we see that to determine the value at any time τ and point x of $\beta(\tau, x)$ as given by its contraction with an arbitrary k-vector, one takes the initial value β_0 and transports it all along the reversed path in time starting at x while contracting it with the k-vector given by the Jacobian process fibered on it determined by the generalized laplacian on k-forms, and then one takes the average over all such paths of the scalar field given by this contraction. This is the solution of the initial-value martingale-problem posed by Eq. (20).

6. NAVIER-STOKES AND THE KINEMATICAL DYNAMO EQUATIONS, AND RCW GRADIENT DIFFUSIONS

The kinematic dynamo equation for a passive magnetic field transported by an incompressible fluid, is the system of equations[10] for the time-dependant magnetic vectorfield $B(\tau, x) = B_\tau(x)$ on M defined by $i_{B_\tau}\mu(x) = \omega_\tau(x)$ (for $\tau \geqslant 0$), where we recall that μ is the Riemannian volume n-form, $\mu = vol(g) = det(g)^{\frac{1}{2}} dx^1 \wedge \ldots \wedge dx^n$, satisfying the initial-value problem for the "magnetic" $n-1$-form:

$$\partial_\tau \omega + (L_{\hat{u}_\tau} - \nu^m \triangle_{n-1})\omega_\tau = 0, \omega(0, x) = \omega(x), \quad 0 \leqslant \tau, \tag{24}$$

where ν^m is the magnetic diffusivity. Here, the *velocity* 1-form $u_\tau(x) = u(\tau, x)$ satisfies the invariant NS,

$$\frac{\partial u}{\partial \tau} = [\nu \triangle_1 - L_{\hat{u}_\tau}]u_\tau - dp_\tau, \quad \delta u_\tau = 0, \tag{25}$$

where p_τ is a time-dependant function, the pressure, ν is the kinematical viscosity, or either, the Euler equations obtained by setting $\nu = 0$; the second equation in (25) is the incompressibility condition written in invariant form, since $\delta u_\tau = -\mathrm{div}_g \hat{u}_\tau$, where $\hat{u}_\tau$, denotes the velocity vectorfield g-conjugate to u_τ. Note that we can rewrite KDE as

$$\partial_\tau \omega = H_{n-1}(2\nu^m g, -\frac{1}{2\nu^m} u_\tau)\omega_\tau, \omega(0,x) = \omega(x), \quad 0 \leqslant \tau, \tag{26}$$

while NS can be written as the system of equations

$$\frac{\partial u}{\partial \tau} = H_1\left(2\nu g, \frac{-1}{2\nu} u_\tau\right) u_\tau - dp_\tau, \quad \delta u_\tau = 0. \tag{25$'$}$$

By considering the vorticity time-dependant 2-form $\Omega_\tau := du_\tau$, taking in account that $d\triangle_1 u_\tau = \triangle_2 du_\tau = \triangle_2 \Omega_\tau$ and $dL_{\hat{u}_\tau} u_\tau = L_{\hat{u}_\tau} du_\tau = L_{\hat{u}_\tau} \Omega_\tau$ we have equivalent system of equations obtained by applying d to NS,[3,5]

$$\frac{\partial \Omega_\tau}{\partial \tau} = H_2\left(2\nu g, \frac{-1}{2\nu} u_\tau\right) \Omega_\tau, \tag{27}$$

$$H_1(g, 0) u_\tau = -\delta \Omega_\tau, \tag{28}$$

the first one being NS for the vorticity (NSV, in the following) and the second one is the Poisson-de Rham equation, obtained by applying δ to the definition of Ω. Note then that NSV is determined by a RCW connection whose trace-torsion is $Q = \frac{-1}{2\nu} u$ and the metric is $2\nu g$, so that the drift is $2\nu g(\frac{-1}{2\nu u}, \ ) = -g(u, \ ) = -\hat{u}$; a similar result follows for KDE, replacing ν with ν^m.[12] In Refs. (3–7), the geometrical theory of diffusion processes was applied to give exact implicit representations for this system, in terms of stochastic differential equations, and further realize these representations in terms of systems of ordinary differential equations, and still to construct the random symplectic structure. In this article, we shall follow the same line of approach but for KDE, which for $n = 3$ is identical to NS for the vorticity, with ν^m instead of ν, yet we must keep in mind that for KDE we are after B_τ.

[12] Numerical dynamics simulations of NS have indicated that viscous turbulence yields dislocations in fluids;[40] the first motivations to use connections with torsion – still related to the objective of extending General Relativity to account for an angular momentum tensor [28], have been provided by crystals, in which the torsion is related to the dislocation tensor, and curvature to disclinations.[23]

In the following we assume additional conditions on M, namely that it is isometrically immersed in an Euclidean space, so that the diffusion tensor is given in terms of the immersion f by $X = \nabla f$. Thus, let u denote a solution of Eq. (25) (or still, when dealing with KDE exclusively, of the Euler equations with $\nu = 0$) and consider the flow $\{F_\tau : \tau \geqslant 0\}$) of the s.d.e. whose i.g. is $\frac{\partial}{\partial \tau} + H_0(2\nu^m g, \frac{-1}{2\nu^m} u)$; from Eq. (9) and Theorem 1 we know that this is the flow defined by integrating the non-autonomous Ito s.d.e.

$$dx(\tau) = [2\nu^m]^{\frac{1}{2}} X(x(\tau))dW(\tau) - \hat{u}(\tau, x(\tau))d\tau, x(0) = x, \quad 0 \leqslant \tau. \quad (29)$$

We shall assume in the following that X and $\hat{u}$ have the regularity conditions stated in Section 3 so that the random flow of Eq. (29) is a diffeomorphism of M of class C^m. Now if we express the random Lagrangian flow in Stratonovich form

$$dx(\tau) = [2\nu^m]^{\frac{1}{2}} X(x(\tau)) \circ dW(\tau) + b^{-u,X}(\tau, x(\tau))d\tau, \quad (30)$$

with

$$b^{-u,X}(\tau, x(\tau)) = \nu^m \mathrm{tr}(\nabla^g_X X)(x(\tau)) - \hat{u}(\tau, x(\tau))), \quad (31)$$

we can approximate in the group of diffeomorphisms of M this flow by considering the sequence of a.e. o.d.e's

$$\frac{dx_k}{d\tau}(\tau) = [2\nu^m]^{\frac{1}{2}} X(x_k(\tau)) \frac{dW_k}{d\tau}(\tau) + b^{-u,X}(\tau, x_k(\tau)), \quad k \in N, \quad (32)$$

with $\frac{dW_k}{d\tau}$ defined in Eq. (18), and we consider as well the jacobian flow on TM, $\{v(\tau) = (x(\tau), \tilde{v}(\tau))\}$ with $\tilde{v}(\tau)$ satisfying the Stratonovich eqts.

$$\begin{aligned} d\tilde{v}(\tau)(x(\tau)) = {} & [2\nu^m]^{\frac{1}{2}} \nabla^g X(x(\tau))(\tilde{v}(\tau)) \circ dW(\tau) \\ & + \nabla^g b^{-u,X}(\tau, x(\tau))(\tilde{v}(\tau))d\tau, \end{aligned} \quad (33)$$

which can be approximated by $\{x_k(\tau), \tilde{v}_k(\tau))\}_{k\in N}$ given by integrating the a.e. o.d.e.

$$\begin{aligned} \frac{d\tilde{v}_k(\tau)}{d\tau} = {} & [2\nu^m]^{\frac{1}{2}} \nabla^g X(x_k(\tau))(\tilde{v}_k(\tau)) \frac{dW_k}{d\tau}(\tau) \\ & + \nabla^g b^{-u,X}(\tau, x_k(\tau))(\tilde{v}_k(\tau)). \end{aligned} \quad (34)$$

Thus, from Ref. (11) follows that the flow of the system of a.e. o.d.e's given by Eqs. (32) and (34), and under the assumption that u is of class

C^m ($m \geqslant 1$), converges uniformly in probability, in the group of diffeomorphisms of TM of class C^{m-1} to the random diffeomorphism flow given by Eqs. (30) and (33), of the same class, that integrates KDE, as we shall see next.

Let us find the form of the strong solution (whenever it exists) of the initial value problem for KDE, thus we look for a time-dependant $(n-1)$-form $\omega(\tau, x)$ satisfying Eq. (24) where we assume that $\omega(0, x) = \omega_0(x)$ to be of class C^2 (twice differentiable). For this, we run backwards in time the random lagrangian flow Eq. (29): For each $\tau \geqslant 0$ consider the s.d.e. (with $s \in [0, \tau]$):

$$dx^{\tau,s,x} = [2\nu^m]^{\frac{1}{2}} X(x^{\tau,s,x}) \circ dW(s) + b^{-u,X}(\tau - s, x^{\tau,s,x})ds, \quad x^{\tau,0,x} = x. \tag{35}$$

and the derived velocity process $\{v^{\tau,s,v(x)} : v^{\tau,0,v(x)} = v(x) \in T_xM, 0 \leqslant s \leqslant \tau\}$ which in a coordinate system we write as $v^{\tau,s,v(x)} = (x^{\tau,s,x}, \tilde{v}^{\tau,s,v(x)})$ verifying Eq. (35) and the s.d.e.

$$\begin{aligned} d\tilde{v}^{\tau,s,v(x)} &= [2\nu^m]^{\frac{1}{2}} \nabla^g X(x^{\tau,s,x})(\tilde{v}^{\tau,s,v(x)}) \circ dW(s) \\ &\quad + \nabla^g b^{-u,X}(\tau - s, x^{\tau,s,x})(\tilde{v}^{\tau,s,v(x)})ds, \ v_0^{\tau,0,v(x)} \\ &= v(x) \in T_xM. \end{aligned} \tag{36}$$

Let $v^1(x), \ldots, v^{n-1}(x)$ linearly independant vectors in T_xM, be initial conditions for the flow $\tilde{v}^{\tau,x,v(x)}$.

Theorem 3. If there is a $C^{1,2}$ (i.e. continuously differentiable in the time variable $\tau \in [0, T)$, and of class C^2 in the space variable) solution $\tilde{\omega}_\tau(x)$ of the initial value problem, it is

$$\tilde{\omega}_\tau(v^1(x) \wedge \cdots v^{n-1}(x)) = E_x[\omega_0(x^{\tau,\tau,x})(\tilde{v}^{\tau,\tau,v^1(x)} \wedge \cdots \wedge \tilde{v}^{\tau,\tau,v^{n-1}(x)})], \tag{37}$$

where E_x denotes the expectation value with respect to the measure on $\{x^{\tau,\tau,x} : \tau \geqslant 0\}$.

Proof. It is evident from Theorems 1 and 2. □

Remarks 4. Thus, we see that to determine the magnetic $(n-1)$, we see that on running the process backwards in time τ, the initial magnetic $(n-1)$ is deformed along the way by the symmetric deformation tensor of the fluid and furthermore, by the noise tensor. This mathematical result clearly describes the actual physical picture. If we replace above ν^m by the kinematical viscosity ν and the $(n-1)$ form by the vorticity 2-form, a similar representation corresponding to NSV is obtained. Finally, the above realizations by a.e. o.d.e.'s was set to give a meaning to the s.d.e. given by

$$\frac{dx^{\tau,s,x}}{ds} = [2\nu^m]^{\frac{1}{2}} X(x^{\tau,s,x}) \circ \frac{dW(s)}{ds} + b^{-u,X}(\tau - s, x^{\tau,s,x}), \quad x^{\tau,0,x} = x. \tag{38}$$

There is an alternative approach that consists to view this as a generalized random process defined on a rigged Hilbert space so that $dW(s)/ds$ is the so-called white-noise process.[55] Whatever the approach to give a meaning to this equation is, it is most remarkable that we obtain a a conceptually similar approach than the Reynolds decomposition of the viscous fluid's flow, into the classical velocity and a random term. Yet, there is an important difference with the classical approach to turbulence, since the noise tensor x is not an unkown, so that the closure problem is lifted, which we recall that in turbulence theory has to be imposed by ad-hoc consideration[59].

7. THE REPRESENTATIONS fOR NAVIER–STOKES EQUATIONS

We have already seen that NS is equivalent to the system of Eqs. (27) and (28), for which we have an evolution equation (which is seemingly linear if we adopt the vorticity as an independant variable) and Eq. (28) for the Poisson-de Rham equation for the velocity given the vorticity as the source. Thus, we have by simple substitution in Eqs. (35), (36) and (37) of the solution for KDE of ν^m, the magnetic viscosity, by ν, the kinematic viscosity, and taking now ω_0 and ω_τ the initial and time-dependant vorticities.

For solving Eq. (28), we solve a Dirichlet problem on the sets of a partition of unity on M, and the boundary condition on the closure of an open set U of this partition is $u = \phi$ on $\partial \bar{U}$, where $\phi = \phi(\tau, x) = \phi_\tau(x)$ is a time-dependant 1-form such that $\delta\phi_\tau = 0$. Since we have to solve the Dirichlet problem

$$\begin{aligned} H_1(g,0)u_\tau(x) &= -\delta\Omega_\tau(x), \quad \forall x \in U, \\ u_\tau(x) &= \phi(x), \quad \forall x \in \partial U, \end{aligned} \tag{39}$$

the s.d.e. we have to run is

$$d\tilde{x}(\tau) = X(\tilde{x}(\tau))dW(\tau) = X(\tilde{x}(\tau)) \circ dW(\tau) + \mathrm{tr}(\nabla^g_X X)(\tilde{x}(\tau))d\tau \tag{40}$$

$$\begin{aligned} d\tilde{v}(\tau) &= \nabla^g X(\tilde{x}(\tau))(\tilde{v}(\tau))dW(\tau) = \nabla^g X(\tilde{x}(\tau))(\tilde{v}(\tau)) \circ dW(\tau) \\ &\quad + \nabla^g \mathrm{tr}(\nabla^g_X X)(\tilde{x}(\tau))(\tilde{v}(\tau))d\tau, \end{aligned} \tag{41}$$

with initial conditions $\tilde{x}(0) = x$, $\tilde{v}(0) = v(x)$, then the representation for u is

$$\tilde{u}_\tau(x)(v(x)) = E_x^B\left[\phi(\tilde{x}(\tau_e))(\tilde{v}(\tau_e)) + \int_0^{\tau_e} \frac{1}{2}\delta\Omega_\tau(\tilde{x}(s))(\tilde{v}(s))ds\right] \quad (42)$$

where the expectation value is with respect to $p(s, x, y)$ the transition density of the s.d.e. (40) whose i.g. is $H_0(g, 0)$, i.e.the fundamental solution of the heat equation on M:

$$\partial_\tau p(y) = 1/2\triangle_g p(y) \quad (43)$$

with $p(s, x, -) = \delta_x$ as $s \downarrow 0$, where τ_e is the first-exit time of U, i.e. $\tau_e = \inf\{\tau : x_\tau^g \notin U\}$.

Returning to KDE (and NSV), we can approximate Eqs. (35) and (36) by taking the jacobian flow $\{(x_k^{\tau,s,x}, \tilde{v}_k^{\tau,s,v(x)})\}_{k\in N}$ on TM given by

$$\frac{dx_k^{\tau,s,x}}{ds}(s) = [2\nu^m]^{\frac{1}{2}} X(x_k^{\tau,s,x})\frac{dW_k(s)}{ds} + b^{-u,X}(\tau - s, x_k^{\tau,s,x}), \quad x_k^{\tau,0,x} = x, \quad (44)$$

$$\begin{aligned}\frac{d\tilde{v}_k^{\tau,s,v(x)}}{ds}(s) &= [2\nu^m]^{\frac{1}{2}} \nabla^g X\left(x_k^{\tau,s,x}\right)\left(\tilde{v}_k^{\tau,s,v(x)}\right)\frac{dW_k(s)}{ds} \\ &\quad +\nabla^g b^{-u,X}\left(\tau - s, x_k^{\tau,s,x}\right)\left(\tilde{v}_k^{\tau,s,v(x)}\right)ds, \\ \tilde{v}_k^{\tau,0,v(x)} &= v(x) \in T_x M \quad (45) \\ \frac{dW_k(s)}{ds} &= 2^k\left\{W\left(\frac{[2^k s/\tau]+1}{2^k}\right) - W\left(\frac{[2^k s/\tau]}{2^k}\right)\right\}, \quad s \in [0, \tau], (\tau > 0), \quad (46)\end{aligned}$$

with $[z]$ the integer part of $z \in (0, 1]$, is the Stroock & Varadhan polygonal approximation[11]. Thus, we can write the expression:

$$\tilde{\omega}_\tau(v^1(x)\wedge\cdots v^{n-1}(x)) = \lim_{k\to\infty} E_x[\omega_0(x_k^{\tau,\tau,x})(\tilde{v}_k^{\tau,\tau,v^1(x)}\wedge\cdots\wedge\tilde{v}_k^{\tau,\tau,v^{n-1}(x)})]. \quad (47)$$

By replacing ν^m by ν and setting $n = 3$ we have the approximations of the representations of NSV. We can proceed identically for the Poisson-de Rham equation, for which in account of Eqs. (40) and (41) we have to substitute $2\nu^m X$ by X and $b^{-u,X} \equiv b^{0,X}$, the latter being the Stratonovich correction term.

8. KDE AND RANDOM SYMPLECTIC DIFFUSIONS

Starting with a general RCW diffusion of 1-forms generated by $H_1(g, Q)$, we introduce a family of Hamiltonian functions, $\mathcal{H}_k (k \in N)$ defined on the cotangent manifold $T^*M = \{(x, p)/p : T_x M \to R \text{ linear}\}$ by

$$\mathcal{H}_k = \mathcal{H}_{X,k} + \mathcal{H}_Q, \tag{48}$$

with (in the following $\langle -, - \rangle$ denotes the natural pairing between vectors and covectors)

$$\mathcal{H}_{X,k}(x, p) = \left\langle \langle p, X(x) \rangle, \frac{dW_k}{d\tau} \right\rangle, \tag{49}$$

where the derivatives of W_k are given in Eq. (18), and

$$\mathcal{H}_{\hat{Q}}(x, p) = \langle p, b^{Q,X}(x) \rangle. \tag{50}$$

Now, we have a sequence of a.a. classical Hamiltonian flow, defined by integrating for each $k \in N$ the a.a. system of o.d.e.'s

$$\frac{dx_k(\tau)}{d\tau} \equiv \frac{\partial \mathcal{H}_k}{\partial p_k} = X(x_k(\tau)) \frac{dW_k}{d\tau} + b^{Q,X}(x_k(\tau)), \tag{51}$$

$$\begin{aligned} \frac{dp_k(\tau)}{d\tau} = -\frac{\partial \mathcal{H}_k}{\partial x_k} &= -\left\langle \langle p_k(\tau), \nabla^g X(x_k(\tau)) \rangle, \frac{dW_k(\tau)}{d\tau} \right\rangle \\ &\quad -\langle p_k(\tau), \nabla^g b^{Q,X}(\tau, x_k(\tau)) \rangle. \end{aligned} \tag{52}$$

which preserves the canonical 1-form $p_k dx_k = (p_k)_\alpha d(x_k)^\alpha$ (no summation on k!), and then preserves its exterior differential, the canonical symplectic form $S_k = dp_k \wedge dx_k$. We shall denote this flow as $\phi^k_{\cdot}(\omega, .)$; thus $\phi^k_\tau(\omega, .) : T^*_{x_k(0)}M \to T^*_{x_k(\tau)}M$, is a symplectic diffeomorphism, for any $\tau \in R_+$ and $\omega \in \Omega$. Furthermore, if we consider the contact 1-form[14] on $R \times T^*M$ given by $\gamma_k := p_k dx_k - \mathcal{H}_{X,k} d\tau - \mathcal{H}_{\hat{Q}} d\tau, \forall k \in N$, we obtain a classical Poincaré–Cartan integral invariant: Let two smooth *closed* curves σ_1 and σ_2 in $T^*M \times \{\tau = \text{constant}\}$ encircle the same tube of trajectories of the Hamiltonian equations for $\mathcal{H}_k$, i.e. Eqs. (51) and (52); then $\int_{\sigma_1} \gamma_k = \int_{\sigma_2} \gamma_k$. Furthermore, if $\sigma_1 - \sigma_2 = \partial \rho$, where ρ is a piece of the vortex tube determined by the trajectories of the classical Hamilton's equations, then it follows from the Stokes theorem[14] that

$$\int_{\sigma_1} \gamma_k - \int_{\sigma_2} \gamma_k = \int_{\sigma_1} p_k dx_k - \int_{\sigma_2} p_k dx_k = \int_\rho d\gamma_k = 0. \tag{53}$$

Returning to our construction of the random Hamiltonian system, we know already that for X and $\hat{Q}$ smooth, the Hamiltonian sequence of flows described by Eqs. (51) and (52) converges uniformly in probability in the group of diffeomorphisms of T^*M, to the random flow of the system given by Eqs. (30) with Eq. (31) and

$$dp(\tau) = -\langle\langle p(\tau), \nabla^g X(x(\tau))\rangle, \circ\, dW(\tau)\rangle - \langle p(\tau), \nabla^g b^{Q,X}(\tau, x(\tau))d\tau\rangle. \tag{54}$$

Furthermore this flow of diffeomorphisms is the mapping: $\phi_\tau(\omega, ., .)(x, p) = (F_\tau(\omega, x), F_\tau^*(\omega, x)p)$, where $F_\tau^*(\omega, x)$ is the adjoint mapping of the jacobian transformation. This map preserves the canonical 1-form pdx, and consequently preserves the canonical symplectic 2-form $S = d(pdx) = dp \wedge dx$, and thus $\phi_\tau(\omega, .) : T^*_{x(0)}M \to T^*_{x(\tau)}M$ is a flow of symplectic diffeomorphisms on T^*M for each $\omega \in \Omega$. [11] Consequently, $\Lambda^n S$ is preserved by this flow, and thus we have obtained the Liouville measure invariant by a random symplectic diffeomorphism. We shall write onwards, the formal Hamiltonean function on T^*M defined by this approximation scheme as

$$\mathcal{H}(x, p) := \left\langle \langle p, X(x)\rangle, \frac{dW_\tau}{d\tau} \right\rangle + \mathcal{H}_{\hat{Q}}(x, p). \tag{55}$$

We proceed now to introduce the random Poincaré–Cartan integral invariant for this flow. Define the formal 1-form by the expression

$$\gamma := pdx - \mathcal{H}_{\hat{Q}}d\tau - \langle p, X\rangle \circ dW(\tau), \tag{56}$$

and its formal exterior differential (with respect to the $\mathcal{N} = T^*M$ variables only)

$$d_{\mathcal{N}}\gamma = dp \wedge dx - d_{\mathcal{N}}\mathcal{H}_{\hat{Q}} \wedge d\tau - d_{\mathcal{N}}\langle p, X\rangle \circ dW(\tau). \tag{57}$$

Clearly, we have a random differential form whose definition was given by Bismut.[7,11] Let a smooth r-simplex with values in $R_+ \times T^*M$ be given as

$$\begin{aligned} &\sigma : s \in S_r \to (\tau_s, x_s, p_s),\\ &\quad \text{where } S_r = \{s = (s_1, \ldots, s_r) \in [0, \infty)^r, s_1 + \ldots + s_r \leqslant 1\}, \end{aligned} \tag{58}$$

with boundary $\partial\sigma$ the $(r-1)$-chain $\partial\sigma = \sum_{i=1}^{r+1}(-1)^{i-1}\sigma^i$, where σ^i are the $(r-1)$-singular simplexes given by the faces of σ. σ can be extended by linearity to any smooth singular r-chains. We shall now consider the *random continuous* r-simplex, c, the image of σ by the flow of symplectic diffeomorphisms ϕ, i.e. the image in $R \times T^*M$

$$\phi(\tau_s, \omega, x_s, p_s) = (\tau_s, F_\tau(\omega, x_s), F_\tau^*(\omega, x_s)p_s), \text{ for fixed } \omega \in \Omega, \tag{59}$$

where $F_\tau(\omega, x)$ and $F_\tau^*(\omega, x)p$ are defined by Eqs. (30), (31) and (54), respectively.

Then, given α_0 a time-dependant 1-form on $\mathcal{N}$, $\beta_0, \ldots, \beta_m$ functions defined on $R \times \mathcal{N}$, the meaning of a random differential 1-form

$$\gamma = \alpha_0 + \beta_0 d\tau + \beta_i \circ dW^i(\tau), \quad i = 1, \ldots, m, \tag{60}$$

is expressed by its integration on a *continuous* 1-simplex

$$c \colon s \to (\tau_s, \phi_{\tau_s}(\omega, n_s)), \quad \text{where} \quad n_s = (x_s, p_s) \in T^*M, \tag{61}$$

the image by $\phi_.(\omega, .)$, $(\omega \in \Omega)$ the random flow of symplectomorphisms on T^*M, of the smooth 1-simplex $\sigma : s \in S_1 \to (\tau_s, (x_s, p_s))$. Then, $\int_c \gamma$ is a measurable real-valued function defined on the probability space Ω in Ref. (11,7). Now we shall review the random differential 2-forms. Let now $\tilde{\alpha}_0$ be a time-dependant 2-form on $\mathcal{N}$, thus $\tilde{\alpha}_0(\tau, n)$ which we further assume to be smooth. Furthermore, let $\tilde{\beta}_0(\tau, n), \ldots, \tilde{\beta}_m(\tau, n)$ be smooth time-dependant 1 forms on $\mathcal{N}$ and we wish to give a meaning to the random differential 2-form

$$\gamma = \tilde{\alpha}_0 + d\tau \wedge \tilde{\beta}_0 + dW^1(\tau) \wedge \tilde{\beta}_1 + \cdots + dW^m(\tau) \wedge \tilde{\beta}_m. \tag{62}$$

on integrating it on a continuous 2-simplex $c : s \to (\tau_s, \phi_{\tau_s}(\omega, n_s))$, or which we define it as a measurable real valued function on Ω in Ref. (11,7). To obtain the random Poincaré–Cartan invariant we need the following results on the approximations of random differential 1 and 2-forms by classical differential forms. Given as before $\tilde{\alpha}_0$ a time dependant smooth 2-form on $\mathcal{N}$ and time-dependant smooth 1-forms $\tilde{\beta}_1, \ldots, \beta_m$ on $\mathcal{N}$, there exists a subsequence k_i and a zero-measure $\hat{\Omega}$ subset of Ω dependant on $\tilde{\alpha}_0, \tilde{\beta}_1, \ldots, \tilde{\beta}_m$ such that for all $\omega \notin \hat{\Omega}$, $\phi_.^{k_i}(\omega, .)$ converges uniformly on any compact subset of $R_+ \times R^{2n}$ to $\phi_.(\omega, .)$ as well as all its derivatives $\frac{\partial^l \phi^{k_i}}{\partial n^l}(\omega, .)$ with $|l| \leqslant m$, converges to $\frac{\partial^l \phi_.}{\partial n^l}(\omega, .)$,and for any smooth 2-simplex, $\sigma : s \to (\tau_s, n_s)$ valued on $R_+ \times \mathcal{N}$, if

$$\gamma_k = \tilde{\alpha}_0 + d\tau \wedge \left(\tilde{\beta}_0 + \tilde{\beta}_1 \frac{dW_k^1}{d\tau} + \cdots \tilde{\beta}_m \frac{dW_k^m}{d\tau} \right) \tag{63}$$

and if c^k is the 2-simplex given by the image of a smooth 2-chain by the a.a. smooth diffeomorphism $\phi^k_{\cdot}(\omega, .)$ defined by integration of Eqs. (51) and (52): $c^k : s \to (\tau_s, \phi^k_{\tau_s}(\omega, n_s))$, and c is the continuous 2-chain $s \to (\phi_{\tau_s}(\omega, n_s))$, then $\int_{c^{k_i}} \gamma^{k_i}$ converges to $\int_c \gamma$. If instead we take a time-dependant 1-forms α_0 and time-dependant functions $\beta_0, \ldots, \beta_m$ on $\mathcal{N}$ and consider the time-dependant 1-form on $\mathcal{N}$ given by

$$\gamma_k = \alpha_0 + \left(\beta_0 + \beta_1 \frac{dW^1_k}{d\tau} + \cdots \beta_m \frac{dW^m_k}{d\tau}\right) d\tau \tag{64}$$

and for any a.e. smooth 1-simplex $c^k : s \to (\tau_s, \phi^k_{\tau_s}(\omega, n_s))$ then there exists a subsequence k_i and a zero-measure set $\hat{\Omega}$, dependant of $\alpha_0, \beta_0, \ldots, \beta_m$, such that for all $\omega \notin \hat{\Omega}$, $\phi^{k_i}_{\cdot}(\omega, .)$ converges uniformly over all compacts of $R^+ \times R^{2n}$ with all its derivatives of order up to m to those of $\phi_{\cdot}(\omega, .)$, and if c is the continuous 1-simplex $c : s \to (\tau_s, \phi_s(\omega, n_s))$, then $\int_{c^{k_i}} \gamma^{k_i}$ converges to $\int_c \gamma$, with γ defined in Eq. (60).

Then, we can state the fundamental theorem of Stokes for this random setting, which is due to Bismut, Ref. (11), Theorem 3.4). Let c be a random continuous 2-simplex image of an arbitrary smooth 2-simplex by the flow $\phi_{\cdot}(\omega, .)$. There exists a zero-measure set $\tilde{\Omega} \subset \Omega$ such that for any $\omega \notin \Omega$, then $\int_c d\gamma = \int_{\partial c} \gamma$, for any differential random 1-form γ.

In the following in the case defined by KDE, for which $\hat{Q} = -\hat{u}$ with u a solution of NS or Euler equations, so that we set

$$\alpha_0 = pdx, \quad \beta_0 = -\mathcal{H}_{-\hat{u}} \equiv \mathcal{H}_{\hat{u}}, \quad \beta_i = -(2\nu^m)^{\frac{1}{2}} \langle p, X_i \rangle \equiv (2\nu)^{\frac{1}{2}} p_\alpha X^\alpha_i, \\ i = 1, \ldots, m, \tag{65}$$

where $X : R^m \to TM$ with $X(x) = \operatorname{grad} f$ with $f : M \to R^d$ is an isometric immersion of M, then

$$\begin{aligned} \gamma_{\text{KDE}} &= pdx + \mathcal{H}_{\hat{u}} d\tau - (2\nu^m)^{\frac{1}{2}} \langle p, X \rangle_i \circ dW^i(\tau) \\ &= p_\alpha (dx^\alpha + (b^{u,X})^\alpha d\tau - (2\nu^m)^{\frac{1}{2}} X^\alpha_i \circ dW^i(\tau)), \end{aligned} \tag{66}$$

is the random Poincaré-Cartan 1-form defined on $R^+ \times \mathcal{N}$ for KDE. The Hamiltonian function for KDE is

$$\mathcal{H}(x, p) := [2\nu^m]^{\frac{1}{2}} \left\langle \langle p, X(x) \rangle, \frac{dW_\tau}{d\tau} \right\rangle + \mathcal{H}_{-\hat{u}}(x, p), \tag{67}$$

with

$$\mathcal{H}_{-\hat{u}}(x, p) = p_\alpha (b^{-u,X})^\alpha = g^{\alpha\beta} p_\alpha (-u_\beta + \nu^m X^\alpha_i \nabla^g_{\frac{\partial}{\partial x^\beta}} X^\beta_i) \tag{68}$$

so that the Hamiltonian system is given by the system

$$dx(\tau) = [2\nu]^{\frac{1}{2}} X(x(\tau)) \circ dW(\tau) + b^{-u,X}(\tau, x(\tau))d\tau, \tag{69}$$

with

$$b^{-u,X}(\tau, x(\tau)) = \nu \nabla^g_X X(x(\tau)) - \hat{u}(\tau, x(\tau))) \tag{70}$$

$$\begin{aligned} dp(\tau) = &-(2\nu^m)^{\frac{1}{2}})\langle\langle p(\tau), \nabla^g X(x(\tau))\rangle, \circ\, dW(\tau)\rangle \\ &-\langle p(\tau), \nabla^g b^{-u,X}(\tau, x(\tau))d\tau\rangle. \end{aligned} \tag{71}$$

As in the general case, we then obtain a Liouville invariant measure produced from the n-th exterior product of the canonical symplectic form. Substituting ν^m by ν we obtain the random Poincaré–Cartan invariant γ_{NSV} for NSV.

To obtain the invariants of the full Navier–Stokes equations, we have to consider in addition, the random Hamiltonian flow corresponding to the invariant Poisson-de Rham equation, i.e. Eq. (40) which we rewrite here

$$d\tilde{x}(\tau) = X(\tilde{x}(\tau)) \circ dW(\tau) + S(X(\tilde{x}), g) \circ dW(\tau), \tag{72}$$

and

$$\begin{aligned} d\tilde{p}(\tau) = &-\langle\langle \tilde{p}(\tau), \nabla^g X(\tilde{x}(\tau))\rangle, \circ\, dW(\tau)\rangle \\ &-\langle \tilde{p}(\tau), \nabla^g S(\nabla^g, X)(\tau, \tilde{x}(\tau))d\tau\rangle. \end{aligned} \tag{73}$$

Furthermore this flow of diffeomorphisms preserves the canonical 1-form $\tilde{p}d\tilde{x}$, and consequently preserves the canonical symplectic 2-form $S = d(\tilde{p}d\tilde{x}) = d\tilde{p} \wedge d\tilde{x}$, and thus $\phi_\tau(\omega, .) : T^*_{x(0)}M \to T^*_{x(\tau)}M$ is a flow of symplectic diffeomorphisms on T^*M for each $\omega \in \Omega$.[11] Consequently, $\Lambda^n S$ is preserved by this flow, and thus we have obtained the Liouville measure invariant by a random symplectic diffeomorphism. We shall write onwards, the formal Hamiltonian function on T^*M defined by the approximation scheme for the formal Hamiltonian function

$$\mathcal{H}(\tilde{x}, \tilde{p}) := \left\langle \langle \tilde{p}, X(\tilde{x})\rangle, \frac{dW_\tau}{d\tau} \right\rangle + \langle \tilde{p}, S(\nabla^g, X)(\tilde{x})\rangle. \tag{74}$$

We now proceed to introduce the random Poincaré–Cartan integral invariant for this flow. It is the 1-form

$$\gamma_{\mathrm{Poisson}} := \tilde{p}d\tilde{x} - S(\nabla^g, X)(\tilde{x})d\tau - \langle \tilde{p}, X\rangle \circ dW(\tau). \tag{75}$$

This completes the construction of the random invariants for NS.

9. THE EUCLIDEAN CASE

To illustrate with an example, consider $M = R^n$, $f(x) = x$, $\forall x \in M$, and then $X = \nabla f \equiv I$, the identity matrix, as well as $g = XX^\dagger = I$ the Euclidean metric, and $\nabla^g = \nabla$, is the gradient operator acting on the components of differential forms. Consequently, the Stratonovich correction term vanishes since $\nabla_X X = 0$ and thus the drift in the Stratonovich s.d.e's. is the vector field $b^{-u,X} = -\hat{u} = -u$ (we recall that $\hat{u}$ is the g-conjugate of the 1-form u, but here $g = I$).

We shall write distinctly the cases $n = 2$ and $n = 3$. In the latter case we have that both the vorticity and the magnetic form, say $\Omega(\tau, x)$ are a 2-form on R^3, or still by duality has an adjoint 1-form, or still a R^3-valued function, which with abuse of notation we still write as $\tilde{\Omega}(\tau, .): R^3 \to R^3$. Consider the flows which integrates KDE (for NSV we simply substitute ν^m by ν) is given by integrating the system of equations ($s \in [0, \tau]$)

$$dx^{\tau,s,x} = [2\nu^m]^{\frac{1}{2}} \circ dW(s) - u(\tau - s, x^{\tau,s,x})ds, x^{\tau,0,x} = x, \qquad (76)$$

$$d\tilde{v}^{\tau,s,v(x))} = -\nabla u(\tau - s, x^{\tau,s,x})(\tilde{v}^{\tau,s,v(x)})ds, \tilde{v}^{\tau,0,v(x)} = v(x) \qquad (77)$$

the second being an ordinary differential equation (here, in the canonical basis of R^3 provided with Cartesian coordinates (x^1, x^2, x^3), ∇u is the matrix $(\frac{\partial u^i}{\partial x^j})$ for $u(\tau, x) = (u^1(\tau, x), u^2(\tau, x), u^3(\tau, x))$, which in account that since $\int_0^\tau \circ dW(s) = W(\tau) - W(0) = W(\tau)$, we integrate

$$x^{\tau,s,x} = x + [2\nu^m]^{\frac{1}{2}} W(s) - \int_0^s u(\tau - r, x^{\tau,r,x})dr, \quad s \in [0, \tau], \qquad (78)$$

and

$$\tilde{v}^{\tau,s,v(x)} = e^{-s\nabla u(\tau - s, x^{\tau,s,x})} v(x) \qquad (79)$$

so that the analytical representation for KDE (and alternatively for NSV) in R^3 is

$$\tilde{\Omega}(\tau, x) = E_x[\tilde{v}^{\tau,\tau,I} \Omega_0(x^{\tau,\tau,x})], \qquad (80)$$

where E_x denotes the expectation value with respect to the measure (if it exists) on $\{x_\tau^{\tau,x} : \tau \geqslant 0\}$, for all $x \in R^3$, which is a Gaussian function albeit not centered in the origin of R^3 due to the last term in Eq. (78) and in the r.h.s. of Eq. (80) we have matrix multiplication Thus, in this case, we have that the deformation tensor acts on the initial vorticity along the random paths. This action is the one that for $3D$ might produce the singularity of the solution of NS for 3D.

We finally proceed to present the random symplectic theory for KDE (and alternatively, NSV) on R^3. In account of Eq. (55) with the above choices, the formal random Hamiltonian function is

$$\mathcal{H}(x,p) := [2\nu^m]^{\frac{1}{2}}\left\langle p, \frac{dW(\tau)}{d\tau}\right\rangle + \mathcal{H}_{-\hat{u}}(x,p), \tag{81}$$

with

$$\mathcal{H}_{-\hat{u}}(x,p) = -\langle p, u\rangle. \tag{82}$$

The Hamiltonian system is described by specializing Eqs. (69), (70) and (71), so that we obtain the Stratonovich s.d.e. for $x(\tau) \in R^3, \forall\tau \geqslant 0$:

$$dx(\tau) = [2\nu^m]^{\frac{1}{2}} \circ dW(\tau) - u(\tau, x(\tau))d\tau, \tag{83}$$

and the o.d.e

$$dp(\tau) = -\langle p(\tau), \nabla u(\tau, x(\tau))\rangle d\tau. \tag{84}$$

If we further set $x(0) = x$ and $p(0) = p$, the Hamiltonian flow preserving the canonical symplectic form $S = dp \wedge dx$ on R^6 is given by

$$\begin{aligned}\phi_\tau(.,.)(x,p) = (x(\tau), p(\tau)) = (x + [2\nu^m]^{\frac{1}{2}} W(\tau)\\ - \int_0^\tau u(r, x(r))dr, e^{-\tau\nabla u(\tau,x(\tau))}p).\end{aligned} \tag{85}$$

Finally, the Poincaré–Cartan 1-form takes the form

$$\gamma_{\mathrm{KDE}} = \langle p, dx - ud\tau - (2\nu^m)^{\frac{1}{2}} \circ dW(\tau)\rangle, \tag{86}$$

and the Liouville invariant is $S \wedge S \wedge S$. This, completes the implementation of the general construction on $3D$, for KDE (alternatively, for NSV).

In the case of R^2, the representations for KDE and for NSV are formally different. We start by NSV, for which the vorticity is now a symplectic form, and still can be thought as a pseudoscalar, since $\Omega_\tau(x) = \tilde{\Omega}_\tau(x)dx^1 \wedge dx^2$, with $\tilde{\Omega}_\tau : R^2 \to R$, and being the curvature identically equal to zero, NSV is (a *scalar* diffusion equation)

$$\frac{\partial\tilde{\Omega}_\tau}{\partial\tau} = H_0\left(2\nu I, \frac{-1}{2\nu}u_\tau\right)\tilde{\Omega}_\tau \tag{87}$$

so that for $\tilde{\Omega}_0 = \tilde{\Omega}$ given, the solution of the initial value problem is

$$\tilde{\Omega}(\tau, x) = E_x[\tilde{\Omega}(x^{\tau,\tau,x})] \tag{88}$$

This solution is qualitatively different from the $3D$ case. Due to a geometrical duality argument, for $2D$ we have factored out completely the derived process in which the action of the deformation tensor on the initial vorticity is present. Furthermore, the solution of Eq. (40) is (recall that $X = I$)

$$\tilde{x}(\tau) = x + W_\tau, \tag{89}$$

and since $\nabla X = 0$, the derived process (see Eq. (41)) is constant

$$\tilde{v}(\tau) = v(x) \in T_x R^2, \quad x = \tilde{x}(0), \ \ \forall \tau \in [0, T]. \tag{90}$$

so that its influence on the velocity of the fluid can be factored out in Eq. (42). Indeed, we have

$$\begin{aligned}\tilde{u}_\tau(x)(v(x)) &= E_x^B\left[\int_0^\infty \frac{1}{2}\delta\Omega_\tau(x + W_s)(\tilde{v}^{g,x,v(x)}(s))ds\right]\\ &= E_x^B\left[\int_0^\infty \frac{1}{2}\delta\Omega_\tau(x + W_s)ds(v(x))\right]\end{aligned}$$

for any tangent vector $v(x)$ at x, and in particular (we take $v(x) = I$) we obtain

$$\tilde{u}_\tau(x) = E_x^B\left[\int_0^\infty \tfrac{1}{2}\delta\Omega_\tau(x + W_s)ds\right]. \tag{91}$$

In this expression we know from Eq. (43) that the expectation value is taken with respect to the standard Gaussian function, $p(s, x, y) = (4\pi s)^{\frac{-n}{2}} \exp\left(-\frac{|x-y|^2}{4s}\right)$. For KDE, the magnetic $n-1$-form is a 1-form $\tilde{\omega}_\tau$ defined on R^2, and the expression is

$$\tilde{\omega}(\tau, x)(v(x)) = E_x[\tilde{\omega}_0(x^{\tau,\tau,x})(\tilde{v}^{\tau,\tau,v(x)})] \tag{92}$$

where $(x^{\tau,\tau,x}, \tilde{v}^{\tau,\tau,v(x)})$ are given by integrating Eqs. (76) and (77), now in R^2 (So that ∇u is a two by two matrix.).

Let us describe in further detail the solution of the Poisson-de Rham, separately for each dimension. We note first that if $\Omega_\tau \in L^1 \cap C_b^1$ (where C_b^1 means continuously differentiable, bounded with bounded derivatives)

$$E^B[\delta\Omega_\tau(x + W_s)] = \delta E^B[\Omega_\tau(x + W_s)] \tag{93}$$

In the case $n = 2$, for a 2-form $\tilde{\beta}$ on M we have

$$\delta\tilde{\beta} = \delta(\tilde{\beta}dx^1 \wedge dx^2) = -(\partial_2\tilde{\beta}dx^1 - \partial_1\tilde{\beta}dx^2) \equiv -\nabla^{\perp}\tilde{\beta}. \tag{94}$$

In the case $n = 3$, for a vorticity described by the 1-form (or a vector-valued function) $\tilde{\Omega}_\tau : R^3 \to R^3$ adjoint to the vorticity 2-form Ω_τ, we have that

$$\delta\Omega_\tau = -d\tilde{\Omega}_\tau = -\text{rot } \tilde{\Omega}_\tau. \tag{95}$$

Therefore, we have the following expressions for the velocity: When $n = 2$ we have from Eq. (94)

$$u_\tau(x) = \int_0^\infty -\tfrac{1}{2}\nabla^{\perp}E_x^B[\tilde{\Omega}_\tau(x + W_s)]ds \tag{96}$$

while for $n = 3$ we have from Eq. (95)

$$u_\tau(x) = \int_0^\infty \frac{-1}{2}\text{rot } E_x^B[\tilde{\Omega}_\tau(x + W_s)]ds. \tag{97}$$

Now we can obtain an expression for the velocity which has no derivatives of the vorticity: consider the semigroup generated by $H_0(I, 0) = \frac{1}{2}\triangle$, i.e. $P_s\tilde{\Omega}_\tau(x) = E[\tilde{\Omega}_\tau(x + W_s)]$ (in the case $n = 3$ this means the semigroup given on each component of $\tilde{\Omega}$). From the Elworthy–Bismut formula valid for scalar fields (see Ref. (13)) we have that (in the following $e_i, i = 1, 2, 3$ denotes the canonical base in R^2 or R^3)

$$\begin{aligned}\partial_i P_s\tilde{\Omega}_\tau(x) \equiv \langle dP_s\tilde{\Omega}(x), e_i\rangle &= \frac{1}{s}E_x^B\left[\tilde{\Omega}(x + W_s)\int_0^s \langle e_i, dW_r\rangle\right]\\ &= \frac{1}{s}E_x^B\left[\tilde{\Omega}(x + W_s)\int_0^s dW_r^i\right] = \frac{1}{s}E^B\left[\tilde{\Omega}_\tau(x + W_s)W_s^i\right].\end{aligned} \tag{98}$$

Therefore, for $n = 2$ we have from Eqs. (96) and (98)

$$u_\tau(x) = -\int_0^\infty \frac{1}{2s}E_x^B[\tilde{\Omega}_\tau(x + W_s)W_s^{\perp}]ds \tag{99}$$

where $W_s^{\perp} = (W_s^1, W_s^2)^{\perp} = (W_s^2, -W_s^1)$. Instead, for $n = 3$ we have from Eqs. (97) and (98) that

$$u_\tau(x) = -\int_0^\infty \frac{1}{2s}E_x^B[\tilde{\Omega}_\tau(x + W_s) \times W_s]ds \tag{100}$$

where $\times$ denotes the vector product and $W_s = (W_s^1, W_s^2, W_s^3) \in R^3$ a Wiener process. Thus, for NSV we have obtained for $2D$ and $3D$ the precise form of the random Poincaré–Cartan invariants.

To complete our symplectic representations for NS, we still have to give the symplectic structure associated to Eq. (39) (Poisson-de Rham) for both R^2 and R^3. This structure is the same in both cases, the only difference is in the form of the random Liouville invariant. Indeed, the random Hamiltonean system for Poisson-de Rham is given by Eqs. (72) and (73), which in the Euclidean case the former yields Eq. (89), while the latter is $d\tilde{p}(\tau) = 0$, so that if $\tilde{p}(0) = p$, then the random symplectic flow for Poisson-de Rham equation is given by

$$\phi_\tau(.,.)(x, p) \equiv (\tilde{x}(\tau), \tilde{p}(\tau)) = (x + W(\tau), p), \tag{101}$$

and the Liouville invariant is $\tilde{S} \wedge \tilde{S}$ for $n = 2$, and $\tilde{S} \wedge \tilde{S} \wedge \tilde{S}$ for $n = 3$, where $\tilde{S} = d\tilde{p} \wedge d\tilde{x}$ is the canonical symplectic form for both cases, for the Poisson-de Rham equation. In distinction with the random symplectic invariants for NSV, here the momentum is constant, and of course, the position variable does no longer depend manifestly on u.

Remarks. Geometrical–topological invariants in magnetohydrodynamics and hydrodynamics have been widely studied.[10,16,18] We have followed the presentation in Refs. (7,35) which lead to the random symplectic invariants of NS, hitherto unkown. The present approach applies as well to the random quantization of quantum mechanics through stochastic differential equations, as we shall present in the accompanying article, and thus we shall have random phase invariants which have been unnoticed till today.

10. DERIVATION OF THE SYMPLECTIC STRUCTURE FOR PERFECT FLUIDS

We have seen that NS has an associated Hamiltonian function and a Liouville invariant, and thus we have in principle the basic elements to develop a statistical mechanics approach to NS. The purpose of this section, is to obtain the symplectic structure for the Euler equations from our perspective. Indeed, note that if we set $\nu = 0$ in our random Hamiltonian system we have a classical limit whose dynamics is described by the characteristics curves defined by the integral curves of $-\hat{u}$, i.e. (minus) the velocity vector-field. Indeed, if we set the kinematical viscosity ν to zero in Eqs. (69), (70) and (71) we obtain

$$\frac{dx(\tau)}{d\tau} = -\hat{u}(\tau, x(\tau)), \tag{102}$$

$$\frac{dp(\tau)}{d\tau} = -\langle p(\tau), \nabla^g u(\tau, x(\tau))\rangle. \tag{103}$$

Now, on integrating Eq. (102) with some given initial condition $x(0)$, we obtain a family (indexed by time) of classical diffeomorphisms of M which to $x(0)$ associates the position $x(\tau)$ of the fluid particles with velocity vector field given by $-\hat{u}(\tau, x(\tau))$; in fact for each τ this diffeomorphisms preserves the Riemannian volume since $\hat{u}$ is divergenceless. Thus, it follows from our particular case for a perfect incompressible fluid obeying the Euler equations (set $\nu = 0$ in Eq. (25)), that the configuration space is given by the volume preserving diffeomorphisms of M, which we denote by SDiff(M) which is nothing else than the starting point the AEM theory; by contrast in the present approach the configuration space for NS are the random diffeomorphisms defined by the lagrangian flow described above, which is *not* volume preserving but in the special case of Euclidean space for which $X = \mathrm{Id}$.

Now SDiff(M) is an infinite-dimensional Lie group, and we are interested – in following Arnold – in its Lie algebra, which is the set of divergenceless vector fields on M, SVect(M) provided with the usual commutator. Arnold further considered the orbits of the coadjoint action of this group on the dual of the Lie algebra, as a Hamiltonian system whose Hamiltonian function is (c.f. definition 7.20 and Lemma-definition 7.21 in Ref.(10)) (following the notation after Eq. (6) above)

$$\tfrac{1}{2}\big([u_\tau], [u_\tau]\big) = \tfrac{1}{2}\int_M g\big([\hat{u}_\tau], [\hat{u}_\tau]\big)\mathrm{vol}_g, \tag{104}$$

where $[u_\tau]$ denotes the equivalence class of all 1-forms on M of the type $u_\tau + df$, with $\delta u_\tau = 0$ and some function $f{:}M \to R$, which is nothing else than

$$-\tfrac{1}{2}\int_M \mathcal{H}(x, [u_\tau])\mathrm{vol}_g := \tfrac{1}{2}\int_M \mathcal{H}_{[\hat{u}_\tau]}(x, [u_\tau])\mathrm{vol}_g, \tag{105}$$

which coincides with Arnold's energy function on SVect(M)*, the dual Lie algebra of SVect(M). From the minimal action principle Arnold obtained finally the geodesic equation in SDiff(M). But we can obtain these equations directly in our setting if we further set $p \equiv u$ in Eq. (103) , so that Eqs. (102) and (103) turn to be the geodesic equation on SDiff(M):

$$\frac{d^2x(\tau)}{d\tau} + \nabla^g_{\hat{u}_\tau(x)} u_\tau(x(\tau)) = 0, \tag{106}$$

which in account of the identity

$$\nabla^g_{\hat{u}_\tau(x)} u_\tau(x) = L_{\hat{u}_\tau(x)} u_\tau(x) - \tfrac{1}{2} d(|u_\tau|^2), \tag{107}$$

we get the Euler equation (see p. 37, 38 in Ref. (10))

$$\frac{\partial u}{\partial \tau} + L_{\hat{u}_\tau(x)} u_\tau(x) = \tfrac{1}{2} d(|u_\tau|^2) \tag{108}$$

identically to set $\nu = 0$ in Eq. (25). Note here that the pressure function $\tilde{p}$ reduces to be (modulo an additive constant) $-\frac{1}{2}|u_\tau|^2$, minus the kinetic energy term of u_τ, and the non-appearance of itself the $-d\tilde{p}$ term in the r.h.s. of Eq. (108) is produced by the fact that our random flows for NS have been constructed for the vorticity equation, for which there is no pressure term since $d^2\tilde{p} = 0$; otherwise stated, to obtain the Euler equation we have taken $u_\tau \in [u_\tau]$ such that $f \equiv 0$, and thus the total pressure is

$$f - \tfrac{1}{2}(|u_\tau|^2) = -\tfrac{1}{2}|u_\tau|^2$$

(see comments in first paragraph after Remark 7.22 in Ref. 10). Thus, we have proved that the random symplectic approach to NS yields the classical symplectic approach to the Euler equation, in the case of null viscosity, as a particular result of the kinematics of the random viscous flow. We may remark that Arnold's approach stops short of discussing analytical representations for NS, yet his symplectic approach has been extended by the addition of Wiener processes, to give the representations of NS for the flat torus, by Gliklikh.[32] Probably the present work could be seen as a natural addendum to the joint work by Arnold and Khesin,[10] in which prior to the introduction of the (random) symplectic geometry, one has to introduce first the stochastic differential geometry from which it stems, both aspects being absent in this beautiful treatise.

11. FINAL COMMENTS

We have derived through the association between RCW connections and generalized Brownian motions, the most general implicit analytical representations for NS, in the case of manifolds without boundaries. The case with smooth boundaries and Euclidean semi-space has been treated completely in Ref.(6). Furthermore, in the case without boundary, we have proved that the interaction representation of the solutions of NS, and in general of diffusion processes, in which the trace-torsion plays the major role of describing the average motions, can be gauged away (for any dimension other then 1) and transformed into an equivalent representations in which the trace-torsion enters in the definition of the noise-tensor, as if the random motion would be completely free![5] Yet, concerning NS this article is still unsatisfying, since the representations are implicit, since we have not presented a theory in which we would decouple the velocity 1-form (the gauge potential) and the vorticity 2-form (the 'curvature' field strength). We would like to suggest that if this problem might have a solution, then it should be approached through the application of Clifford algebras and Clifford analysis, in which through the Dirac operator whose square is the NS laplacian, we could integrate the theory in terms of the vorticity alone. This would be similar to the Maxwell equation as a single equation for the electromagnetic field strength (a 2-form, and not in terms of the electromagnetic potential 1-form), as we shall describe in the accompanying article that follows the present one. In forthcoming articles, we shall present the relations between fluid-dynamics and turbulence, electrodynamics and quantum mechanics.

ACKNOWLEDGMENTS

The author would like to express his deep gratitude to Prof. Larry Horwitz and the Organizing Committee of IARD 2004, for their kind invitation to contribute to the Conference, and for sending preprints of his articles, and for the editorial work. As well, our gratitude to the referees for suggesting a number of improvements to the original submission.

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